Chapter 10: Connecting to the Standard Model

We have came a long way in our understanding and delineation of the quantum aspects of our world. It is now time to begin to apply our quantum description to nature on the smallest scale that can be determined with present technology. We will find a rich, detailed set of interactions and particles which make up the bricks and mortar of our physical world.

We have covered some of the important dynamical principles involved, but we have not yet met the particles. We will do so momentarily, but first we need a way to distinguish or categorize the particles that exist in our world. We already know about the dynamics associated with some of the properties of particles: mass, parity, and spin. However, there are also quantum properties which derive from discrete symmetries which characterize a particle and its interactions. There are three such properties I would like to discuss at this point: parity, time reversal and charge conjugation. First, we will become reacquainted with the property known as parity, from a transformation, not quantum, point of view.

I. Discrete Symmetries

Parity

We have of course already discussed the parity operation in this text. The emphasis here will be on the effect of parity on the transformation properties of quantities of quantum fields. Start with the idea of scalar and pseudoscalar fields. Under any orthogonal transformations:

$$\phi'(\vec{x}\,') = \phi(\vec{x}). \quad \text{(a scalar field)} \qquad (10.1)$$

Example: dot product. Proof: $\left(A_i' = \sum_j a_{ij} A_j,\ B_i' = \sum_k a_{ik} B_k\right)$

$$\vec{A}\,'\cdot\vec{B}\,' = \sum_i A_i' B_i' = \sum_{i,j,k} a_{ij} A_j a_{ik} B_k,$$

$$= \sum_{j,k} \underbrace{\left(\sum_i a_{ij} a_{ik}\right)}_{\delta_{jk}} A_j B_k = \vec{A}\cdot\vec{B}\ . \quad (10.2)$$

Another example: $\vec{\nabla}\cdot\vec{A}$ (if $\vec{A}$ is a vector). There are also pseudoscalars. They transform as

$$\phi'(\vec{x}\,') = (\det a)\ \phi(\vec{x}\,). \qquad (10.3)$$

The transformation of vector and pseudovector fields can be characterized by their behavior under rogations. Component statement:

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j \begin{pmatrix}\text{rotate coord: } \underline{\text{passive}}\\ \text{rotate system: } \underline{\text{active}}\end{pmatrix}. \qquad (10.4)$$

<u>Matrix</u> statement:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad (10.5)$$

$$\text{or}\quad x' = ax. \qquad (10.6)$$

So, anything transforming as

$$V_i'(\vec{x}\,') = \sum_j a_{ij}V_j(\vec{x}\,)\ \left(V(\vec{x})\ \text{a vector}\ \underline{\text{field}}\right), \qquad (10.7)$$

under an active rotation is a vector.

Conditions on the a_{ij}? Must preserve the length of vectors.

$$\Rightarrow \quad \sum_i a_{ij}a_{ik} = \delta_{jk}, \qquad (10.8)$$

or

$$a^T a = 1, \qquad (10.9)$$

$$\Rightarrow \quad a^T = a^{-1} \quad \text{"orthogonal".} \qquad (10.10)$$

The above condition represents 6 eq$\underline{^{n}}$s in 9 unknowns
=> 3 a_{ij}'s are free (rotations about x,y,z axes). Determinant:

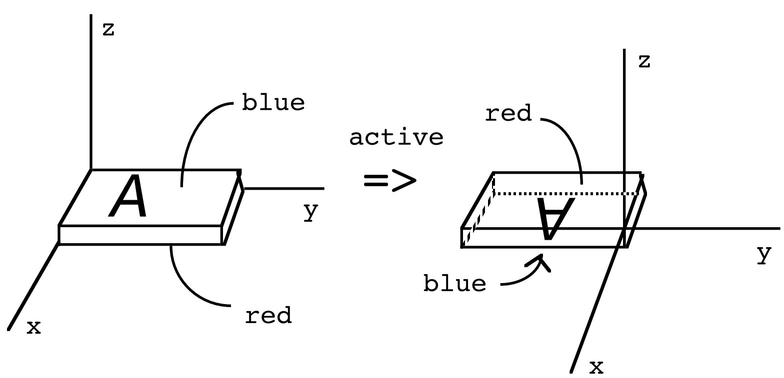
$$\det A = \underbrace{\overset{\Downarrow \text{completely antisymmetric}}{\varepsilon_{ijk\ldots}}}_{n \text{ indices}} \; \underbrace{a_{1i}a_{2j}a_{3k}\ldots}_{n \text{ factors}} \;\;] \qquad (10.11)$$

Take determinant of $a^T a = 1$:

$$\Rightarrow \quad (\det a)^2 = 1, \qquad (10.12)$$

$$\Rightarrow \quad \det a = \pm 1 \quad \underline{\text{only}}. \qquad (10.13)$$

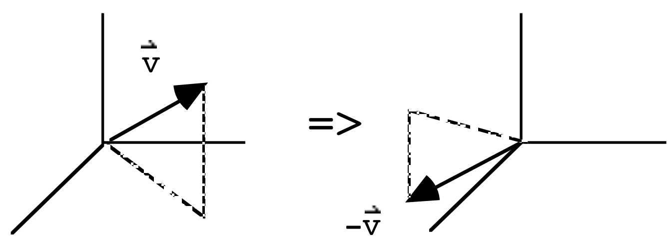
det a = 1 describes pure rotations. What does det a = -1 describe? Example (makes a new object):



Given by:

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Can not be done by a rotation. It is an <u>inversion</u> (also orthogonal). Vectors change sign under a complete inversion:



There is another type of vectorial quantity which, however, is distinguished by its behavior under inversion.
Consider ($\vec{A}$, $\vec{B}$ vectors)

$$(\vec{A} \times \vec{B})_i = \sum_{j,k} \varepsilon_{ijk} A_j B_k. \qquad (10.14)$$

Need result (problem):

$$\sum_{j,k} \varepsilon_{ijk} a_{j\ell} a_{km} = (\det a) \sum_n \varepsilon_{n\ell m} a_{in}. \qquad (10.15)$$

Thus

$$(\vec{A}' \times \vec{B}')_i = \sum_{j,k} \varepsilon_{ijk} A'_j B'_k = \sum_{j,k,\ell,m} \varepsilon_{ijk} a_{j\ell} A_\ell a_{km} B_m$$

$$= \sum_{j,k,\ell,m} \varepsilon_{ijk} a_{j\ell} a_{km} A_\ell B_m = (\det a) \sum_{n,\ell,m} \varepsilon_{n\ell m} a_{in} A_\ell B_m$$

$$= (\det a) \sum_n a_{in} (\vec{A} \times \vec{B})_n. \qquad (10.16)$$

This type of vector (<u>pseudovector</u>) does not change sign under an inversion. It's transformation law is simply

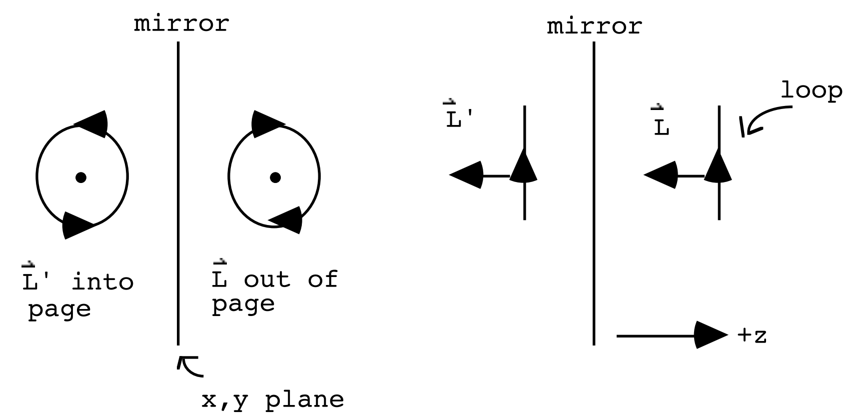
$$V_i'(\vec{x}\,') = (\det a)\sum_j a_{ij}V_j(\vec{x}) \quad \left(V(\vec{x}) \text{ a vector } \underline{\text{field}}\right). \quad (10.17)$$

Example: angular momentum ($\vec{L} = (\vec{x} \times \vec{p})$). Choose:

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Under this a:

$$L_x' = -L_x, \quad L_y' = -L_y, \quad L_z' = +L_z \ .$$



Notice that since $\vec{L}$ is a pseudovector, the lack of change of sign on the z-component under this transformation is what distinguishes it from a vector under the same transformation. Particle spin also behaves as a pseudovector under coordinate transformations.

A problem at the end of the chapter illustrates the use of parity in the properties and "selection rules" for electromagnetic transitions. This discrete symmetry is contained in the more general set of relative linear coordinate transformations, called Lorentz transformations, which preserve only the "proper length", $\Delta s^2 \equiv c^2\Delta t^2 - \Delta\vec{x}^2$, between events, rather than space and time intervals separately. We will learn in an upcoming section that the strong interaction, which conserves parity, is responsible for producing bound states from several particles. For example, meson states are bound states of particles known as quarks. These are classified by their parity, just like atomic states. However, not all interactions in nature conserve parity. We will see that the weak interactions intrinsically do not conserve parity. This is not a small violation for this interaction, but in some sense it is "maximally" violated, and was difficult to confirm in the laboratory only because of the extreme weakness of the interaction involved. This realization was one of the most profound shocks to particle physics in the 20th century.

Time Reversal

A better name for this discrete symmetry is reversal of motion. All non-velocity dependent static forces in classical mechanics are time-reversal invariant. This can be seen from Newton's law:

$$\left.\begin{array}{l} \vec{F} = m\vec{a}, \\ \vec{\nabla}V(\vec{x}) = m\ddot{\vec{x}} \end{array}\right\} \text{invariant under } t \to -t \qquad (10.18)$$

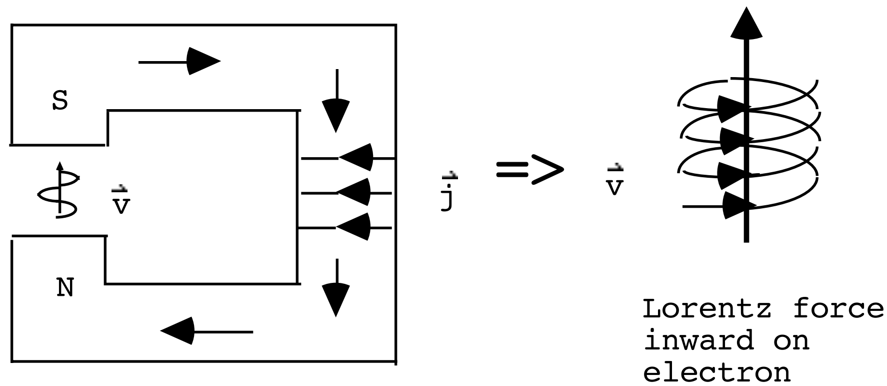
$\Rightarrow$ if $\vec{x}(t)$ is a possible trajectory (solution), then so is $\vec{x}(-t)$

Also, if all one had in the world were electric fields any particle trajectory would be invariant under:

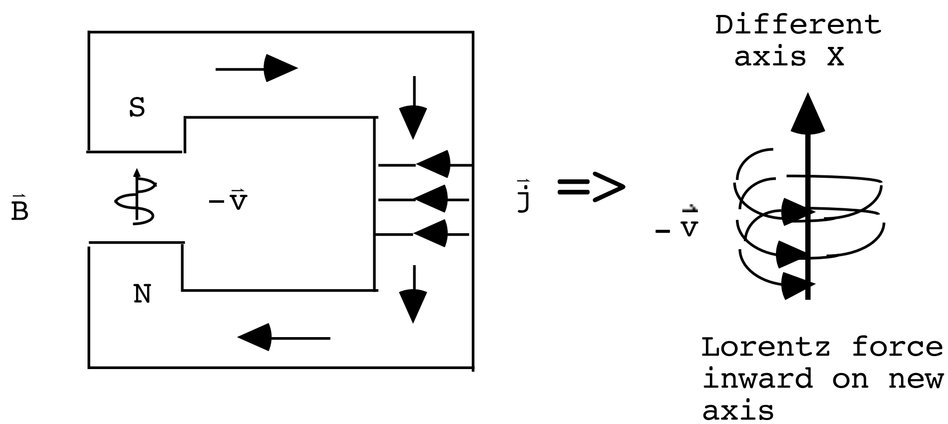
$$\vec{E} \rightarrow \vec{E},\ t \rightarrow -t. \qquad (10.19)$$

$$\text{Since } \vec{F} = q\vec{E} = -q\vec{\nabla}\Phi(\vec{x}) \qquad (10.20)$$

However, magnetic fields are a little trickier. Imagine a world where (electron charge, e<0):



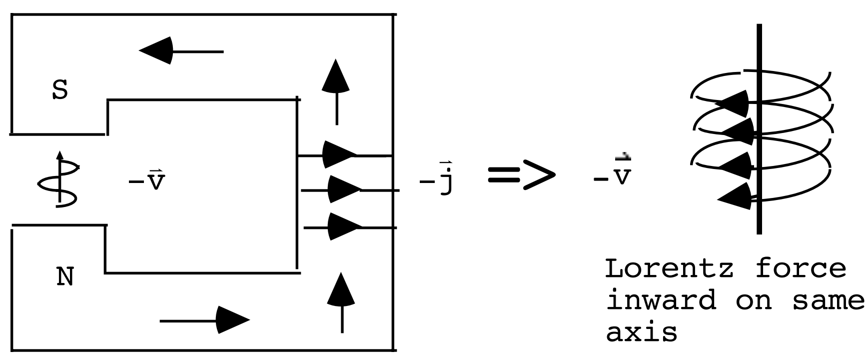
Imagine just letting $t \to -t$, $\vec{v} \to -\vec{v}$.



