

## Chapter 8

### Plane waves in dielectric media

We will now consider plane waves in a non-conducting (dielectric) medium. We will assume  $\epsilon$ ,  $\mu$  to be independent of  $\omega$  for right now. We have (no sources)

$$\vec{\nabla} \cdot \vec{E} = 0, \quad (8.1)$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0, \quad (8.2)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (8.3)$$

$$\vec{\nabla} \times \vec{B} - \frac{\mu\epsilon}{c} \frac{\partial \vec{E}}{\partial t} = 0. \quad (\vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = 0) \quad (8.4)$$

We now have that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t}) = 0, \quad (8.5)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = 0, \quad (8.6)$$

$$\Rightarrow (\nabla^2 - \underbrace{\frac{\mu\epsilon}{c} \frac{\partial}{\partial t}}_{\frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2}}) \vec{E} = 0. \quad (8.7)$$

Likewise

$$(\nabla^2 - \frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2}) \vec{B} = 0. \quad (8.8)$$

The solutions are plane waves,

$$\left. \begin{aligned} \vec{E} &= \text{Re} \left( \vec{\mathcal{E}} e^{i\vec{k}\cdot\vec{x}-i\omega t} \right) \\ \vec{B} &= \text{Re} \left( \vec{\mathcal{B}} e^{i\vec{k}\cdot\vec{x}-i\omega t} \right) \end{aligned} \right\} \quad (8.9)$$

(There can be arbitrary phases also in  $\vec{\mathcal{E}}$ ,  $\vec{\mathcal{B}}$ .) These waves travel in the  $\vec{k}$  direction, the magnitude of which ("wave number") is given by

$$k^2 = \mu\epsilon \frac{\omega^2}{c^2}. \quad (\text{a "dispersion relation"}) \quad (8.10)$$

$k$  can turn complex in phenomenological descriptions. If  $k$  is real, we can ride on the crest of a wave,

$$\begin{aligned} \vec{k}\cdot\vec{x} - \omega t &= \text{const.}, \\ \Rightarrow \vec{k}\cdot\frac{d\vec{x}}{dt} &= \omega. \end{aligned} \quad (8.11)$$

Set  $\hat{\eta} = \vec{k}/k$  (can possibly be complex also). Again for  $k$  real, in the direction of  $\vec{k}$ ,  $\vec{x} = v_{\text{phase}}\hat{\eta}t$ , we have

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{c}{\sqrt{\mu\epsilon}}. \quad (8.12)$$

$\vec{\nabla}\cdot\vec{E} = 0$  demands ( $k$  complex again)

$$\vec{\nabla}\cdot\vec{E} = \text{Re} \left( i\vec{\mathcal{E}}\cdot\vec{k} e^{i\vec{k}\cdot\vec{x}-i\omega t} \right) = 0, \quad (8.13)$$

$$\Rightarrow \text{Re}(i\vec{\mathcal{E}}\cdot\vec{k}) = \text{Im}(i\vec{\mathcal{E}}\cdot\vec{k}) = 0, \quad (8.14)$$

$$\Rightarrow \vec{\mathcal{E}}\cdot\hat{\eta} = 0. \quad (8.15)$$

From  $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ , we then get

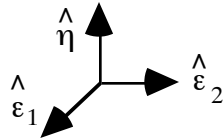
$$\vec{\beta} = \sqrt{\mu\epsilon} \hat{\eta} \times \vec{\mathcal{E}}, \quad (8.16)$$

$$\Rightarrow \vec{\beta} \cdot \hat{\eta} = 0 \text{ also.} \quad (8.17)$$

Oscillating motion of  $\vec{E}$  &  $\vec{B}$  is transverse to the direction of motion. Since there are 2 directions  $\perp$  to a given  $\vec{k}$ , there can be two linearly indep. states of  $\vec{E}$  or  $\vec{B}$ .

Setup:

( $\hat{\epsilon}_{1,2}$  pointing in some fixed spatial directions)



$$\begin{aligned} \hat{\epsilon}_1 \times \hat{\epsilon}_2 &= \hat{\eta}, \\ \hat{\epsilon}_{1,2}^2 &= 1 \end{aligned}$$

So, in general we have

$$\vec{E}_i = \text{Re}(\mathcal{E}_i \hat{\epsilon}_i e^{i\vec{k} \cdot \vec{x} - i\omega t}), \quad (8.18)$$

↑ Still possibly complex

( $i = 1, 2$ ) with

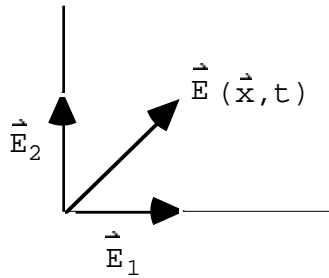
$$\vec{B}_i = \sqrt{\mu\epsilon} \hat{\eta} \times \vec{E}_i. \quad (8.19)$$

Then, the most general plane waves can be written

$$\vec{E}(\vec{x}, t) = \sum_{i=1,2} \vec{E}_i(\vec{x}, t), \quad (8.20)$$

$$\vec{B}(\vec{x}, t) = \sum_{i=1,2} \vec{B}_i(\vec{x}, t). \quad (8.21)$$

Let us choose  $\mathcal{E}_{1,2}$  real for the moment. Then  $\vec{E}$  is given instantaneously at a single point in space by:



$\hat{\eta}$  out of page  
 ( $\vec{E}_1, \vec{E}_2$  oscillate  
 together in time)

A complex phase difference in the  $\mathcal{E}_{1,2}$  translates into a time phase difference in time between the two components and the resulting  $\vec{E}$  can change in both magnitude & direction at a given  $\vec{x}$ .