Electron does <u>not</u> retrace former trajectory. Let

$$t \to -t, \ \vec{v} \to -\vec{v}, \ \vec{j} \to -\vec{j}, \ \vec{B} \to -\vec{B}.$$

↑ Makes sense from point of view of reversal of motion.



Formally, Maxwell's equations are invariant under

$$t \rightarrow -t,\ \vec{E} \rightarrow -\vec{E},\ \vec{B} \rightarrow -\vec{B},\ f \rightarrow f,\ \vec{j} \rightarrow -\vec{j},\ \vec{v} \rightarrow -\vec{v}. \qquad (10.21)$$

What about the Schrödinger equation in this context? It is a first order differential equation in time. We have

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = [\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x})]\psi(\vec{x},t). \qquad (10.22)$$

$\psi^*(\vec{x},-t)$ (complex conjugate) is also a solution:

$$i\hbar\frac{\partial\psi^*(\vec{x},-t)}{\partial t} = [\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x})]\psi^*(\vec{x},-t). \qquad (10.23)$$

So we learn that if $<\vec{x}|\alpha>$ is the t = 0 wavefunction, the time-reversed (motion-reversed) wavefunction is given by $<\vec{x}|\alpha>^* = <\alpha|\vec{x}|>$.

$$<\vec{x}|\alpha>^* \underset{\text{time-reversal}}{\Rightarrow} <\alpha|\vec{x}>. \qquad (10.24)$$

From this very simple observation, we deduce that if we are to introduce an operator which represents time reversal, it is a very unusual object. The reason is that an operator only takes bras into bras and kets into kets. For example:

$$X = \sum_{i,j} |a_i><a_j|X_{ij}, \qquad (10.25)$$

$$\Rightarrow X|a_k> = \sum_i X_{ik}|a_i>, \quad <a_k|X = \sum_i X_{kj}<a_j|. \qquad (10.26a,b)$$

What we want to do now is more akin to an <u>operation</u>, rather than an operator. For example, we had for Hermitian conjugation (see Ch.1, Eq.(188)),

$$(|\alpha\rangle)^+ = \langle\alpha|. \tag{10.27}$$

Under $^+$:

$$(\langle\vec{x}|\alpha\rangle)^+ = \langle\alpha|\vec{x}\rangle. \tag{10.28}$$

However, now consider the momentum space time-reversed wavefunction. We have (t=0),

$$\tilde{\psi}_\alpha(\vec{p}\,') = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x\; e^{-i\vec{p}\,'\cdot\vec{x}/\hbar}\, \tilde{\psi}_\alpha(\vec{x}\,'), \tag{10.29}$$

↑ time-reversed <u>position</u> wavefunction

$$\tilde{\psi}_\alpha(\vec{x}\,') = \tilde{\psi}^*_\alpha(\vec{x}\,'), \tag{10.30}$$

$$\tilde{\phi}_\alpha(\vec{p}\,') = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x\; e^{-i\vec{p}\,'\cdot\vec{x}/\hbar}\, \tilde{\psi}^*_\alpha(\vec{x}\,'),$$

$$= \tilde{\phi}^*_\alpha(-\vec{p}\,'). \tag{10.31}$$

We have found that

$$\langle\vec{p}\,'|\alpha\rangle^* \underset{\text{time-reversal}}{\Rightarrow} \langle\alpha|-\vec{p}\rangle. \tag{10.32}$$

⇒ Time-reversal is not equivalent to "$^+$" (dagger).

We will take the following point of view. Define the effect of a <u>anti-unitary</u> operation:

$$\text{base kets} \begin{cases} (\langle\alpha|)^A = \langle\tilde{\alpha}| = U|\alpha\rangle, \\ (|\beta\rangle)^A = \langle\tilde{\beta}| = \langle\beta|U^+. \end{cases} \tag{10.33a,b}$$

Hermitian conjugation is one such example with U = 1. However, we also define

No star!

$$\left(C_1\langle\alpha| + C_2\langle\beta|\right)^A = \overset{\downarrow}{C_1}|\tilde{\alpha}\rangle + \overset{\downarrow}{C_2}|\tilde{\beta}\rangle, \tag{10.34}$$

$$\left(\langle\alpha|\beta\rangle\right)^{A} = \langle\alpha|\beta\rangle = \left\langle\tilde{\beta}|\tilde{\alpha}\right\rangle. \qquad (10.35)$$

Notice these rules imply

$$\left(\langle\alpha|U\right)^{T} = \left(\langle\alpha|\right)^{T}U^{T}, \qquad (10.36)$$

$$\left(U|\beta\rangle\right)^{T} = (U)^{T}|\beta\rangle^{T}, \qquad (10.37)$$

and that

$$U\left(|\alpha\rangle\right)^{T} = \langle\tilde{\alpha}|U. \qquad (10.38)$$

We now define the effect of the time-reversal <u>operation</u> on $\vec{x}'$, $\vec{p}'$ eigenstates:

$$\left(\langle\vec{x}'|\right)^{T} = |\vec{x}'\rangle, \quad \left(|\vec{x}'\rangle\right)^{T} = \langle\vec{x}'| \qquad (10.39)$$

$$\left(\langle\vec{p}'|\right)^{T} = |-\vec{p}'\rangle, \quad \left(|\vec{p}'\rangle\right)^{T} = \langle-\vec{p}'|. \qquad (10.40)$$

Effect on $\vec{x}, \vec{p}$?

$$\left(\langle\vec{x}'|\vec{x}'\right)^{T} = \left(\langle\vec{x}'|\right)^{T}\vec{x}' = \vec{x}'|\vec{x}'\rangle = \vec{x}^{T}|\vec{x}'\rangle \qquad (10.41)$$

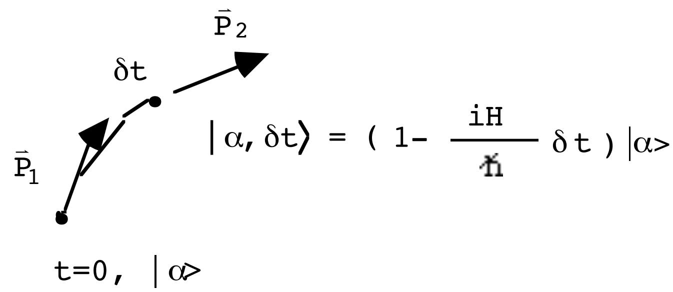
$$\Rightarrow \vec{x}^{T} = \vec{x}. \qquad (10.42)$$

Similarly $\vec{p}^{T} = -\vec{p}$.

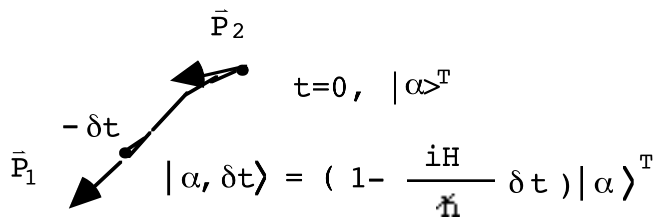
Let us examine the effect of the time reversal operation on particle states using the above formalism. The standard way of time evolving a system for a simple time independent Hamiltonian is

$$|\alpha, \delta t> = \left(1-\frac{iH}{\hbar}\delta t\right)|\alpha>. \qquad (10.43)$$

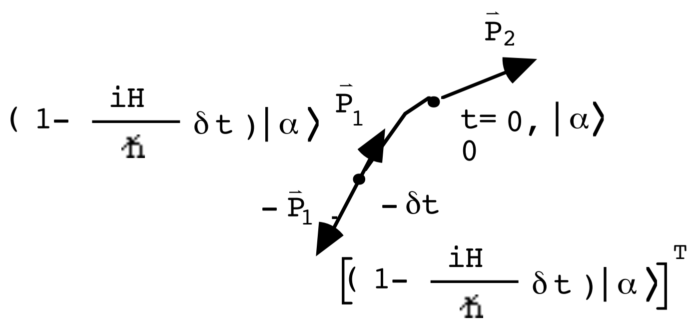
Visualization (pictures after Sakurai, Fig. 4.11):



Change the time origin, motion reverse, and evolve backward in time:



Evolve backward, then motion reverse:



If the system is motion reversal invariant, then we have

$$\langle\tilde{\alpha}| \left(1 + \frac{iH}{\hbar}\delta t\right)^T = \langle\tilde{\alpha}| \left(1 + \frac{iH}{\hbar}\delta t\right) . \qquad (10.44)$$

For any $\langle\tilde{\alpha}|$, this implies $(H)^T = H$.

We are using the special rule that the complex number "i" does not have any special transformational properties under time reversal. There is an immediate consequence of this property, which I will state as a theorem.

Theorem: $H^T = H$ and the energy eigenkets $|n\rangle$ are nondegenerate. Then we may always choose

$$\langle \bar{x}'|n\rangle = \langle \bar{x}'|n\rangle^*. \tag{10.45}$$

Proof:

Given: $H|n\rangle = E_n|n\rangle$. Take ^{T}: $(|n\rangle)^T = \langle \tilde{n}|$.

$$\Rightarrow \langle \tilde{n}|H^T = E_n\langle \tilde{n}|, \qquad \Rightarrow \langle \tilde{n}| = e^{i\delta}\langle n| \text{ (choose } \delta=0\text{)},$$

$$\Rightarrow \langle \bar{x}'|n\rangle = \langle \bar{x}'|n\rangle^T = (\langle \bar{x}'|)^T \cdot (|n\rangle)^T = \langle \tilde{n}|\bar{x}'\rangle,$$

↑ complex number

$$\Rightarrow \langle \bar{x}'|n\rangle = \langle \bar{x}'|n\rangle^*.$$

My treatment of time reversal as an operation, rather than an operator, is not standard. However, much of the Dirac notation formalism developed in earlier chapters must be modified or abandoned if we take the operator point of view. For example, the operator point of view has $\langle\alpha|(\Theta|\beta\rangle) \neq (\langle\alpha|\Theta)|\beta\rangle$ where Θ is the time reversal operator. In addition, even though $(H)^T = H$ characterizes a time reversal invariant Hamiltonian, unlike parity there is no "conservation of time reversal quantum number." The operation point of view makes this explicit since there is no operator that commutes with the Hamiltonian!

Charge Conjugation

Charge conjugation refers to the act of changing particle into anti-particles. we have not yet talked about anti-particles, which are particle states with all additive quantum numbers reversed in sign. (Additive quantum numbers consist of electric charge, the various quark "flavors," such as strangeness, and the three types of lepton number.) Calling this operator C, it's affect on a state with additive quantum numbers A, B, C, would be

$$C|A,B,C\rangle = |-A,-B,-C\rangle. \qquad (10.46)$$

Given that the electric charge, Q, is such a quantum number, one can immediately see that these operators have the property that they ant-commute:

$$\{C,Q\} = 0. \qquad (10.47)$$

This last equation implies that no state of non-zero charge can be an eigenstate of C. In general, this applies for any additive charge. In addition, applying charge conjugation twice gives back the same state, which says that

$$C^2 = I. \qquad (10.48)$$

This equation implies that the eigenvalues of C are ±1, for non-degenerate states, just like parity. It is a multiplicative quantum number, just like parity.

One particle which has no additive charge is the photon. Changing the sign of the charge will change the sign of the photon field, A_μ. We learn in quantum field theory that this field may be used to create or destroy a photon. Thus, each additional photon in a given state changes the sign of the charge conjugation state. This quantum number will be conserved if C commutes with the electromagnetic Hamiltonian causing the decay, and thus the charge conjugation of the initial and finial states are the same. The neutral pion decay,

$$\pi^0 = \gamma + \gamma, \qquad (10.49)$$

implies that $C_{\pi^0} = 1$. This particle is never seen to decay into 3 photons. The neutral pion state is formed from the strong interaction, and thus this is a very strong hint that the strong interactions also conserve charge conjugation. Other neutral states for which charge conjugation conservation are applicable are positronium (e^+e^-) and so-called quarkonium states, such as $c\bar{c}$ (charm/anti-charm).

The following table summarizes the transformation properties of states under the various discrete operations described above.

Table I: Summary of described discrete transformations

Parity: $\vec{J} \rightarrow \vec{J}$ ($\vec{J} = \vec{L}$ or $\vec{S}$), $\vec{p} \rightarrow -\vec{p}$, $\vec{x} \rightarrow -\vec{x}$.

Time Reversal: $\vec{J} \rightarrow -\vec{J}$, $\vec{p} \rightarrow -\vec{p}$, $\vec{x} \rightarrow \vec{x}$.

Charge conjugation: $\vec{J} \rightarrow \vec{J}$, $\vec{p} \rightarrow \vec{p}$, $\vec{x} \rightarrow \vec{x}$,
(All additive quantum numbers such as electric charge, baryon number and the various lepton numbers change sign.)

II. Particle Zoo

In this section we will be briefly introduced to the particles of the so-called "Standard Model" of particle physics. Our world is remarkably and intricately made from a collection of 61 particles, some stable from decay into other fundamental particles and some not. All of these 61 particles are now known directly from experiment with one notable exception to be described later. These particles can be classified according to their types of interactions: the quarks feel the EM, strong, and weak interactions, and the leptons participate in EM and weak interactions. In addition, there are the particle mediators of these interactions, the so-called gauge bosons. For a quick picture of the types of interactions in which these particles can participate, see the pictorial list of allowed particle vertices in Appendix A of the present chapter.

Let us begin first with the quarks. Quarks have spin 1/2 and come in 6 different "flavors" as far as the strong interactions are concerned - these have come to be called "down" (d), "up" (u), "strange" (s), "charmed" (c), and "bottom" (b), and "top" (t). These are presented along with their electric charges (in units of the proton's electric charge) in Table II. (There are also anti-quarks, $\bar{d}$, $\bar{u}$, $\bar{s}$, $\bar{c}$, $\bar{b}$, and $\bar{t}$ with the opposite electric, color charges of their particle partners.) The reason for the grouping of two flavors is that each combination (d,u), (s,c) and (b,t) is considered a different "generation" or "family." Note from the table that the electric charges in each generation are repeated. Each flavor of quark has a unique mass, but specifying their mass values is difficult because, as we will see, quarks are only detected in bound states; individual quarks, such as u or d, are never detected in the laboratory. At high enough energies, one can define the quark masses by analyzing certain experiments. From this method, one finds that the up quark mass is anywhere from about 2 MeV to 4 MeV, the down quark has a mass of from 4 MeV to 8 MeV, and the strange quark has a mass of about 80 MeV to 130 MeV. The first three quarks are considered "light", the last three are "heavy". The mass of the charm quark is about 1.2 GeV, the b-quark has a mass of approximately 4.26 GeV, while the top quark's mass is a whopping 174 GeV. Each quark also has a "baryon number", arbitrarily assigned as 1/3, so that three quark combinations, such as the proton or neutron, have unit baryon number. This quantum number is conserved in all particle interactions as far as is known. Counting up the number of quarks gives a counting of 6 (flavors) x 3 (colors) x 2 (particle/anti-particle) = 36 particles of the 61 standard model particles.

Table II- Quark Additive Quantum Numbers

Flavor	Charge	Baryon Number
u (up)	$^2/_3$	$^1/_3$
d (down)	$-^1/_3$	$^1/_3$
c (charmed)	$^2/_3$	$^1/_3$
s (strange)	$-^1/_3$	$^1/_3$
t (top)	$^2/_3$	$^1/_3$
b (bottom)	$-^1/_3$	$^1/_3$

As I said above, the group of particles known as leptons feel only the EM and weak forces. There are 6 of these particles, just as there are 6 flavors of quarks, and they also have spin 1/2. Also, just as the (d,u), (s,c) and (b,t) combination of quarks forms a different "generation" or "family", the leptons are similarly grouped, but of course their electric charges are different. There are electron and electron neutrinos, (e,ν_e), the muon and muon neutrono, (μ,ν_μ), and the tau and tau neutrino, (τ,ν_τ). (Again, there are also the anti-particles for each of these.) And just like the quarks, the masses of each generation increase ($m_e > m_\mu > m_\tau$), with the possible exception of the neutrinos, which for many years have been thought of as being massless. It is now known that all these particles very likely have small, but non-zero masses. This is a subject of on-going research. Since the neutrinos are neutral, they participate only in weak interactions.

All the leptons in Table III are fundamental (not made up of other particles), but not all of them are stable. For example, the muon and tau leptons (masses of 105.6 MeV and 1.78 GeV, respectively) decay. The primary decay mode for the muon is $\mu \rightarrow e\ \bar{\nu}_e\ \nu_\mu$. The lifetime associated with the muon is $2.2\text{x}10^{-6}$ sec. Notice that in this decay the lepton numbers defined in the table below are conserved. The muon has $L_\mu=1$; this decays into particles having $L_e=1$ (e), $L_e=-1$ ($\bar{\nu}_e$), and $L_\mu=1$ (ν_μ).

The counting of leptons is: 6 (types) x 2 (particle/anti-particle) = 12 of the 61. We are up to 36 + 12 = 48 of 61 particles.

All forces in nature are mediated by other particles, known as gauge bosons. Bosons have integer spin and gauge bosons all have spin 1. The gauge boson in electrodynamics is the photon, the particle of light; the gauge boson in QCD is called the gluon. These two particles are massless and travel at the speed of light in vacuum. There are 8 such particles (see the below discussion under QCD particle interactions). The gauge bosons of the weak interactions are the charged $W^{\pm}$ and the neutral Z^0 bosons. Both of these particles are massive; in fact, the $W^{\pm}$ has a mass of about 86 times that of a proton (80.4 GeV as opposed to .938 GeV), and the other has a mass of about 97 times times a proton (91 GeV).

The counting of gauge bosons is: 8 (gluons) + 2 ($W^{\pm}$) + 1 (photon) + 1 (Z) = 12. We are up to 60 of 61 particles.

Table III: Lepton Additive Quantum Numbers

Lepton	Charge	L_e	L_μ	L_τ
e (electron)	-1	1	0	0
ν_e (electron neutrino)	0	1	0	0
μ (muon)	-1	0	1	0
ν_μ (muon neutrino)	0	0	1	0
τ (tau)	-1	0	0	1
ν_τ (tau neutrino)	0	0	0	1

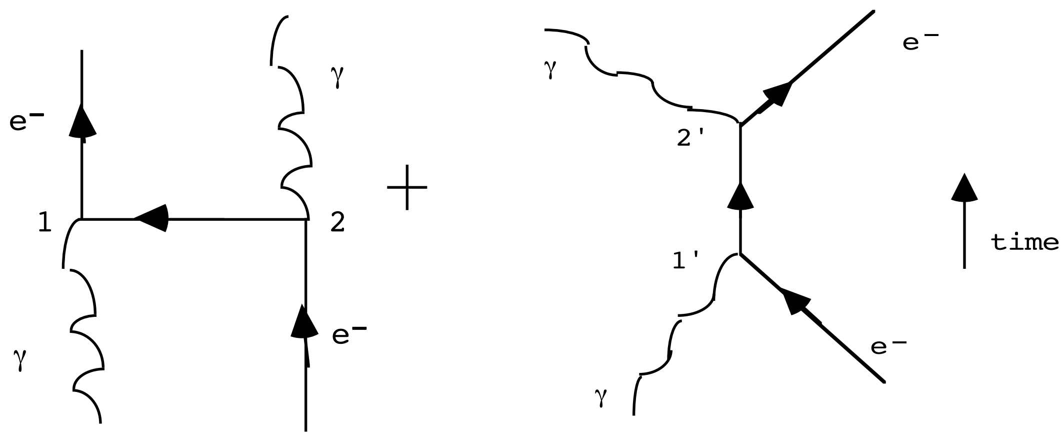
The only particle we have not mentioned yet in the standard model is the Higgs boson. It is a spin 0 neutral scalar particle. It's mass is not determined by the Standard model. It has not been observed, but it's mass is now constrained by various experiments to lie in the range of from about 65 GeV to about 114 GeV. We will have more to say about the Higgs boson in the weak interaction section of the following particle interactions section. There is supposedly only one Higgs boson.

These are the 61 particles of the Standard Model of particle physics.

III. Particle Interactions

Quantum Electrodynamics (QED)

The basic interaction vertex in electrodynamics is shown in the Appendix A. Quantum electrodynamics, QED, is the best knows and most studied aspect of the Standard Model. Using Feynman diagrams, the structure and exact properties of many processes are straightforward to compute. For example, the Feynman diagrams for $e^- \gamma \rightarrow e^- \gamma$ all given to lowest order as



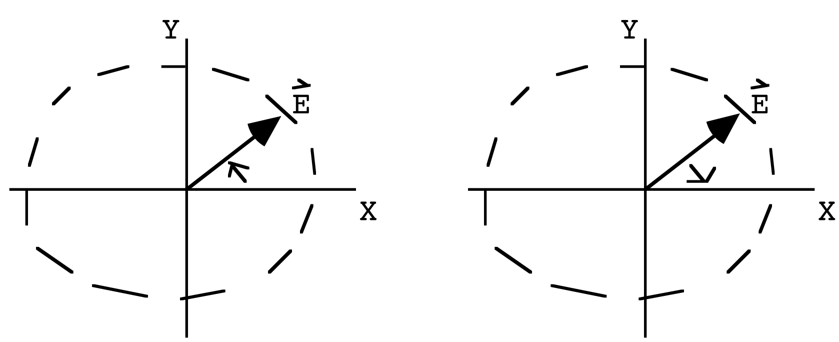
Again, remember that the vertices can come in any time ordering, the only thing invariant is the structure or topology of the diagram. For example,

If "1" is before "2"; e+ line
If "2" is before "1"; e- line

A similar statement can be made about the vertices 1′ and 2′ in the right diagram. The particles which are exchanged in these diagrams, the internal lines in the Feynman diagrams, are called "virtual" particles. We already know a little about the range and lifetime of such particles from the discussion in chapter 7 regarding the pion. There we learned that the Heisenberg uncertainty principle determined these quantities, see Equations

(7.112) and (7.113). It is similar here with the electron; it is exchanged over an approximate range of $\hbar/(m_e c) = 3.9\text{x}10^{-11}$ cm (the so-called electron Compton wavelength) with a time uncertainty of $\hbar/(m_e c^2) = 1.3\text{x}10^{-21}$ sec.

The most fundamental and stringently tested conservation law in nature is the conservation of electric charge. This conservation is intimately tied up with the fact that the photon, the gauge boson of QED, is exactly massless, as we have already pointed out. Massless particles have and important property related to spin: the have only two degrees of freedom, rather than the expected. 2s+1=3 m values of s=1. Formally, the range of massless photons is infinite. It turns out that relativistic field theory requires the spin of massless particles to only point along the direct of motion, $\vec{p}$, as in the opposite direction. That is, the photon's <u>helicity</u>, $\frac{\vec{s}\cdot\vec{p}}{|\vec{p}|}$, can take only the values + 1 or – 1. This particle property is familiar to us as the 2 polarizations of light we know about from classical electrodynamics. These two polarizations can be pictured a follows. Let us say that the motion of a photon is out of the page. Then, the two independent polarizations may be pictured as



The left diagram has an $\vec{E}$ field vector which is instantaneously rotating in a circular path in a counterclockwise fashion. This is called left circular polarization and is associated with negative helicity. Of course, we may use a <u>linear</u> polarization basis to describe light beams; these are just linear combinations of these two polarizations. We also learned in the above section on charge conjugation that the photon has no additive quantum numbers. This implies that there is no such thing as an anti-photon. The photon can be considered it's own anti-particle.

Electrodynamics provides the “glue” which makes atoms possible and essentially all the forces which make up the material around us. QED is so well understood that it can be used to test or understand other forces. For example, in the early 60’s a series of experiments were done at the Stanford Linear Acceleration Center (SLAC) involving the scattering of electrons off of protons and neutrons. ($e^- N \rightarrow e^- X$, where "X" is anything. This is called “inclusive” scattering). These scattering just involved exchanging photons, are one would expect from the theory. What physicists discovered from a careful study is that the cross sections behaved as is the electrons were hitting point objects within the nucleus, just as Rutherford long ago discovered the atomic nucleus from the scattering of b-particles (electrons). This type of interaction is called <u>deep inelastic scattering</u>, and by this physicists discovered quarks. The quarks could not be knocked out of their environment (this is called <u>confinement</u>), but the electromagnetic interaction made it clear that there were a number of point objects in photons and neutrons. Although the quarks could hide themselves within a hardon, they could not escape detection by the infinite-ranged photon!

Quantum Chromodynamics (QCD)

As I said before, the theory of strong interactions is described by QCD. The force is mediated my the gluons and is extremely strong compared, for example, to electromagnetism. This force is Coulombic ($\sim \frac{1}{r^2}$) at small distances and distance independent (~ constant) at large distances. This constant force, or tension, equivalent to about 15 tons, is what is responsible for quark confinement. This string can break at large enough distances, but instead of getting two pieces of string, another quark/anti-quark pair emerges from the vacuum to terminate the string ends, much like what happens to a magnet which is broken in half. Quarks are only seen confined, in hadrons. The energy scales and sizes of hardons are determined by the strength and range of this force. They are the major building blocks, by mass, of our known physical world. Of these, the up and down quarks dominate; most of hadronic matter is made up of neutrons and protons, which are composed of two down-quarks and an up-quark, or by two up-quarks and a down-quark, respectively*. The small mass differences between up and down quarks, down being slightly more massive than the up quarks, is responsible for neutrons being slightly more massive than protons. (There is also a smaller electromagnetic effect due to the charge on the proton which raises it's energy. This also affects other particles, like charged and neutral pions.) Just as electrons carry electric charge, which is absolutely conserved, quarks carry three color charges, which are also absolutely conserved (call them green, blue, and red). The name "color" is arbitrary, but suggestive. Just as white light is composed of a mixture of all the colors in the spectrum, one can make white or colorless combinations of quarks. This can be done in two ways: combining a color with its opposite, making a quark-anti-quark pair (mesons) or by combining three different colored quarks (baryons). Hadronic physics is the study of the properties and interactions of these composite particles.