To understand what else it can do, let us introduce another basis set:

$$\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}} (\hat{\epsilon}_1 \pm i \hat{\epsilon}_2). \quad (8.22)$$

They are orthonormal in a complex sense:

$$\hat{\epsilon}_{-}^* \cdot \hat{\epsilon}_{+} = 0, \quad (8.23)$$

$$\hat{\epsilon}_{+}^* \cdot \hat{\epsilon}_{+} = 1. \quad (8.24)$$

Most general  $\vec{E}$  can also be written:

$$\vec{E}_{\pm} = \text{Re}(\mathcal{E}_{\pm} \hat{\epsilon}_{\pm} e^{i\vec{k}\cdot\vec{x} - i\omega t}), \quad (8.25)$$

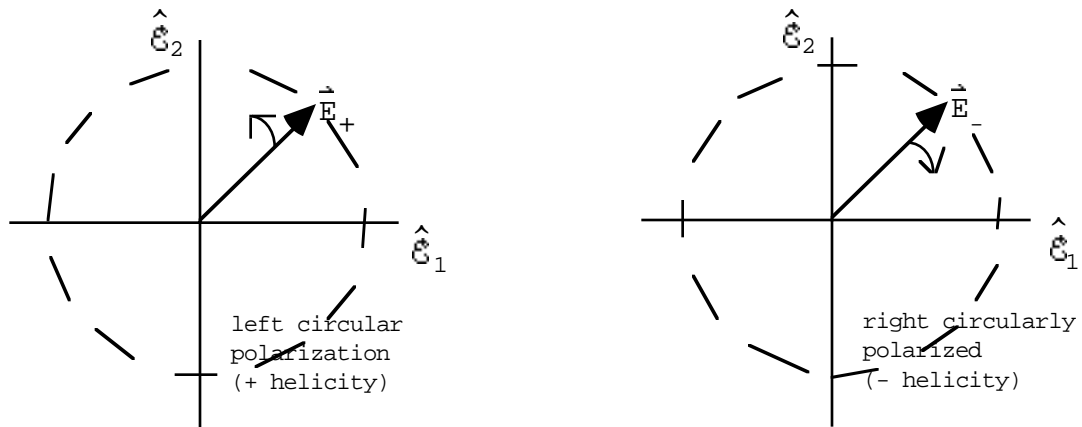
$$\vec{E}(\vec{x}, t) = \vec{E}_{+} + \vec{E}_{-}. \quad (8.26)$$

Let's say  $\mathcal{E}_{\pm}$  are real again. What do  $\vec{E}_{\pm}$  describe?

$$\vec{E}_{+} = \frac{\mathcal{E}_{+}}{\sqrt{2}} (\hat{\epsilon}_1 \cos(\vec{k}\cdot\vec{x} - \omega t) - \hat{\epsilon}_2 \sin(\vec{k}\cdot\vec{x} - \omega t)), \quad (8.27)$$

$$\vec{E}_{-} = \frac{\mathcal{E}_{-}}{\sqrt{2}} (\hat{\epsilon}_1 \cos(\vec{k}\cdot\vec{x} - \omega t) + \hat{\epsilon}_2 \sin(\vec{k}\cdot\vec{x} - \omega t)). \quad (8.28)$$

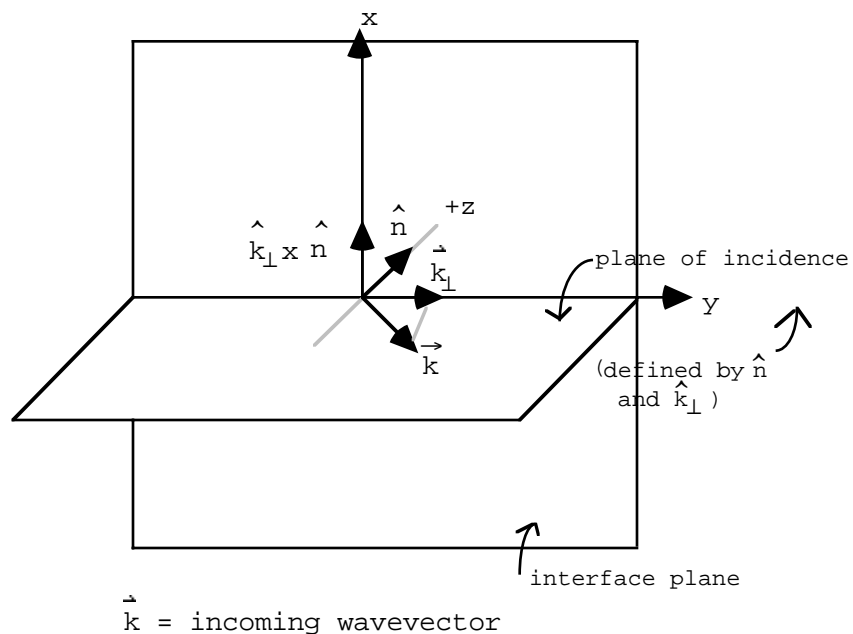
Pictures: ( $\vec{x}$  fixed;  $\hat{\eta}$  out of page)



The fact that only two independent polarizations of the electric (and magnetic) fields are possible is due to the photon nature of light. Photon spin can only point parallel/anti-parallel to  $\hat{\eta}$ .