The theory of the strong interactions is called Quantum Chromodynamics, QCD for short. As pointed out above, in all particle theories, forces are mediated by other particles known as gauge bosons. Bosons have integer spin - gauge bosons all have spin 1. The gauge boson in electrodynamics is the photon, the particle of light; the gauge boson in QCD is called the "gluon".

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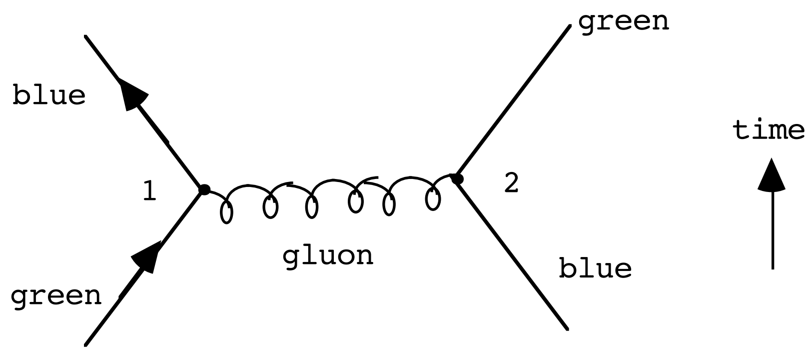
* There is also an admixture of all other types of quarks in protons and neutrons. Quantum field theory tells us that quark/anti-quark pairs of all species are continuously appearing and disappearing in the vicinity of hadrons. These are called "sea quarks", and they can have dramatic physical effects, although these flavors are "hidden".

These two particles are massless. There is one all-important different between protons and gluons. Photons are chargeless, whereas gluons also carry color. To see why, consider the diagram:



The gluon changes the quark color from one vertex to the other and thus carries color also. This gluon carries the color green/anti-blue ($g\overline{b}$) if vertex 1 comes before vertex 2, and blue/anti-green ($b\overline{g}$) for the opposite ordering. (Remember Feynman diagrams are agnostic on time ordering of vertices, so this specification is really not necessary.) There are 9 such combinations of 3 colors; however, the colorless combination, $g\overline{g} + b\overline{b} + r\overline{r}$, does not correspond to a particle, so there are only 8 gluons. The basic vertices for quarks and gluons are shown in Appendix A, where we see that unlike photons, gluons can couple to themselves. They are also confined inside hadrons, like quarks.

The theory of strong interactions can not be formulated and solved in the usual field theory way of using Feynman diagrams, as can electrodynamics or QED. QED interactions are characterized by a particle/photon coupling strength given by the square root of the fine structure constant, $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$, which is small compared to one and allows the Feynman diagrams to be summed to very high order. For QCD, the quark-gluon vertex has an analogous coupling strength, α_{strong}, which is close to one, making the series of Feynman diagrams divergent. This coupling strength is a function of the interaction energy and becomes smaller at high energies due to sea quark contributions (see the footnote on sea quarks in proton, neutrons). Thus for high energy interactions, Feynman diagrams are again useful. This property of the QCD coupling strength is known as "asymptotic freedom." (This property is used to define quark masses at high energies.)

Let us consider an unphysical world where all quarks are "heavy". Then their motions inside hadrons would be nonrelativistic. This is called the "quark model." This model is an incredibly good guide to overall properties, like magnetic moments and mass orderings. QCD does not change any of the flavors into any of the other flavors, so the u, d, etc. quantum numbers are conserved in strong interactions. Since the u, d quarks are considered degenerate in mass in this model, this gives an effective SU(2) in flavor space which is called "isospin." The mathematics of isospin are exactly the same as the two-valuedness property of Chapter 1, and different isospins may be added exactly as we learned in Chapter 8. The fact that this quantum number is conserved in the strong interactions gives us the ratio of certain matrix elements involved in particle interactions (see the problems at the end of the chapter). Tables IV, V, and VI shows the flavor - spin wavefunctions of the lowest mass particles expected in the quark model, along with their isospin classification. (The various K mesons also have a strangeness quarntum number which distinguishes them.)

There is a numerical method for solving QCD, called "lattice QCD", which solves the theory directly and does not depend on the summation of Feynman diagrams. In fact, the degrees of freedom in quantum field theories are not the number of particles involved (like in nonrelativistic quantum mechanics), but the points of space and time themselves. Thus, the simplest field theory already has an infinite number of degrees of freedom.

Table IV: Pseudoscalar Meson Table

	Mass (MeV)	Isospin	$\text{Spin}^{\text{Parity}}$	wave function
$\pi^{\pm}$	139.6	$I=1,\ I_3=\pm 1$	0^-	$u\bar{d}$ $(d\bar{u})$ x (singlet spin)
π^0	135.0	$I=1,\ I_3=0$	0^-	$(u\bar{u}-d\bar{d})/\sqrt{2}$
$K^{\pm}$	493.7	$I=\frac{1}{2},\ I_3=\pm\frac{1}{2}$	0^-	$u\bar{s}$ $(s\bar{u})$
$K^0(\overline{K^0})$	497.3	$I=\frac{1}{2},\ I_3=\mp\frac{1}{2}$	0^-	$d\bar{s}$ $(s\bar{d})$
η	548	$I=0$	0^-	$(2s\bar{s}-u\bar{u}-d\bar{d})/\sqrt{6}$
η'	958	$I=0$	0^-	$(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$

Table V: Vector Meson Table

	Mass (MeV)	Isospin	$\text{Spin}^{\text{Parity}}$	wave function
$\rho^{\pm}$	776	$I=1,\ I_3=\pm 1$	1^-	$u\bar{d}$ $(d\bar{u})$ x (triplet spin)
ρ^0	776	$I=1,\ I_3=0$	1^-	$(u\bar{u}-d\bar{d})/\sqrt{2}$
$K^{*\pm}$	892	$I=\frac{1}{2},\ I_3=\pm\frac{1}{2}$	1^-	$u\bar{s}$ $(s\bar{u})$
$K^{*0}(\overline{K^{*0}})$	899	$I=\frac{1}{2},\ I_3=\mp\frac{1}{2}$	1^-	$d\bar{s}$ $(s\bar{d})$
ω	782	$I=0$	1^-	$(u\bar{u}-d\bar{d})/\sqrt{2}$
ϕ	1020	$I=0$	1^-	$s\bar{s}$

Table VI: Baryon Octet Table

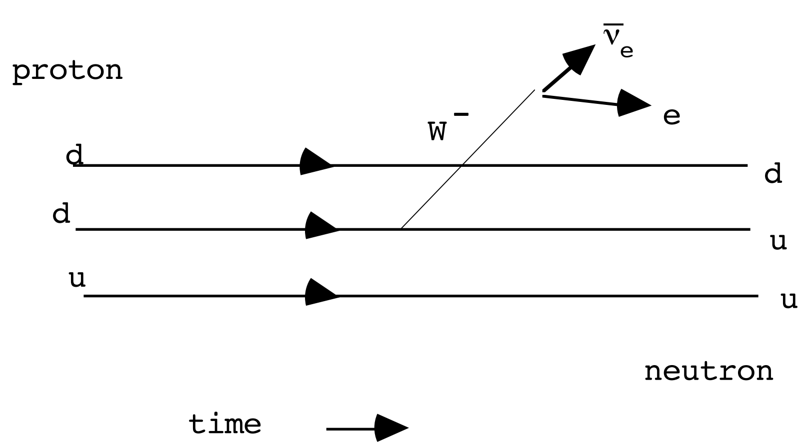
	Mass (MeV)	Isospin	wave function(+cyclic permutations)
p	938.3	$I=\frac{1}{2},\ I_3=+\frac{1}{2}$	$\lvert uud\rangle(2\lvert ++-\rangle-\lvert +-+\rangle-\lvert -++\rangle)/3\sqrt{2}$
n	939.6	$I=\frac{1}{2},\ I_3=-\frac{1}{2}$	$-\lvert ddu\rangle(2\lvert ++-\rangle-\lvert +-+\rangle-\lvert -++\rangle)/3\sqrt{2}$
Λ^0	1115	$I=0$	$(\lvert uds\rangle-\lvert dus\rangle)(\lvert +-+\rangle-\lvert -++\rangle)/2\sqrt{3}$
Σ^+	1189	$I=1,\ I_3=1$	$\lvert uus\rangle(2\lvert ++-\rangle-\lvert +-+\rangle-\lvert -++\rangle)/3\sqrt{2}$
Σ^0	1192	$I=1,\ I_3=0$	$(\lvert uds\rangle+\lvert dus\rangle)(2\lvert ++-\rangle-\lvert +-+\rangle-\lvert -++\rangle)/6$
Σ^-	1197	$I=1,\ I_3=-1$	$\lvert dds\rangle(2\lvert ++-\rangle-\lvert +-+\rangle-\lvert -++\rangle)3\sqrt{2}$
Ξ^0	1314	$I=\frac{1}{2},\ I_3=+\frac{1}{2}$	$\lvert uss\rangle(\lvert ++-\rangle+\lvert +-+\rangle-2\lvert -++\rangle)3\sqrt{2}$
Ξ^-	1321	$I=\frac{1}{2},\ I_3=-\frac{1}{2}$	$\lvert dss\rangle(\lvert ++-\rangle+\lvert +-+\rangle-2\lvert -++\rangle)3\sqrt{2}$

It is this infinitude of degrees of freedom which cause many of the divergences encountered in field theories. In order to control this situation, imagine restricting the number of points to a finite "lattice" of space-time points - the quark degrees of freedom then reside at these points and interact with one another via the gluon fields, which can be considered connections or "links" between the points. It turns out that the entire theory may be formulated in terms of such site-specific quark fields and gluon links. This lattice theory may be put into computer language and numerical methods used to solve for specific quantities. One important numerical technique used in this program is called Monte Carlo. Formally, the quantities being measured in the lattice simulations can formally be expressed as an integral over all the degrees of freedom of the lattice system. The dimensions of this integral are now very large (10's of millions in current simulations) but finite. The Monte Carlo technique allows an estimation of these integrals by simply averaging over N likely values of the integrand. It also allows an estimation of the likely variation in this value if the simulation were repeated many times. Thus, lattice simulations give values and error bars on physical quantities. The error bars may be reduced of course for a larger, more computer-time intensive simulation, but they only fall $1/\sqrt{N}$, where N refers to the number of values in the integrand which have been used in the average.

Weak Interactions

We now come to the most interesting and complicated set of particle interactions, the weak interactions. This theory is also a gauge theory, like E&M (QED) and strong interactions (QCD). In fact, it is considered unified with the electromagnetic interaction and is often referred to as the weak-electromagnetic gauge theory. However, because the forces are mediated by massive particles, the $W^{\pm}$ and the Z^0, the force is extremely short ranged from Heisenberg's uncertainty principle. Although the weak interactions are a gauge theory, they have no associated exactly conserved quantities like electric charge or color charge. The reason for this is quite subtle. We will get back to this point momentarily.

The weak interactions involve both the quarks and leptons, and cause many types of decays of hadronic states. For example, consider the decay $p \to n\, e\, \bar{\nu}_e$.



Here we see a proton entering from the left and a neutron, an electron, and an anti-electron neutrino exiting on the right. This occurs because of the coupling of the W^- particle to both quarks and leptons. This is called a "charged current" interaction because of the difference in the charge of the two other particles at the two W^- vertices. The W^- particle in this diagram is virtual, the other particles can be detected in the laboratory. The effects of the Z^0 particle, on the other hand, are much more subtle. We learned above that the photon has no additive quantum numbers. The Z^0 is also such a particle/anti-particle combination. This means it is present where ever photons are created or destroyed, but because of it's great mass it's range and possible detection are extremely limited.

Although both quarks and leptons participate in the weak interactions, they do so very differently. The quarks are of course all massive, but the neutrinos, for all intents and purposes, are massless. Remember our discussion of particle helicity while discussing photons above. For a massless paricle, whether it is a fermion or a boson (excluding spin 0), the only allowed physical states have helicity values $\pm$ 1, representing spin pointed along or anti-parallel to the direction of motion of the particle. This is very different from an electron, which also has only 2 spin degress of freedom, but these can point "up" or "down" relative to *any* coordinate axis. It turns out that all neutrinos (electron, muon, and tau) participate in interactions as if they were completely left-handed, i.e., their spin is pointed anti-parallel to their direction of motion. Anti-neutrinos are right-handed. This association of left-handedness with neutrinos and right-handedness with anti-neutrinos is intrinsically and maximally parity-violating. In fact, the weak interactions are known to violate all of the discrete symmetries we learned about at the beginning of this chapter. The only symmetry that survives is called CPT, a combined charge conjugation, parity and time reversal.

There is a coupling constant in the weak interactions which is analogous to the electromagnetic (proportional to $\sqrt{\alpha}$) and strong coupling constants ($\sqrt{\alpha_{strong}}$). Let us call it g_W. However, because the charged current interactions the W's are all virtual, the *effective* coupling constant is actually ~ $\frac{g_W}{M_W^2}$, where M_W is the mass of the $W^{\pm}$ particles. This is known as Fermi's coupling constant (I have left out some numerical coefficients in the actual value). The reason the mass of the $W^{\pm}$ particle appears squared in the denominator is due to the fact that 1) Feynman diagrams take place in momentum space and 2) that instead of Coulomb's law, ~1/r, which in momentum space is proportional to $1/(\vec{q}^2)$, an extremely massive particle behaves like a point in space, i.e., a Dirac delta function. The Fourier transform of a Dirac delta function is a constant in momentum space. This constant needs to have the same physical dimensions as our $1/(\vec{q}^2)$ function because they represent different limits of the same function*.