### Reflection and refraction of plane waves from dielectric interfaces I: $E_{\perp}$ polarization

This next section is adapted from a treatment in Schwinger's book, Chapter 41. We wish now to study the reflection and refraction of plane waves at a plane interface between dielectrics. (We will use parts of this model again to find the fields inside an imperfect conductor in the next Chapter.) First, a big picture and some vector algebra.



Take the dielectric constant

$$\epsilon(z, \omega) = \begin{cases} \epsilon_2(\omega) & , z > 0, \\ \epsilon_1(\omega) & , z < 0. \end{cases} \quad (8.29)$$

Thus, we are assuming linear, homogeneous but possibly dispersive media. (More on the physical consequences of dispersion later.) For any vector field, we may use  $\hat{n}$ ,  $\hat{k}_\perp$ ,  $\hat{k}_\perp \times \hat{n}$  to decompose it. For example

$$\vec{E} = E_z \hat{n} + E_{||} \hat{k}_\perp + E_\perp (\hat{k}_\perp \times \hat{n}). \quad (8.30)$$

(However, remember  $\vec{k} \cdot \vec{E} = 0$ , so only 2 indep. components.) The notation here is tricky, but logical.  $E_\perp$  is the component of  $\vec{E}$  perpendicular to the plane of incidence (defined by  $\vec{k}$  and  $\hat{n}$ ), and  $E_{||}$  is the component parallel to the same plane. The relevant Maxwell equations are:

$$\vec{\nabla} \times \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{J}, \quad (8.31)$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t}. \quad (8.32)$$

We will concentrate on examining the behavior of em waves of a definite frequency near the interface. (Will assume a steady state situation). Fourier transform the above equations using

$$\int_{-\infty}^{\infty} dt e^{i\omega t} F(\vec{x}, t) \equiv F(\vec{x}, \omega), \quad (8.33)$$

( $F^*(\vec{x}, -\omega) = F(\vec{x}, \omega)$ ),  $\omega$  real.) Then the above become ( $\frac{\partial}{\partial t} \rightarrow -i\omega$ )

$$\vec{\nabla} \times \vec{H} = -\frac{i\omega}{c} \vec{D} + \frac{4\pi}{c} \vec{J}, \quad (8.34)$$

$$\vec{\nabla} \times \vec{E} = \frac{i\omega}{c} \vec{B}, \quad (8.35)$$

where  $\vec{H}$ ,  $\vec{E}$ , etc. have arguments  $(\vec{x}, \omega)$ . Also, assume  $\mu=1$  (removed in HW problem 2), so that (assume  $\epsilon(z, \omega)$  is real)

$$\vec{D}(\vec{x}, \omega) = \epsilon(z, \omega) \vec{E}(\vec{x}, \omega), \quad (8.36)$$

$$\vec{H}(\vec{x}, \omega) = \vec{B}(\vec{x}, \omega). \quad (8.37)$$

We now Fourier transform in coordinate space (use translational indep. in x and y directions)

$$\int d^2x_{\perp} e^{-i\vec{k}_{\perp} \cdot \vec{x}_{\perp}} F(\vec{x}, \omega) \equiv F(z, \vec{k}_{\perp}, \omega). \quad (8.38)$$

In the Maxwell eq<sup>n</sup>s, this means effectively

$$\vec{\nabla} \rightarrow i\vec{k}_{\perp} + \hat{n} \frac{\partial}{\partial z}, \quad (8.39)$$

and the eq<sup>n</sup>s become

$$\begin{aligned} & (i\vec{k}_{\perp} + \hat{n} \frac{\partial}{\partial z}) \times (B_z \hat{n} + B_{\parallel} \hat{k}_{\perp} + B_{\perp} (\hat{k}_{\perp} \times \hat{n})) \\ &= \frac{-i\omega\epsilon}{c} (E_z \hat{n} + E_{\parallel} \hat{k}_{\perp} + E_{\perp} (\hat{k}_{\perp} \times \hat{n})) + \frac{4\pi}{c} (J_z \hat{n} + J_{\parallel} \hat{k}_{\perp} + J_{\perp} (\hat{k}_{\perp} \times \hat{n})), \end{aligned} \quad (8.40)$$

and

$$\begin{aligned} & -(i\vec{k}_{\perp} + \hat{n} \frac{\partial}{\partial z}) \times (E_z \hat{n} + E_{\parallel} \hat{k}_{\perp} + E_{\perp} (\hat{k}_{\perp} \times \hat{n})) \\ &= \frac{-i\omega}{c} (B_z \hat{n} + B_{\parallel} \hat{k}_{\perp} + B_{\perp} (\hat{k}_{\perp} \times \hat{n})). \end{aligned} \quad (8.41)$$

Match components. Curl  $\vec{H}$  one becomes:

$$\hat{n}: \quad k_{\perp} B_{\perp} = \frac{\omega\epsilon}{c} E_z + \frac{4\pi i}{c} J_z, \quad (8.42)$$

$$\hat{k}_{\perp}: \quad \frac{\partial}{\partial z} B_{\perp} = -\frac{i\omega\epsilon}{c} E_{\parallel} + \frac{4\pi}{c} J_{\parallel}, \quad (8.43)$$

$$\hat{k}_\perp \times \hat{n}: \quad \frac{\partial}{\partial z} B_\parallel - iB_z k_\perp = \frac{i\omega\epsilon}{c} E_\perp - \frac{4\pi}{c} J_\perp. \quad (8.44)$$

Again, for the curl  $\vec{E}$  eq<sup>n</sup>:

$$\hat{n}: \quad k_\perp E_\perp = -\frac{\omega}{c} B_z, \quad (8.45)$$

$$\hat{k}_\perp: \quad \frac{\partial}{\partial z} E_\perp = \frac{i\omega}{c} B_\parallel, \quad (8.46)$$

$$\hat{k}_\perp \times \hat{n}: \quad \frac{\partial}{\partial z} E_\parallel - iE_z k_\perp = -\frac{i\omega}{c} B_\perp. \quad (8.47)$$

(These give  $\vec{B}$  if  $\vec{E}$  is known.) Get 2nd order eq<sup>n</sup>s for  $E_\perp, B_\perp$ :

$$\begin{aligned} \frac{\partial^2}{\partial z^2} E_\perp &= \frac{i\omega}{c} \frac{\partial}{\partial z} B_\parallel \\ &= \frac{i\omega}{c} \left( iB_z k_\perp + \frac{i\omega\epsilon}{c} E_\perp - \frac{4\pi}{c} J_\perp \right), \\ &= -\frac{\omega^2\epsilon}{c^2} E_\perp - i\frac{4\pi\omega}{c^2} J_\perp - \frac{\omega}{c} k_\perp \left( -\frac{c}{\omega} k_\perp E_\perp \right). \end{aligned} \quad (8.48)$$

Differential eq<sup>n</sup> is therefore

$$\left[ \frac{\partial^2}{\partial z^2} - k_\perp^2 + \frac{\omega^2\epsilon}{c^2} \right] E_\perp = -i\frac{4\pi\omega}{c^2} J_\perp. \quad (8.49)$$

Likewise (remember  $\epsilon = \epsilon(z, \omega)$ )

$$\begin{aligned} \frac{\partial}{\partial z} \frac{1}{\epsilon} \frac{\partial}{\partial z} B_\perp &= -\frac{i\omega}{c} \frac{\partial}{\partial z} E_\parallel + \frac{4\pi}{c} \frac{\partial}{\partial z} \left( \frac{1}{\epsilon} J_\parallel \right), \\ &= -\frac{i\omega}{c} \left( iE_z k_\perp - i\frac{\omega}{c} B_\perp \right) + \frac{4\pi}{c} \frac{\partial}{\partial z} \left( \frac{1}{\epsilon} J_\parallel \right), \\ &= -\frac{\omega^2}{c^2} B_\perp + \frac{4\pi}{c} \frac{\partial}{\partial z} \left( \frac{1}{\epsilon} J_\parallel \right) + \frac{\omega}{c} k_\perp \left( \frac{k_\perp c}{\omega\epsilon} B_\perp - \frac{4\pi i}{\omega\epsilon} J_z \right). \end{aligned} \quad (8.50)$$

$$\Rightarrow \left[ \frac{\partial}{\partial z} \frac{1}{\epsilon} \frac{\partial}{\partial z} - \frac{k_\perp^2}{\epsilon} + \frac{\omega^2}{c^2} \right] B_\perp = -\frac{4\pi i}{c\epsilon} k_\perp J_z + \frac{4\pi}{c} \frac{\partial}{\partial z} \left( \frac{1}{\epsilon} J_\parallel \right). \quad (8.51)$$

Notice that both reduce to the wave eq<sup>n</sup> (with sources) if  $\epsilon = \text{const}$ . Given the sources  $J_{\perp}, J_{\parallel}$  and  $J_z$ , we can solve these eq<sup>n</sup>s for  $E_{\perp}, B_{\perp}$ . Can then get the expressions for  $E_{\parallel}, B_{\parallel}, E_z$  and  $B_z$  from the previous set.

Let's construct and solve the reduced Green function for these equations for a plane interface. The G.f. for  $E_{\perp}$  is of the usual (reduced) wave eq<sup>n</sup> form,

$$-\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2 \epsilon(z, \omega)}{c^2} - k_{\perp}^2\right) g(z, z') = \delta(z - z'). \quad (8.52)$$

Let's put this in perspective. The spacetime field  $E_{\perp}(\vec{x}, t)$  is given by

$$E_{\perp}(\vec{x}, t) = \int \frac{d^2 k_{\perp}}{(2\pi)^2} \int \frac{d\omega}{2\pi} e^{i\vec{k}_{\perp} \cdot \vec{x}_{\perp} - i\omega t} E_{\perp}(z, \vec{k}_{\perp}, \omega), \quad (8.53)$$

with (infinite space)

$$E_{\perp}(z, \vec{k}_{\perp}, \omega) = \frac{4\pi i \omega}{c^2} \int_{-\infty}^{\infty} dz' g(z, z') J_{\perp}(z', \vec{k}_{\perp}, \omega). \quad (8.54)$$

When  $z \neq z'$ , the solutions to  $g(z, z')$  are of the form used before (see (7.179); only the  $\epsilon(\omega)$  here is new)

$$g(z, z') \sim \begin{cases} \exp\left(\pm \sqrt{k_{\perp}^2 - \frac{\omega^2 \epsilon}{c^2}} z\right), & k_{\perp}^2 - \frac{\omega^2 \epsilon}{c^2} > 0, \\ \exp\left(\pm i S_{\omega} \sqrt{\frac{\omega^2 \epsilon}{c^2} - k_{\perp}^2} z\right), & \frac{\omega^2 \epsilon}{c^2} - k_{\perp}^2 > 0. \end{cases} \quad (8.55)$$