How did the $W^{\pm}$ and Z^0 particles get to be so massive if the weak interactions are just another gauge theory? It is because of an extremely subtle field theory effect known as <u>spontaneous symmetry breaking</u>. This effect depends upon the existence of the Higgs boson, which is another of the particles without any additive quantum numbers. It is thought that as the temperature of the universe cooled, The Higgs particle, ϕ, shifts it's value: $\phi \rightarrow \phi' + v$, where "v" is just a number, called the vacuum expectation value (VEV). This is just like the spntaneous formation of little magnetic domains where the direction of the magnetic fields of the atoms are fixed as the temperature of a

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* This function in field theory is known as the particle propagator, and has is proportional to

$$\frac{1}{q^2+M_W^2},$$

where $|q|$ represents the momentums allowed in a given interaction). For an infinite ranged interaction the mass M_W would be zero and we would have a Coulomb interaction, whereas in the other limit the particle propagator goes is replaced with M_W^2 and represents a point interaction. Actually the q^2 above is a four-dimenensional Lorentz dot product, but the idea is the same.

magnet cools. However, not only does the Higgs field (which is originally a doublet of complex fields having a total of 4 real components) pick out a direction in it's isospin space like a magnet, but also picks up a VEV as it cools. This mechanism is thought to give rise to the masses of the $W^{\pm}$ and Z^0 particles as well as to all the quarks and leptons, via their original couplings (vertices) to the Higgs. In fact, in one of the most colorful phrases in particle physics, the $W^{\pm}$ and Z^0, which were originally massless and therefore had only 2 degrees of freedom (two helicities) are said to "eat" the 3 lost degrees of freedom of the Higgs, acquiring the correct 3 components for massive spin 1 particles. The left over massive particle is the physical Higgs.

One of the most subtle effects in weak interactions is flavor mixing. The weak and strong interactions deal with particle flavor differently. For the quark doublets we saw above that we had the flavor groupings, relevant to the strong interactions:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}. \qquad (10.51)$$

For the weak interactions, the flavor groupings are different:

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}, \begin{pmatrix} t \\ b' \end{pmatrix}. \qquad (10.52)$$

The matrix which connects these particles is called the Cabbibo-Kobayashi-Maskawa, or CKM matrix. It is unitary and three dimensional. That is, one has

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud}, V_{us}, V_{ub} \\ V_{cd}, V_{cs}, V_{cb} \\ V_{td}, V_{ts}, V_{tb} \end{pmatrix} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}. \qquad (10.53)$$

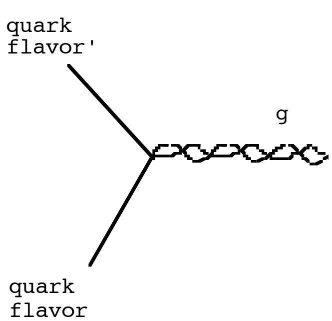
The effects of this mixing on the interactions is too involved to discuss here, but see Appendix B on weak flavor mixing. The Standard Model offers no explanation for the values of the parameters in this matrix. In fact, it is now thought that the same sort of mixing occurs for neutrinos as well, except there the strong interactions are not involved. There is strong evidence that neutrinos have small but nonzero masses from the observation of neutrino flavor change, which can only take place if at least one of the neutrinos is massive. The proliferation of parameters in these matrices is one of the reasons most physicists do not think of the Standard Model as the final theory in particle physics, but simply another step toward a more fundamental theory.

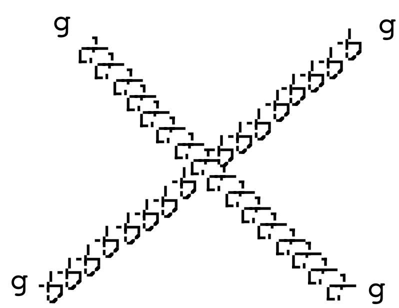
That concludes our very short excursion into particle physics. We have learned about both the principles of quantum mechanics and how they apply to the particles around us. I hope you now have a better appreciation for the mathematical beauty and physical structure the natural world, which seems anything but random.

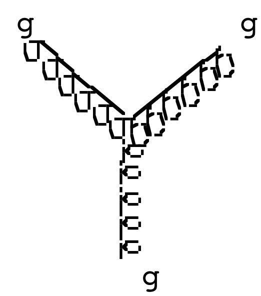
Appendix A: Allowed Standard Model Interactions

The generic allowed interaction vertices in Feynman diagrams for the various Standard model interactions are shown below. All lines can be considered as "real" (external) or virtual (internal). These are just the primitive "vertices" - one must put them together to make complete Feynman diagrams representing physical processes. Of course, individual quark, gluon lines are confined and can not be external. The Higgs boson couplings are not shown.

Strong:





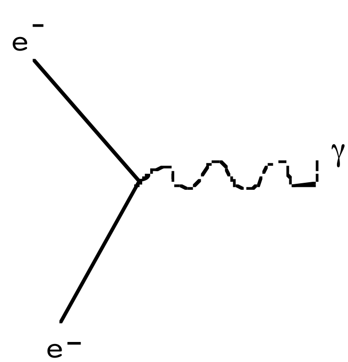


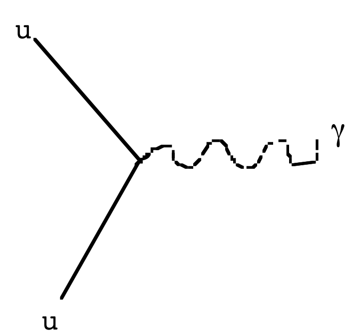
Leptonic

Quarkish

EM

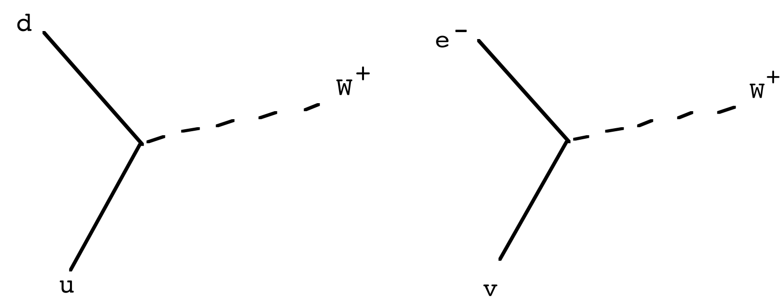
(any charged particle):





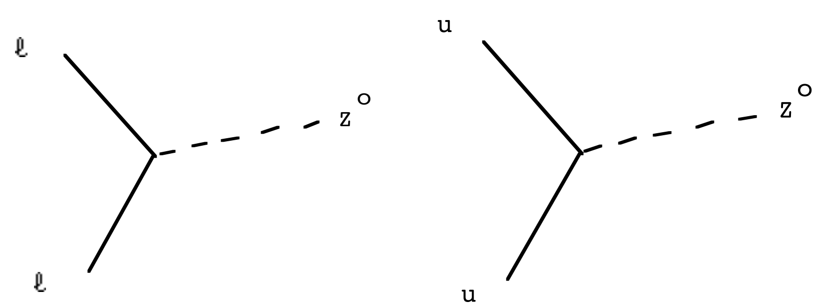
Leptonic   Quarkish

Weak (Charged):

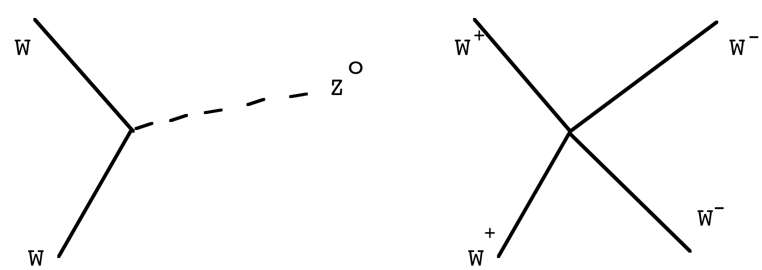


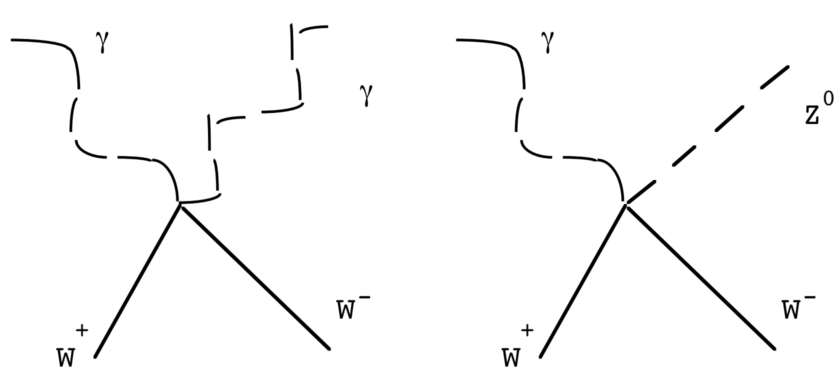
Leptonic   Quarkish

Weak (Neutral):



Other Weak:





Appendix B: Weak Flavor Mixing

In this Appendix, I will explain more about the significance of flavor mixing in the weak interactions. I will base this on a very insightful article by Howard Georgi in a Physics Today article which appeared back in April of 1988. The interesting point made here is that the structure of flavor interactions can be understood from a straightforward analogy with coupled harmonic oscillators. I will try to keep my explanation mostly in line with the concepts and ideas already introduced in these notes, so the emphases and presentation will differ a little from Georgi's article. I am presenting this in an Appendix because the text does not require this coverage for the flow of ideas encountered there and also because it is not guaranteed that all the concepts encountered here will have had an appropriate pedagogical introduction. Georgi's article is much more refined than my poor presentation, and I highly recommend the original for a more complete and expert point of view.

Our starting Hamiltonian is just

$$H_0 = \Omega\hbar \sum_j A_j^+ A_j \,, \quad \text{(Drop const. term; called “normal ordering”)}$$

which we recognize as Hamiltonian for the three dimensional harmonic oscillator. There are symmetries here. Let

$$A_j \rightarrow \sum_k U_{jk} A_k \,, \tag{B.1}$$

$$\Rightarrow \; A_j^+ \rightarrow \sum_k A_k^+ U_{jk}^+ = \sum_k A_k^+ U_{kj}^+ . \tag{B.2}$$

$$H_0 \rightarrow \Omega\hbar \sum_{j,k,\ell} A_k^+ U_{kj}^+ U_{j\ell} A_\ell = \Omega\hbar \sum_{k,\ell} A_k^+ A_\ell \left(\sum_j U_{kj}^+ U_{j\ell} \right) . \tag{B.3}$$

U is unitary:

$$U^{+}U = 1, \quad \Rightarrow \sum_j U^{+}_{kj}U_{j\ell} = \delta_{k\ell}, \tag{B.4}$$

$$\Rightarrow H_0 \rightarrow \Omega\hbar \sum_{k,\ell} A^{+}_{k}A_{\ell}\delta_{k\ell} = H_0 \text{ : unchanged} \tag{B.5}$$

U is a unitary, 3x3 matrix $\Rightarrow$ the group known as SU(3). Actually, there is a further symmetry here. Let

$$A_j \rightarrow e^{i\alpha}A_j, \quad \Rightarrow A^{+}_{j} \rightarrow e^{-i\alpha}A^{+}_{j}, \tag{B.7}$$

$$H_0 \rightarrow \sum_j A^{+}_{j}A_j e^{-i\alpha}e^{i\alpha} = H_0 \text{ : unchanged}, \tag{B.8}$$

Real Symmetry: SU(3)xU(1). Consequence is that N_k is conserved:

$$[H_0, N_k] = \hbar\Omega \sum_j \left[A^{+}_{j}A_j, A^{+}_{k}A_k\right]. \tag{B.9}$$

We have shown in the last chapter that

$$\left[N_j, N_k\right] = \left[A^{+}_{j}A_j, A^{+}_{k}A_k\right] = 0. \tag{B.10}$$

Ch.7, prob.1 shows that the degeneracy factor for the state $n^{\underline{th}}$ quantum state is $\frac{1}{2}(n+1)(n+2)$. Let's go through it here to refresh ourselves.

$$E = \hbar\omega(n_1 + n_2 + n_3 + \tfrac{3}{2}) = \hbar\omega(n + \tfrac{3}{2}),$$
$$n_1 + n_2 + n_3 = \{0, 1, 2 \ldots\}.$$

Set $n_1=0$	How many ways?	n+1
Set $n_1=1$	"	n
.	.	.
.	.	.
.	.	.
Set $n_1=1$	How many ways?	1

$$\sum_{i=1}^{n+1} i = \tfrac{1}{2}(n+1)(n+2). \qquad \left(\text{Given} \sum_{i=1}^{n} i = \tfrac{1}{2} n(n+1)\right)$$