(Top part describes a quasi-static solution and the bottom part static and traveling waves; see prob. 15 of Ch.7. We can assume that  $\omega > 0$  without loss of generality from prob.19, Ch.7.) We will work with the branch for which  $\frac{\omega^2 \epsilon(\omega)}{c^2} - k_{\perp}^2 > 0$ ; if we need

to reconstruct the full Green function, the other branch follows by analytic continuation. Let us call

$$k_{1,2} \equiv \left( \frac{\omega^2 \epsilon_{1,2}}{c^2} - k_{\perp}^2 \right)^{1/2} > 0. \quad (8.56)$$

Then, putting the exponential behavior of  $e^{i\vec{k}_{\perp} \cdot \vec{x}_{\perp} - i\omega t}$  together with the  $e^{\pm i k_{1,2} z}$  behavior in  $g(z, z')$ , we see that  $E_{\perp}(\vec{x}, t)$  is just a superposition of waves of the form

$$e^{i\vec{k}_{\perp} \cdot \vec{x}_{\perp}} e^{\pm i k_{1,2} z} e^{-i\omega t}, \quad (8.57)$$

or

$$e^{i\vec{k}_{1,2} \cdot \vec{x} - i\omega t}, \quad (8.58)$$

where

$$\vec{k}_{1,2} = (\vec{k}_{\perp}, \pm k_{1,2}), \quad (8.59)$$

$\downarrow$  ——— effective z - component  
notation: not the magnitude of  $\vec{k}_{1,2}$ .

represents the direction of propagation. That is, we are assuming a dispersion relation

$$\vec{k}_{1,2}^2 = \frac{\omega^2 \epsilon_{1,2}}{c^2}, \quad (8.60)$$

where, however,  $\epsilon_{1,2}$  are unknown (but positive) functions of  $\omega$ . If we take the + sign above, this represents plane waves with a positive component of propagation along  $z$  (or  $\hat{n}$ ), and similarly for the - sign. Let us imagine that the sources are in region 2, that is  $z' > 0$ . (This is a choice of B.C. on  $g(z, z')$ ; there is also a G.f. solution to describe waves originating in region 1.) Given this choice, the solution for  $g(z, z')$  must be of the form:

$$g = \begin{cases} A e^{ik_2 z}, & z > z', \\ B e^{ik_2 z} + C e^{-ik_2 z}, & z' > z > 0, \\ D e^{-ik_1 z}, & z < 0. \end{cases} \quad (8.61)$$

The other BC's on  $g$  are

$$g(z, z') \Big|_{z=0^-}^{z=0^+} = 0, \quad (E_{\perp} \text{ is continuous across interface}) \quad (8.62)$$

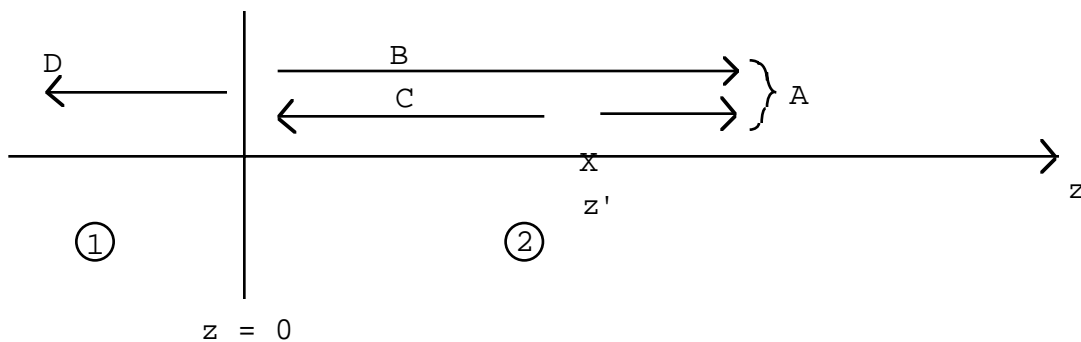
$$\frac{\partial g(z, z')}{\partial z} \Big|_{0^-}^{0^+} = 0. \quad (\text{from } \frac{\partial E_{\perp}}{\partial z} = \frac{i\omega}{c} B_{\parallel}) \quad (8.63)$$

( $B_{\parallel}$  is continuous across interface for zero free surface current; remember  $\mu = 1$  on both sides.) At  $z = z'$ , the usual conditions (from the differential equation) are

$$g(z, z') \Big|_{z=z'^-}^{z=z'^+} = 0, \quad (8.64)$$

$$-\frac{\partial}{\partial z} g(z, z') \Big|_{z=z'^-}^{z=z'^+} = 1. \quad (8.65)$$

What's happening is :



These equations give

$$B + C = D, \quad (8.62)$$

$$B - C = -\frac{k_1}{k_2} D, \quad (8.63)$$

$$Ae^{ik_2 z'} = Be^{ik_2 z'} + Ce^{-ik_2 z'}, \quad (8.64)$$

$$ik_2(-Ae^{ik_2 z'} + Be^{ik_2 z'} - Ce^{-ik_2 z'}) = 1. \quad (8.65)$$

Won't bore you with the details. Solution is (4 eq<sup>n</sup>s in 4 unknowns)

$$A = \frac{k_2 - k_1}{k_2 + k_1} \frac{i}{2k_2} e^{ik_2 z'} + \frac{i}{2k_2} e^{-ik_2 z'}, \quad (8.66)$$

$$B = \frac{k_2 - k_1}{k_2 + k_1} \frac{i}{2k_2} e^{ik_2 z'}, \quad (8.67)$$

$$C = \frac{i}{2k_2} e^{ik_2 z'}, \quad (8.68)$$

$$D = \frac{2k_2}{k_2 + k_1} \frac{i}{2k_2} e^{ik_2 z'}. \quad (8.69)$$

This means (strictly only for  $\frac{\omega^2 \epsilon}{c^2} - k_{\perp}^2 > 0$ ; can combine the forms for  $z > z'$  and  $z' > z > 0$  into one for  $z > 0$ )

$$g(z, z') = \begin{cases} \frac{k_2 - k_1}{k_2 + k_1} \frac{i}{2k_2} e^{ik_2(z + z')} + \frac{i}{2k_2} e^{ik_2|z - z'|}, & z > 0, \\ \frac{2k_2}{k_2 + k_1} \frac{i}{2k_2} e^{-ik_1 z} e^{ik_2 z'}, & z < 0. \end{cases} \quad (8.70)$$

Notice, if  $\epsilon_{1,2} = 1$ , then ( $k_1 = k_2 = 0$ )

$$g(z, z') \rightarrow \frac{i}{2\theta} e^{i\theta|z - z'|} \quad \forall z, \quad (8.71)$$

which is the same form as the  $g(z, z')$  found in Ch. 7 for traveling waves when  $\frac{\omega^2}{c^2} > k_{\perp}^2$ ,  $\omega > 0$ . Also note that for  $\epsilon_1 \rightarrow \infty$  or  $k_1 \rightarrow \infty$  (conductors)

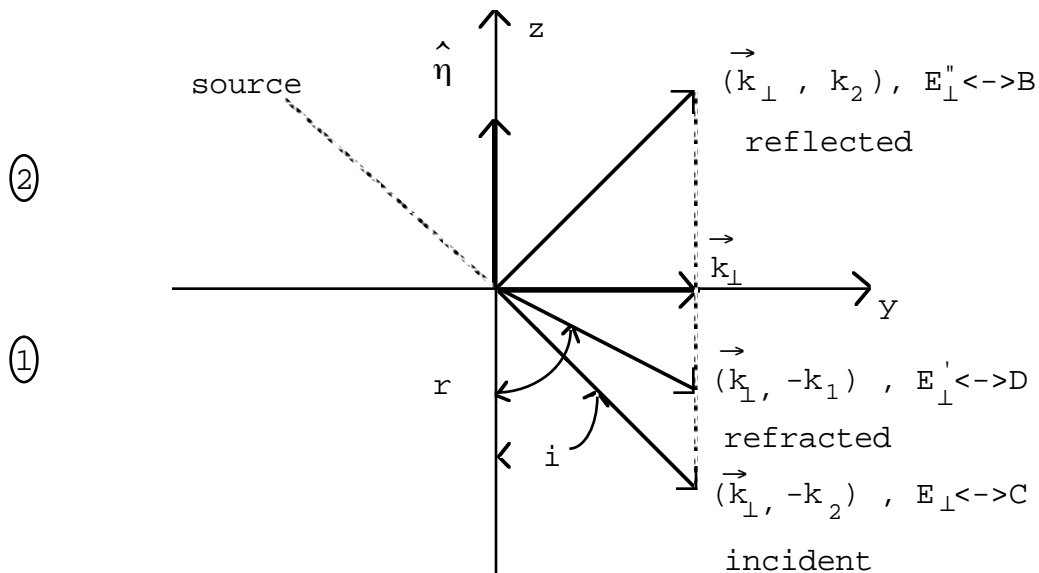
$$g(z, z') \rightarrow \begin{cases} \frac{i}{2k_2} (e^{ik_2|z - z'|} - e^{ik_2(z + z')}) & , z > 0, \\ 0, & z < 0, \end{cases} \quad (8.72)$$

which is the same form for the reduced G.f. for the conducting plane for  $\frac{\omega^2}{c^2} > k_{\perp}^2$  ( $\epsilon_2 = 1$ , say) when  $\omega > 0$ . The ratios

$$\frac{D}{C} = \frac{2k_2}{k_1 + k_2'} \quad (8.73)$$

$$\frac{B}{C} = \frac{k_2 - k_1}{k_1 + k_2'} \quad (8.74)$$

are the transmission and reflection coefficient, respectively. What is their significance in terms of fields? Connect up with book's notation (diagram slightly different from book):



In the above figure, the  $E_\perp$  components are measured with respect to the  $x$  axis ( $\hat{k}_\perp \times \hat{n}$  direction).