First few energies:

$$6\begin{cases} n_1 = 2, n_2 = 2, n_3 = 2 \quad (\text{others} = 0) \\ n_1 = n_2 = 1, n_3 = 0 \\ n_1 = n_3 = 1, n_2 = 0 \\ n_2 = n_3 = 1, n_1 = 0 \qquad E = \tfrac{7}{2}\hbar\omega \end{cases}$$

$$3\begin{cases} n_1 = 1, n_2 = n_3 = 0 \\ n_2 = 1, n_1 = n_3 = 0 \\ n_3 = 1, n_1 = n_2 = 0 \qquad E = \tfrac{5}{2}\hbar\omega \end{cases}$$

$$1\{ n_1 = 1, n_2 = n_3 = 0 \qquad E = \tfrac{3}{2}\hbar\omega$$

Now cease to think of these as energy states. Consider n_1, n_2, n_3 to be particle occupation numbers of a B.E. system. Remember, I showed in the last chapter that the algebra of

a_i and a_i^+ (raising and lowering operators)

for the harmonic oscillator is identical to

A and A^+ (creation and annihilation operators)

for the multi-particle bosonic states. If instead n_1, n_2, n_3 were the particle occupation numbers of a Fermi-Dirac system, we would have the Cartesian classification:

		FD deg.	BE deg.
$E = \frac{9}{2}\hbar\omega$	$\|1,1,1\rangle$	1	10
$E = \frac{7}{2}\hbar\omega$	$\|1,1,0\rangle$ $\|0,1,1\rangle$ $\|1,0,1\rangle$	3	6
$E = \frac{5}{2}\hbar\omega$	$\|1,0,0\rangle$ $\|0,1,0\rangle$ $\|0,0,1\rangle$	3	3
$E = \frac{3}{2}\hbar\omega$	$\|0,0,0\rangle$	1	1

Now introduce an interaction:

$$H_{int} = \vec{B}\cdot\left(\vec{p}\times\vec{r}\right) = -\vec{B}\cdot\vec{L} \tag{B.11}$$

Why is such a term reasonable? I argued in Ch.8 (based upon the Ch.1 discussion) that a magnetic field interacts with a magnetic dipole according to

$$H_{int} = \vec{m}\cdot\vec{B}. \tag{B.12}$$

Given a gyromagnetic ratio such that

$$\vec{m} = \gamma\,\vec{L}, \tag{B.13}$$

($\gamma = \frac{q}{2mc}$ classically) then

$$H_{int} = \gamma\,\vec{B}\cdot\vec{L}. \tag{B.14}$$

Given the harmonic oscillator raising and lowering operators $\left(K = m\Omega^2\right)$,

$$A_j = \frac{1}{\sqrt{2m\hbar\Omega}}\, p_j - i\sqrt{\frac{K}{2\hbar\Omega}} r_j, \tag{B.15}$$

$$A_j^+ = \frac{1}{\sqrt{2m\hbar\Omega}}\, p_j + i\sqrt{\frac{K}{2\hbar\Omega}} r_j, \tag{B.16}$$

we can evaluate $H_{int} = \vec{B}\cdot\left(\vec{p}\times\vec{r}\right)$ as (problem):

$$\vec{B}\cdot\left(\vec{p}\times\vec{r}\right) = i\hbar\sum_{i,j,k} \varepsilon_{ijk} B_i A_j A_k. \tag{B.17}$$

Writing this as

$$H_{int} = \sum_{j,k} A_j^+ M_{j,k} A_k, \tag{B.18}$$

we identify

$$M_{jh} = ik\sum_{i} \varepsilon_{ijk} B_i. \tag{B.19}$$

Notice that

$$M_{jh}^* = M_{kj} \Rightarrow \text{Hermitian.}$$

Also notice (do the transformation again) that under

$$A_j \rightarrow \sum_k U_{jk} A_k, \quad A_j^+ \rightarrow \sum A_k^+ U_{kj}^+, \tag{B.20}$$

we have, $H_0 \rightarrow H_0$ but

$$H_{int} \rightarrow \sum_{j,h} A_i^+ U_{\ell j}^+ M_{jk} U_{km} A_m = \sum_{\ell,m} A_\ell^+ \tilde{M}_{\ell m} A_m, \tag{B.21}$$

where

$$\tilde{M}_{\ell m} = \sum_{j,k} U^{+}_{\ell j} M_{jk} U_{km}. \tag{B.22}$$

As matrices: $\tilde{M} = U^{+}MU$.

Since we don't get $H_{int} \to H_{int}$, we have that the SU(3) symmetry is broken by H_{int}. The consequence is that the individual N_k are no longer conserved, just the total $N = \sum_k N_k$.

That is, we have

$$[H_{int},\ N_j] = \left[i\hbar \sum_{i,k,\ell} \varepsilon_{ik\ell} B_i A^{+}_{k} A_{\ell},\ A^{+}_{j} A_{j}\right]$$

$$= i\hbar \sum_{i,k,\ell} \varepsilon_{ik\ell} B_i \left[A^{+}_{k} A_{\ell},\ A^{+}_{j} A_{j}\right]. \tag{B.23}$$

But

$$\left[A^{+}_{i} A_{j},\ A^{+}_{k} A_{\ell}\right] = \delta_{jk} A^{+}_{i} A_{\ell} - \delta_{i\ell} A^{+}_{k} A_{j}, \tag{B.24}$$

so $(i = j)$

$$\left[A^{+}_{j} A_{j},\ A^{+}_{k} A_{\ell}\right] = \delta_{jk} A^{+}_{j} A_{\ell} - \delta_{j\ell} A^{+}_{k} A_{j}.$$

$$\Rightarrow [H_{int}\ \ N_j] = -i\hbar \sum_{i,k,\ell} \varepsilon_{ik\ell} B_i \left\{\delta_{jk} A^{+}_{j} A_{\ell} - \delta_{j\ell} A^{+}_{k} A_{j}\right\},$$

$$= -i\hbar \sum_{i,\ell} \varepsilon_{ik\ell} B_i \left(A^{+}_{j} A_{\ell} + A^{+}_{\ell} A_{j}\right) \neq 0. \tag{B.25}$$

But notice, however, that

$$\sum_{j} [H_{int}, N_j] = -i\hbar \sum_{i,\ell,j} \varepsilon_{ij\ell} B_i \left(A^{+}_{j} A_{\ell} + A^{+}_{\ell} A_{j}\right) = 0. \tag{B.26}$$

Now let's go back to

$$\tilde{M} = U^{+}MU. \tag{B.27}$$

Can show that we can diagonalize any Hermitian matrix, M, in this fashion. Why do we want to do this? Because if we can find a U such that

$$\tilde{M}_{ij} = \omega_i \delta_{ij}, \tag{B.28}$$

then

$$H_{int} = \sum_{\ell,m} A_\ell^{+} \tilde{M}_{\ell,m} A_m \rightarrow \sum_{\ell,m} \omega_\ell A_\ell^{+} A_\ell. \tag{B.29}$$

Let's find the w's for our explicit case above.

$$M = i\hbar \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}. \tag{B.30}$$

Get the w's from the <u>characteristic</u> <u>equation</u>. (See p.4.11 of this text). We form,

$$\det\left(M - \omega_i\right) = 0, \tag{B.31}$$

$$\Rightarrow \det \begin{pmatrix} \omega & B_3 & -B_2 \\ -B_3 & \omega & B_1 \\ B_2 & -B_1 & \omega \end{pmatrix} = 0, \tag{B.32}$$

$$\Rightarrow \omega^3 - \hbar^2 \omega\left(B_1^2 + B_2^2 + B_3^2\right) = 0. \tag{B.32}$$

Three solutions:

$$\begin{cases} \omega_1 = |\vec{B}|, \\ \omega_2 = -|\vec{B}|, \\ \omega_3 = 0. \end{cases} \tag{B.33}$$

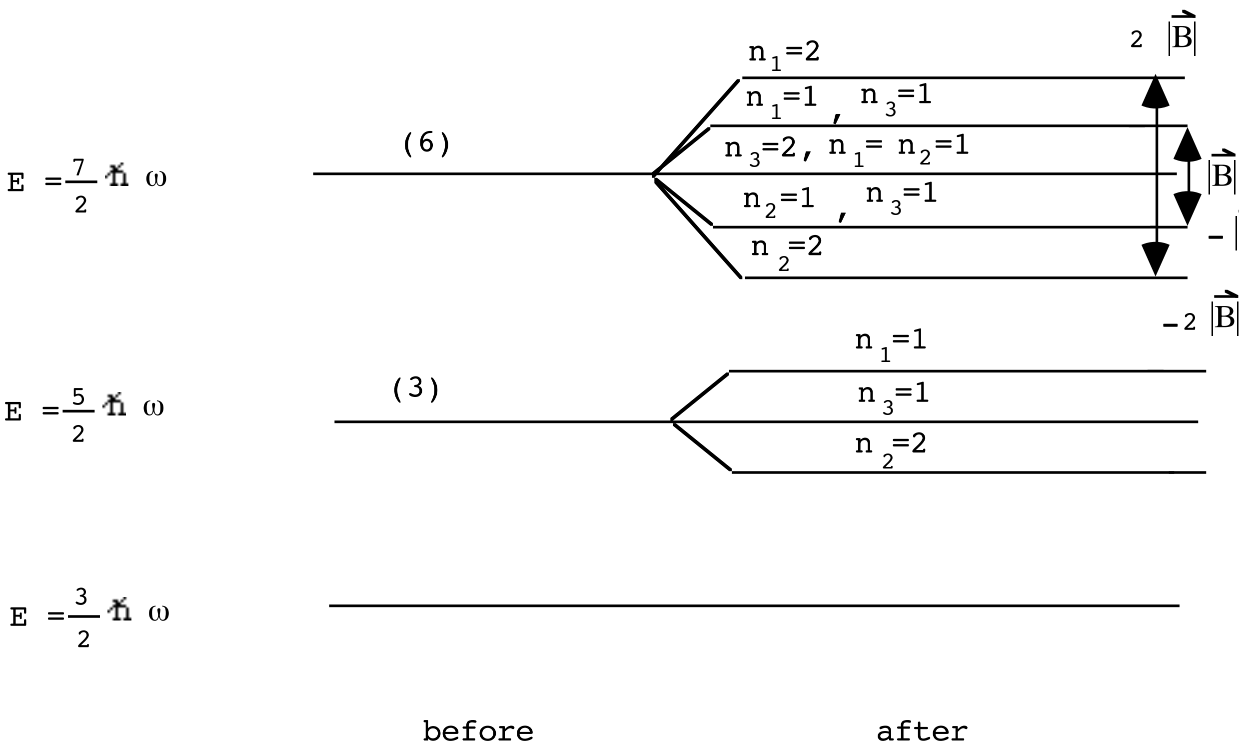
Therefore

$$H_{int} = \sum_i \omega_i A_i^+ A_i = |\vec{B}|\left(A_1^+ A_1 - A_2^+ A_2\right). \tag{B.34}$$

Then using 1st order perturbation theory,

$$\langle H_{int}\rangle = |\vec{B}|\left(n_1 - n_2\right). \tag{B.35}$$

The energy levels now appear as:



In general, the splitting is

$$\langle H_{int} \rangle = \left\langle \sum_i \omega_i A_i^+ A_i \right\rangle = \sum_i \hbar\omega_i N_i = \hbar\omega\left(N_\ell\omega_\ell + N_2\omega_2 + N_3\omega_3\right), \quad (B.36)$$

($N_1 + N_2 + N_3 = N$.) For N = 1, the splittings are

$$\begin{array}{ll} N_1 = 1 & \hbar\,\omega_1 \\ N_2 = 1 & \hbar\,\omega_2 \\ N_3 = 1 & \hbar\,\omega_3 \end{array}$$

For N = 2, the splittings are

$$\begin{array}{ll} N_1 = 2 & 2\hbar\,\omega_1 \\ N_2 = 2 & 2\hbar\,\omega_2 \\ N_3 = 2 & 2\hbar\,\omega_3 \\ N_1 = N_2 = 1 & \hbar\left(\omega_1 + \omega_2\right) \\ N_1 = N_3 = 1 & \hbar\left(\omega_1 + \omega_3\right) \\ N_2 = N_3 = 1 & \hbar\left(\omega_2 + \omega_3\right) \end{array}$$

Thus the new Hamiltonian is

$$H_A = \hbar \sum_j \left(\Omega + \omega_j\right) A_j^+ A_j. \quad (B.37)$$

Add another interaction, H_B, analogous to the first:

$$\left[A_i, B_j\right] = 0, \left[A_i, B_j^+\right] = 0, \quad (B.38)$$

$$H_A + H_B = \hbar \sum_j \left[\left(\Omega_A + \omega_{Aj}\right) A_j^+ A_j + \left(\Omega_B + \omega_{Bj}\right) B_j^+ B_j\right]. \quad (B.39)$$

We will see that we can view the H_A Hamiltonian as representing the charge 2/3 quarks (u, c, and t), and the H_B Hamiltonian as representing the charge -1/3 quarks (d, s, and b), at least when we are talking about the N=1 states. States are represented by:

$$\left| N_{A1}, N_{A2}, N_{A3}, N_{B1}, N_{B2}, N_{B3} \right\rangle. \tag{B.40}$$

Now also add (we're almost there!)

$$H_w = \hbar W \sum_j \left(A_j^+ B_j + B_j^+ A_j \right), \tag{B.41}$$

Notice $\left(H_w^+ = H_w\right)$. If we were to drop the ω_{Aj}, ω_{Bj} terms above, we would have a symmetry, as follows:

$$\left.\begin{array}{l} A_j \rightarrow \sum\limits_k U_{jk} A_k \\ B_j \rightarrow \sum\limits_k U_{jk} B_k \end{array}\right\} \text{ same } U_{jk}. \tag{B.42}$$

Then

$$H_w \rightarrow \hbar W \sum_{j,k} \left[A_k^+ U_{kj}^+ U_{j1} B_1 + B_\ell^+ U_{\ell j}^+ A_k \right] = H_w. \tag{B.43}$$

This is a <u>combined</u> $SU_{A+B}(3)$ symmetry. This has the consequence that now only $\left(N_{Ai} + N_{Bi}\right)$ is conserved. That is (problem)

$$\left[N_{Ai} + N_{Bi}, H_w \right] = 0. \tag{B.44}$$

Now let the ω_{Aj} and ω_{Bj} both be non-zero. The resultant full Hamiltonian can be written

$$H = \hbar \sum_j \left(A_j^+ \;\; B_j^+ \right) \begin{pmatrix} \Omega_A + \omega_{Aj} & W \\ W & \Omega_B + \omega_{Bj} \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix}. \tag{B.45}$$

Now, it is only the <u>total</u> $N_A + N_B = \sum\limits_i \left(N_{Ai} + N_{Bi}\right)$ that is conserved. Let

$$\hat{M} = \begin{pmatrix} \Omega_A + \omega_{Aj} & W \\ W & \Omega_B + \omega_{Bj} \end{pmatrix}. \tag{B.46}$$

Just as before, diagonalize it to get the energies. (Call $\Omega_A + \omega_{Aj} = a_j$, $\Omega_B + \omega_{Bj} = b_j$)

$$\det \begin{pmatrix} a_j - \lambda & W \\ W & b_j - \lambda \end{pmatrix} = 0, \tag{B.47}$$

$$\Rightarrow \lambda^2 - \lambda\left(a_j + b_j\right) - W^2 + a_j b_j = 0, \tag{B.48}$$

$$\lambda = \frac{a_j + b_j}{2} \pm \frac{1}{2}\sqrt{\left(a_j - b_j\right)^2 + 4W^2}. \quad \text{(eigenvalues)} \tag{B.49}$$

What about eigenvectors? Can verify that H can be written as

$$H = \frac{\hbar}{2}\sum_j \left(\alpha_j^+ \ \beta_j^+\right)\begin{pmatrix} a_j + b_j + \sqrt{\left(a_j + b_j\right)^2 + 4W^2} & 0 \\ 0 & a_j + b_j - \sqrt{\left(a_j + b_j\right)^2 + 4W^2} \end{pmatrix}\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}, \tag{B.50}$$

where

$$\alpha_j^+ = \cos\Theta_j A_j^+ + \sin\Theta_j B_j^+, \tag{B.51}$$

$$\beta_j^+ = -\sin\Theta_j A_j^+ + \cos\Theta_j B_j^+. \tag{B.52}$$

The angle Θ is given by

$$\tan 2\Theta = \frac{2W}{\left(\beta_j - \alpha_j\right)}. \tag{B.53}$$

Since

$$\cos^2 2\Theta_j = \frac{1}{1 + \tan^2 2\Theta_j}, \quad \text{(B.54)}$$

we have

$$\cos^2 2\Theta_j = \frac{\left(a_j - b_j\right)^2}{\left(a_j - b_j\right)^2 + 4W^2}, \quad \text{(B.55)}$$

$$\cos 2\Theta_j = \pm \frac{\left(\alpha_j - b_j\right)}{\sqrt{4\omega^2 + \left(\alpha_j - b_j\right)^2}}, \text{ (Choose + sign)} \quad \text{(B.56)}$$

$$\Rightarrow \sin 2\Theta_j = \mp \frac{2\omega}{\sqrt{4\omega^2 + \left(\alpha_j - b_j\right)^2}}. \text{ (Choose - sign)} \quad \text{(B.57)}$$

(Must choose the above signs for the form of H written above. Can see this in the $W \to 0$ limit). Verify this form:

$$\sum_j \left\{\alpha_j^+\alpha_j\left(a_j + b_j + \sqrt{\left(a_j - b_j\right)^2 + 4W^2}\right) + B_j^+B_j\left(a_j + b_j - \sqrt{\left(a_j - b_j\right)^2 + 4W^2}\right)\right\}, \quad \text{(B.58)}$$

$$\alpha_j^+\alpha_j = \cos^2 \Theta_j A_j^+A_j + \sin^2 \Theta_j B_j^+B_j + \cos \Theta_j \sin \Theta_j\left(A_j^+A_j + B_j^+B_j\right), \quad \text{(B.59)}$$

$$\beta_j^+\beta_j = \sin^2 \Theta_j A_j^+A_j + \cos^2 \Theta_j B_j^+B_j - \cos \Theta_j \sin \Theta_j\left(A_j^+A_j + B_j^+B_j\right). \quad \text{(B.60)}$$

Finish the rest up as a problem. Should get earlier expression (problem):

$$H = \hbar\sum_j \left\{a_jA_j^+A_j + b_jB_j^+B_j + W\left(A_j^+B_j + B_j^+A_j\right)\right\}. \quad \text{(B.61)}$$

Remember Fermi's Golden Rule. From chapter 9,

$$\text{rate} \sim \left|\langle 2|T|1\rangle\right|^2, \quad \text{(B.62)}$$

for a 2->1 transition. Let's take an example:

$$\left|1_{\alpha j}\right\rangle = \alpha_j^+|0\rangle, \quad \left|1_{Bj}\right\rangle = \beta_j^+|0\rangle, \tag{B.63}$$

$$\left\langle 1_{\alpha j}|T|1_{Bj}\right\rangle =$$

$$\left\langle 0 \,|\, \left(\cos\Theta_j A_j + \sin\Theta_j B_j\right)\sum_k \left(\underbrace{A_k^+ A_k}_{N_{Ak}} - \underbrace{B_k^+ B_k}_{N_{Ak}}\right)\left(-\sin\Theta_j A_j^+ + \cos\Theta_j B_j^+\right)|\, 0\right\rangle. \tag{B.64}$$

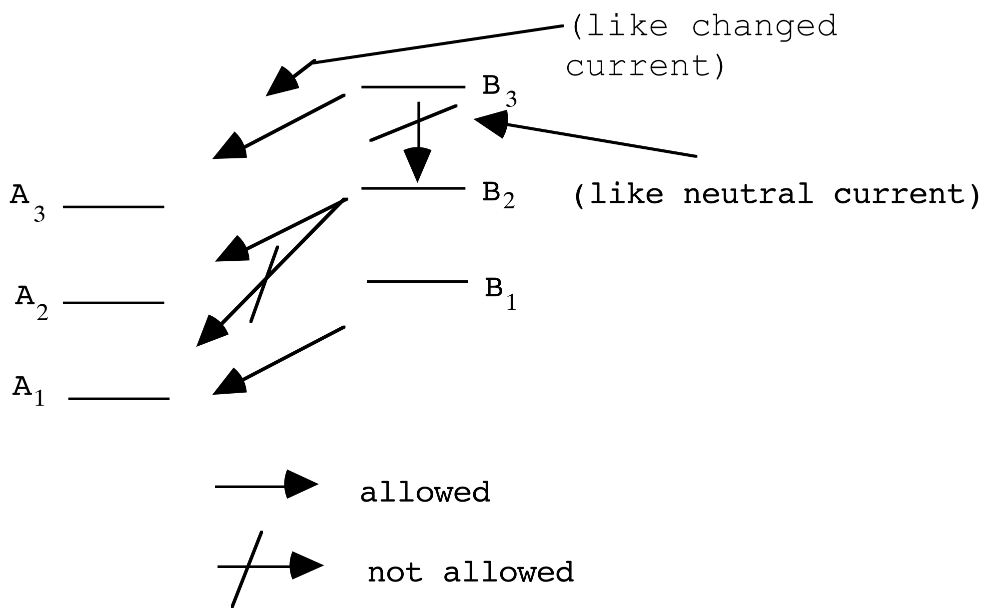
$\left(\left[N_{Ak}, A_j^+\right] = A_j^+\delta_{kj},\ \left[N_{Bk}, B_j^+\right] = B_j^+\delta_{kj}.\right)$ Thus

$$\begin{aligned}\left\langle 1_{\alpha j}|T|1_{Bj}\right\rangle &= \left\langle 0 \,|\, \left(\cos\Theta_j A_j + \sin\Theta_j B_j\right)\left(-\sin\Theta_j A_j^+ - \cos\Theta_j B_j^+\right) \,|\, 0\right\rangle, \\ &= -\cos\Theta_j - \sin\Theta_j\left\langle 0 \,|\, \left(A_j A_j^+ + B_j B_j^+\right) \,|\, 0\right\rangle, \\ &= -2\cos\Theta_j \sin\Theta_j = -\sin\left(2\Theta_j\right).\end{aligned} \tag{B.65}$$

Likewise, can show (problem; all $i \neq j$):

$$\left\langle 1_{A_i}|T|1_{A_j}\right\rangle = 0, \quad \left\langle 1_{B_i}|T|1_{B_j}\right\rangle = 0, \quad \left\langle 1_{A_i}|T|1_{B_j}\right\rangle = 0. \tag{B.66}$$

This gives rise to the picture of interactions (take the N=1 case again):



We see that interactions involving the same generation of quark are allowed (the charged current interactions, mediated by the $W^{\pm}$ bosons). However, not all the charged current interactions are possible at this point, but only those with i=j (within each generation). We do not yet have the small neutral current interactions involving different generations. However, they're not hard to model. We need only make one more change to our Hamiltonian.

Here'e the intersting part. Let us simply change the H_w above to

$$H_W = \hbar W \sum_{j,k} \left(A_j^+ V_{jk} B_k + B_j^+ V_{jk}^+ A_k \right), \qquad \left(H_W^+ = H_W \right) \qquad \text{(B.67)}$$

where V_{jk} is a unitary matrix. We now have mixings of generations involving all the charged current interactions. This would be completely equivalent to old H_w if ω_{Aj}, ω_{Bj} terms in H_A and H_B are dropped. Let

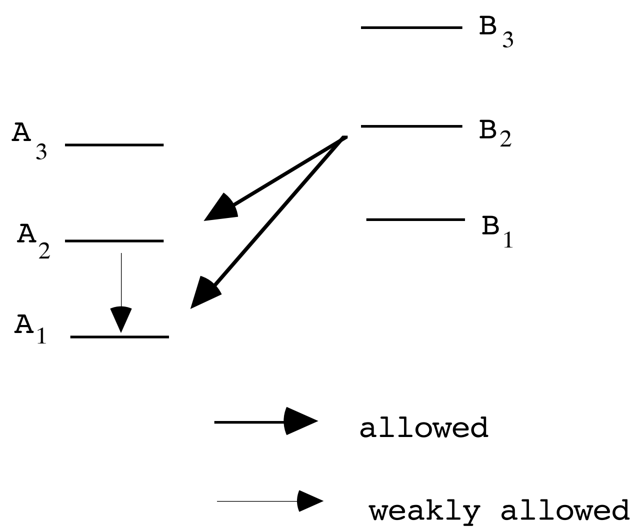
$$A_j \to \sum_k V_{jk} A_k, \; A_j^+ \to \sum_k A_k^+ V_{kj}^+, \qquad \text{(B.68)}$$

then

$$H_W = \hbar W \sum_{j,k} \left(A_\ell^+ \underbrace{V_{\ell j}^+ V_{jk}}_{\delta_{\ell k}} B_k + B_j^+ \underbrace{V_{jk}^+ V_{k\ell}}_{\delta_{j\ell}} A_\ell \right),$$

$$= \hbar W \sum_j \left(A_j^+ B_j + B_j^+ A_j \right) \quad \text{, old form.} \qquad (B.69)$$

That is, without the $SU_A(3)$, $SU_B(3)$ symmetry breaking terms, this new form is completely equivalent to the old and would not add anything new. However, because of the ω_{Aj}, ω_{Bj} terms, we now have all charged transitions on the previous picture, even those between different generations ($i \neq j$). As I said, the analog of the transitions from the β_i's to the α_j's are the charged current interactions, mediated by the $W^\pm$ particles. The V_{jk} matrix is analogous tp the CKM matrix in weak interactions. There are still no direct generation-changing neutral currents (those connecting α_i and α_j or β_i and β_j for $i \neq j$), but now notice the following. We can now have transitions which look like the following;



B_2 is seen to be mixed into A_1 and A_2, which means, from the magic of quantum mechanics, that A_1 and A_2 also are mixed. A "second order transition" would first take A_2 into the higher state B_2, then back down to A_1. However, such transitions are doubly suppressed. First, this transition only occurs "indirectly" by this two step process, and, as we saw above, the effect is proportional to the presence of the off-diagonal elements of V_{jk}. These in fact only contribute because of the presence of the ω_{Aj}, ω_{Bj} terms. However, it's not just that the ω_{Aj}, ω_{Bj} terms are there, but that they are all *different* that allows these small transitions. An example of such a process in the standard Model is $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, which has been observed.

Problems for appendix B

1. Let's see if we can also understand the $\frac{1}{2}(n+1)(n+2)$ degeneracy factor from the radial classification of states. Go back to Ch.7, prob.15. Compared (3-D oscill.)