Points:

1. I have shown angle incidence = angle reflection because

$$(\vec{k}_\perp, -k_2) \rightarrow (\vec{k}_\perp, k_2). \quad (8.75)$$

2. We also have Snell's law since  $|\vec{k}_\perp| = \text{const.}$ :

$$\sqrt{k_\perp^2 + (k_2)^2} \sin i = \sqrt{k_\perp^2 + (k_1)^2} \sin r$$

and

$$k_{\perp}^2 + (k_{1,2})^2 = \frac{\omega^2}{c^2} n_{1,2}^2, \quad (n_{1,2}^2 \equiv \epsilon_{1,2}) \quad (8.76)$$

$$\Rightarrow n_2 \sin i = n_1 \sin r. \quad (8.77)$$

From the picture we identify

$$\frac{D}{C} = \frac{E'_{\perp}}{E_{\perp}}, \quad (8.78)$$

$$\frac{B}{C} = \frac{E''_{\perp}}{E_{\perp}}. \quad (8.79)$$

Thus, since

$$k_1 = \sqrt{k_{\perp}^2 + k_1^2} \cos r = \frac{\omega}{c} n_1 \cos r, \quad (8.80)$$

$$k_2 = \sqrt{k_{\perp}^2 + k_2^2} \cos i = \frac{\omega}{c} n_2 \cos i, \quad (8.81)$$

we have

$$\frac{E'_{\perp}}{E_{\perp}} = \frac{2n_2 \cos i}{n_1 \cos r + n_2 \cos i}, \quad (8.82)$$

$$\frac{E''_{\perp}}{E_{\perp}} = \frac{n_2 \cos i - n_1 \cos r}{n_1 \cos r + n_2 \cos i}. \quad (8.83)$$

( $n_2 \rightarrow n$ ,  $n_1 \rightarrow n'$  in book.) For normal incidence ( $i = r = 0$ ) these become

$$\frac{E'_{\perp}}{E_{\perp}} \rightarrow \frac{2n_2}{n_1 + n_2} \Rightarrow \text{pure reflection if } n_1/n_2 \rightarrow \infty, \quad (8.84)$$

$$\frac{E''_{\perp}}{E_{\perp}} \rightarrow \frac{n_2 - n_1}{n_1 + n_2} \Rightarrow \text{phase reversal if } n_1 > n_2. \quad (8.85)$$

### Reflection and refraction of plane waves from dielectric interfaces II: $B_{\perp}$ polarization

Now let's do the other polarization. The reduced Green function for  $B_{\perp}$  is

$$- \left[ \frac{\partial}{\partial z} \frac{1}{\epsilon} \frac{\partial}{\partial z} - \frac{k_{\perp}^2}{\epsilon} + \frac{\omega^2}{c^2} \right] g(z, z') = \delta(z - z'), \quad (8.86)$$

or

$$- \left[ \frac{\partial}{\partial z} \frac{1}{\epsilon} \frac{\partial}{\partial z} + \frac{k_{1,2}^2}{\epsilon} \right] g(z, z') = \delta(z - z'). \quad (8.87)$$

with  $k_{1,2}$  as before. Given the plane at  $z = 0$ , the traveling wave solutions in the various regions when the source is at  $z' > 0$  are (source as before)

$$g = \begin{cases} A e^{ik_2 z} & , z > z' \\ B e^{ik_2 z} + C e^{-ik_2 z} & , z' > z > 0 \\ D e^{-ik_1 z} & , 0 > z. \end{cases} \quad (8.88)$$

However, the BC's are now

$$g|_{0+}^{0-} = 0, \quad (B_{\perp} \text{ continuous}) \quad (8.89)$$

$$\frac{1}{\epsilon} \frac{\partial g}{\partial z} \Big|_{0+}^{0-} = 0. \quad \left( \frac{ic}{\omega} \frac{1}{\epsilon} \frac{\partial}{\partial z} B_{\perp} = E_{\parallel} \right) \quad (8.90)$$

( $E_{\parallel}$  = continuous.) In addition

$$g \Big|_{z'-}^{z'+} = 0, \quad (8.91)$$

$$- \frac{1}{\epsilon_2} \frac{\partial}{\partial z} g \Big|_{z'-}^{z'+} = 1. \quad (8.92)$$

These become

$$B + C = D, \quad (8.89)$$

$$B - C = - \frac{\epsilon_2 k_1}{\epsilon_1 k_2} D, \quad (8.90)$$

$$Ae^{ik_2 z'} = Be^{ik_2 z'} + Ce^{-ik_2 z'}, \quad (8.91)$$

$$ik_2(-A e^{ik_2 z'} + Be^{ik_2 z'} - Ce^{-ik_2 z'}) = \epsilon_2. \quad (8.92)$$

Solve them:

$$A = \frac{1 - \frac{\epsilon_2 k_1}{\epsilon_1 k_2}}{1 + \frac{\epsilon_2 k_1}{\epsilon_1 k_2}} \frac{i\epsilon_2}{2k_2} e^{ik_2 z'} + \frac{i\epsilon_2}{2k_2} e^{-ik_2 z'}, \quad (8.93)$$

$$B = \frac{1 - \frac{\epsilon_2 k_1}{\epsilon_1 k_2}}{1 + \frac{\epsilon_2 k_1}{\epsilon_1 k_2}} \frac{i\epsilon_2}{2k_2} e^{ik_2 z'}, \quad (8.94)$$

$$C = \frac{i\epsilon_2}{2k_2} e^{ik_2 z'}, \quad (8.95)$$

$$D = \frac{2}{1 + \frac{\epsilon_2 k_1}{\epsilon_1 k_2}} \frac{i\epsilon_2}{2k_2} e^{ik_2 z'}. \quad (8.96)$$