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} - \rho^2 + 2(n+\tfrac{3}{2})\right]R_{n\ell}(\rho) = 0,$$

to (2-D oscill.)

$$\left[\frac{d^2}{d\rho^2} - \frac{(m^2 - 1/4)}{\rho^2} + 2(|m| + 2n_r + 1) - \rho^2\right]\sqrt{\rho}\, P_{n_r m}(\rho) = 0.$$

Should have found: <u>2-D</u> <u>3-D</u>

$$m^2 - \tfrac{1}{4} \rightarrow \ell(\ell+1),$$

$$2n_r + |m| + 1 \rightarrow n + \tfrac{3}{2},$$

or

$$|m| \rightarrow \ell + \tfrac{1}{2}, \quad n_r \rightarrow \frac{n-\ell}{2}. \; (n_r = 0,1,2,3,\ldots)$$

(a) Considering the n even and n odd cases separately, show that the degeneracy is $\frac{1}{2}(n+1)(n+2)$.

(b) Write the radial quantum numbers (n,ℓ,m) of the $E = \frac{3}{2}\hbar\omega$, $\frac{5}{2}\hbar\omega$, $\frac{7}{2}\hbar\omega$ states out explicitly.

2. Show

$$\vec{B}\cdot\left(\vec{p}\times\vec{r}\right) = i\hbar\sum_{i,j,k}\varepsilon_{i,j,h}B_i A_j A_k.$$

3. Show

$$\left[N_{Ai} + N_{Bi}, H_w\right] = 0.$$

4. After the change of variables described in the text, show that the Hamiltonian can still be written as:

$$H = \hbar \sum_j \left\{ a_j A_j^+ A_j + b_j B_j^+ B_j + W\left(A_j^+ B_j + B_j^+ A_j\right)\right\}.$$

5. Explicitly show (before the introduction of the V_{jk} matrix)

$$\left\langle 1_{\alpha i} | T | 1_{\alpha i} \right\rangle = 0 \quad (i \neq j)$$
$$\left\langle 1B_i | T | 1_{Bi} \right\rangle = 0 \quad (i \neq j)$$
$$\left\langle 1_{\alpha i} | T | 1_{Bi} \right\rangle = 0 \quad (i \neq j).$$

6. Show the Feynman diagram for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$.

Problems

1. Starting with the definition of the determinant of a 3x3 matrix,

$$(\det a) = \sum_{\ell,m,n} \varepsilon_{\ell mn}\, a_{1\ell}\, a_{2m}\, a_{3n}\ .$$

show that,

$$\sum_{j,k} \varepsilon_{ijk}\, a_{j\ell}\, a_{km} = \det(a) \sum_{n} \varepsilon_{n\ell m}\, a_{in}\ .$$

where a_{ij} are elements of a general orthogonal transformation.

2. Considering the matrix element $\langle s'|\vec{J}\cdot\vec{p}|s'\rangle$, show that a nonzero value implies parity nonconservation. (Hints; First consider $|s'\rangle$ a good parity state, then a mixture.)

3. For the time reversal operation, show:

(a) $(AB)^T = B^T A^T$.

(b) $(\vec{x} \times \vec{p})^T = -\vec{x} \times \vec{p}$.

4. There is a famous and useful theorem in mathematics called the Wigner-Eckart theorem. It generalizes the spin addition considerations of Ch. 8, and can be applied to spin or isospin matrix elements. (For isospin, see probs. 7, 8, and 9 below.)

$$\langle J',M'|T_{Kq}|J,M\rangle = \langle JK;Mq\ |JK;J'M'\rangle \frac{\langle J'|T_K|J\rangle}{\sqrt{2J+1}},$$

where the $\langle J'|T_K|J\rangle$ is the "reduced matrix element" (independent of M, M'), the T_{Kq} are spherical tensors (see Sakurai's <u>Modern Quantum Mechanics</u>, Ch.3), and the $\langle JK;Mq\ |JK;J'M'\rangle$ are just the

Clebsch-Gordon coefficients of Ch.8. This theorem immediately implies the so-called triangle inequaltiy $|J-K| \leq J' \leq J+K$.

You are given the electromagnetic Hamiltonian density,

$$\mathcal{H}^{em} = \sum_{L,m} \sum_{X=(E,M)} A^{(X)}_{Lm} \quad T^{(X)}_{Lm},$$

where the $A^{(E)}_{Lm}$, $A^{(M)}_{Lm}$ (E=electric, M=magnetic) are expansion coefficients ($A^{(M)}_{00}$ = 0) and the $T^{(E)}_{Lm}$, $T^{(M)}_{Lm}$ are spherical tensors. The operators $T^{(E,M)}_{Lm}$ have the properties

$$\pi T^{(E)}_{Lm} \pi = (-1)^{L} T^{(E)}_{Lm} \quad (L \geq 0) ,$$
$$\pi T^{(M)}_{Lm} \pi = (-1)^{L+1} T^{(M)}_{Lm} \quad (L \geq 1) ,$$

where π is the parity operator ($\pi^{\dagger} = \pi$). Consider the expectation value (modeling static em moments):

$$\langle J,m_J,\pi_J|\mathcal{H}^{em}|J,m_J,\pi_J\rangle,$$

where m_J = J.(The states $|J,m_J,\pi_J\rangle$ have good total angular momentum, J_z, and parity.)

(a) For general J ($J=0,\frac{1}{2},1,\frac{3}{2},2,\ldots$), what is the largest L that can contribute?

(b) For general J, how many terms in $\mathcal{H}^{em}$ give nonzero contributions? Specify them.

Consider the more general matrix element (modeling em transitions) where $\mathcal{H}^{rad}$ is the same as $\mathcal{H}^{em}$ but without the L=0 E term:

$$\langle J_2,m_{J2},\pi_{J2}|\mathcal{H}^{rad}|J_1,m_{J1},\pi_{J1}\rangle,$$

(c) For general J_1, J_2, how many terms in $\mathcal{H}^{rad}$ give nonzero contributions? Specify them.

(d) Assuming the terms with smallest L give the leading contribution, give the transition selection rules for the various $\Delta J = |J_2 - J_1|$ when $\pi_{J1}\pi_{J2}=1$ or $\pi_{J1}\pi_{J2}=-1$.

5. Particle physicists usually express dimensionful quantities in terms of MeV (millions of electron volts), $\hbar$, and c. For example,

$$1\ \text{kg} = 5.61\text{x}10^{29}\ \frac{\text{MeV}}{c^2},$$

$$1\ \text{cm}^{-1} = 1.97\text{x}10^{-11}\ \frac{\text{MeV}}{\hbar c^2},$$

$$1\ \text{sec}^{-1} = 6.58\text{x}10^{-22}\ \frac{\text{MeV}}{\hbar}.$$

Using these values, convert the QCD string tension of 15 tons (1 ton = $8.9\text{x}10^3$ Newtons) into $\frac{\text{MeV}}{\text{fm}}$ units.

6. Using the generalized Pauli principle, stating that the total wavefunction, assumed to be a product of space, spin, and isospin parts, must be anti-symmetric under the interchange of the two nucleons, show that pp scattering can only take place in (^{2S+1}L notation)

$$^1S,\ ^3P,\ ^1D,\ ^3F, \ldots$$

scattering states. What is the total isospin of the above states? Note: space interchamge is the same as parity, $(-1)^{\ell}$, here.

7.(a) Use the Wigner-Eckart theorem from prob. 4 above. Define the isospin states:

$$|\Delta^{++}\rangle = |\tfrac{3}{2}, \tfrac{3}{2}\rangle,$$
$$|\Delta^{+}\rangle = |\tfrac{3}{2}, \tfrac{1}{2}\rangle,$$
$$|p\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle,$$
$$|n\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle,$$
$$|\pi^{+}\rangle = |1, 1\rangle,$$

$$|\pi^0\rangle = |1,0)\rangle.$$

Use equation (8.42) with $I_{\pm}^{\Delta} = I_{\pm}^{\pi} + I_{\pm}^{nuc}$ and

$$|\Delta^{++}\rangle = |p\rangle|\pi^+\rangle$$

to show that

$$\Delta^+ = \sqrt{\frac{1}{3}}\,|n\rangle|\pi^+\rangle + \sqrt{\frac{2}{3}}\,|p\rangle|\pi^0\rangle.$$

(b) Using the given proportionality (good for the isospin parts of wavefunctions in strong interaction decays only)

$$w(\text{initial} \rightarrow \text{final}) \propto |\langle\text{initial}|\text{final}\rangle|^2,$$

find the decay branching ratio:

$$\frac{w(\Delta^+ \rightarrow p + \pi^0)}{w(\Delta^+ \rightarrow n + \pi^+)}.$$

8. From isospin considerations, find the ratio of rates for the decays indicated (see probs. 4, 7):

a) $\dfrac{w(K^{*+} \rightarrow K^0 + \pi^+)}{w(K^{*+} \rightarrow K^+ + \pi^0)} = ?$, b) $\dfrac{w(K^{*0} \rightarrow K^+ + \pi^-)}{w(K^{*0} \rightarrow K^0 + \pi^0)} = ?$.

The K mesons have $I = \frac{1}{2}$; you can consider K^+ the $|\frac{1}{2},\frac{1}{2}\rangle$ state and K^0 the $|\frac{1}{2},-\frac{1}{2}\rangle$.state. The $K^*(892)$ mesons are excited states of the K and also have $I = \frac{1}{2}$. The pion of course has I=1. I will grade you on well you justify your derivation of the necessary isospin coefficients using the machinery of isospin addition.

9.(a) Find the isospin decomposition of the states:

$$|\pi^0 p\rangle, |\pi^+ n\rangle, \text{ and } |\pi^- n\rangle.$$

[Hints: You need only use the concepts of raising and lowering operators and orthogonality of states. See probs. 4, 7.]

(b) Find the matrix elements

$$<\pi^- n|S|\pi^- n>, \ <\pi^0 p|S|\pi^+ n>, \text{ and } <\pi^0 p|S|\pi^0 p>$$

in terms of $<\frac{3}{2}\ |S|\frac{3}{2}> \equiv f_{3/2}$ and $<\frac{1}{2}\ |S|\frac{1}{2}> \equiv f_{1/2}$.

10. Find the magnetic moment of the neutron in the quark model. The normalized "spin up" wavefunction of the neutron can be written as

$$|n,\text{"up"}>=\frac{1}{\sqrt{2}}\left(\frac{(ddu-dud)}{\sqrt{2}}\frac{(|++->-|+-+>)}{\sqrt{2}}+\frac{(ddu+dud-2udd)}{\sqrt{6}}\frac{(|++->+|+-+>-2|-++>)}{\sqrt{6}}\right),$$

where (m is a constituent u,d quark mass)

$$\mu_z = <n,\text{"up"}|(\mu_z)_{op}|n,\text{"up"}>,$$

$$(\mu_z)_{op} = \frac{e\hbar}{2mc}\sum_{i=1,2,3} e_i\sigma_{iz},$$

(the σ_{iz} are the usual z-component $\vec{\sigma}$ matrices working in the i^{th} quark space) and

$$e_i = \begin{cases} \frac{2}{3}, & \text{u quark} \\ -\frac{1}{3}, & \text{d quark} \end{cases} .$$

11. Draw the lowest order (smallest number of vertices) Feynman diagrams for the following processes. You will have to figure out which interaction is responsible for each. Some of them may have more than one topologically distinct lowest order diagram. Consult the list of allowed vertices passed out early in the class. For hadrons draw the Feynman diagrams at the quark level. There are several ringers in here which can't occur; tell me which ones they are and what conservation law they violate. If you have trouble figuring out the nature of a reaction or decay, looking up the rates should tell you which interaction is responsible or dominant. (Make sure you make clear which lines represent which particles.)

a) $e^+ + \gamma \to e^+ + \gamma$

b) $e^+ + e^- \to \nu_e + \bar{\nu}_e$

c) $p + \pi^- \to \Lambda^0 + K^0$

d) $\nu_\mu + p \to \mu^- + \Delta^{++}$

e) $\eta^0 \to \pi^+ + \pi^-$

f) $\Sigma^+ \to n + \pi^+$

g) $\Delta^+ \to n + \pi^+$

h) $\mu^+ \to e^+ + \nu_e + \nu_\mu$

12. The following gives a list of particle decays which do not occur in nature (as far as is known). In each case please give me the physical property or reason why the reaction does not occur. (In some cases there may be more than one reason; in this case give me the "strongest" one.)

a) $n \to \pi^+ + \pi^-$

b) $\mu^+ \to e^+ + \nu_e + \nu_\mu$

c) $\pi^0 \to 3\gamma$

d) $\Sigma^- \to \bar{p} + \pi^0$

e) $\Lambda^0 \to \Xi^0 + \pi^0$

f) $\eta^0 \to \pi^+ + \pi^-$

13. Assuming time reversal invariance, find the ratio of the total (unpolarized) cross sections

$$\frac{\sigma(\bar{p} + p \to \pi^+ + \pi^-)}{\sigma(\pi^+ + \pi^- \to \bar{p} + p)} .$$

Write your answer so that it will be valid in any reference frame.

14. Given

$$\begin{cases} \pi^+ = u\bar{d}, \\ \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), \end{cases}$$

(a) Which particle do you think is more massive, and why?

(b) For the decay pseudoscalar $\pi^0 \rightarrow \gamma + \gamma$, which photon amplitude describes the final state and why? ($\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ are the photon polarization vectars and $\hat{k}$ give the direction of one photon)

(i) $\hat{\varepsilon}_1 \bullet \hat{\varepsilon}_2$

(ii) $\hat{k} \bullet (\hat{\varepsilon}_1 \times \hat{\varepsilon}_2)$