Same as before except  $k_{1,2} \rightarrow \frac{k_{1,2}}{\epsilon_{1,2}}$  everywhere in the coefficients of the exponents. The transmission and reflection coefficients are:

$$\frac{B'_\perp}{B_\perp} = \frac{2k_2/\epsilon_2}{k_1/\epsilon_1 + k_2/\epsilon_2}, \quad (8.97)$$

$$\frac{B''_\perp}{B_\perp} = \frac{k_2/\epsilon_2 - k_1/\epsilon_1}{k_1/\epsilon_1 + k_2/\epsilon_2}, \quad (8.98)$$

or ( $k_1 = \frac{\omega}{c} n_1 \cos r$ ,  $k_2 = \frac{\omega}{c} n_2 \cos i$ )

$$\frac{B'_{\perp}}{B_{\perp}} = \frac{2n_1 n_2 \cos i}{n_1 n_2 \cos i + n_2^2 \cos r}, \quad (8.99)$$

$$\frac{B''_{\perp}}{B_{\perp}} = \frac{n_1^2 \cos i - n_1 n_2 \cos r}{n_1^2 \cos i + n_1 n_2 \cos r}. \quad (8.100)$$

Note: The  $B_{\perp}$  results are equivalent to Jackson's results (Section 7.4) for  $E_0$  for the " $\vec{E}$  parallel to plane of incidence case" since ( $\vec{B} = \sqrt{\epsilon} \hat{k} \times \vec{E}$ )

$$\frac{B'_{\perp}}{B_{\perp}} = \left( \frac{E'_0}{E_0} \right) \frac{n_1}{n_2}, \quad (8.101)$$

↓ Jackson quantity

$$\frac{B''_{\perp}}{B_{\perp}} = \left( \frac{E''_0}{E_0} \right). \quad (8.102)$$

From the  $E_{\perp}$  and  $B_{\perp}$  results, we can find results for the other components of  $\vec{E}$  and  $\vec{B}$ . Also, for normal incidence

$$\frac{B'_{\perp}}{B_{\perp}} = \frac{2n_1}{n_1 + n_2}, \quad \left( \frac{E'_{\perp}}{E_{\perp}} = \frac{2n_2}{n_1 + n_2} \right) \quad (8.103)$$

$$\frac{B''_{\perp}}{B_{\perp}} = \frac{n_1 - n_2}{n_1 + n_2}. \quad \left( \frac{E''_{\perp}}{E_{\perp}} = \frac{n_2 - n_1}{n_1 + n_2} \right) \quad (8.104)$$

These results suggest that for conductors (at low frequencies,  $\omega$ ), which are described phenomenologically by  $\epsilon(\omega) \sim 4\pi i\sigma(0)/\omega$  where  $\sigma(0)$  is the static conductivity, the transmitted fields inside the conductor are mainly magnetic.

### **Brewster's angle and total internal reflection**

It is interesting to ask when the reflection coefficient vanishes. In the  $E_{\perp}$  case, this happens only when  $k_1 = k_2$  or  $\epsilon_1 = \epsilon_2$  (no interface). For  $B_{\perp}$  we have

$$\frac{k_1}{\epsilon_1} = \frac{k_2}{\epsilon_2}. \quad (8.105)$$

However

$$\frac{k_{1,2}^2}{\epsilon_{1,2}^2} = \frac{\omega^2}{c^2} \frac{1}{\epsilon_{1,2}} - \frac{k_{\perp}^2}{\epsilon_{1,2}^2}, \quad (8.106)$$

$$\Rightarrow \frac{(\epsilon_2 - \epsilon_1)}{\epsilon_2 \epsilon_1} \left[ \frac{\omega^2}{c^2} - k_{\perp}^2 \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \right] = 0. \quad (8.107)$$

Can vanish when

$$\frac{\omega}{c} = \sqrt{\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2}} |\vec{k}_{\perp}|. \quad (8.108)$$

But  $|\vec{k}_{\perp}| = \frac{\omega}{c} n_2 \sin i$ , so

$$\sin i_B = \sqrt{\frac{\epsilon_1}{\epsilon_1 + \epsilon_2}},$$

$$\text{or } \tan i_B = \sqrt{\epsilon_1/\epsilon_2} = \frac{n_1}{n_2}. \quad (\text{"Brewster's angle"}) \quad (8.109)$$

At this angle  $B_{\perp}$  is completely transmitted, so the reflected wave is completely polarized with  $\vec{E} \perp$  to the plane of incidence. There is also the phenomenon of total internal reflection if  $n_2 > n_1$ . Snell's law says

$$n_1 \sin r = n_2 \sin i, \quad (8.110)$$

$$\frac{n_2}{n_1} > 1 \Rightarrow r > i.$$

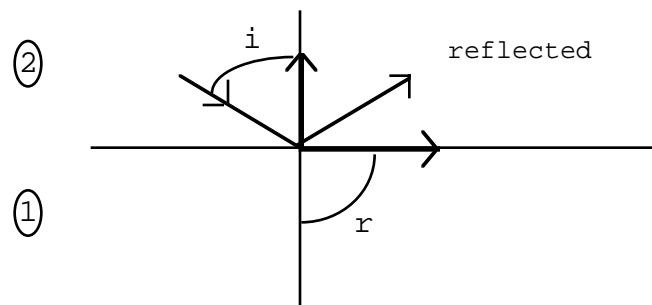
Thus, there are values of  $i$  such that

$$\sin r = \frac{n_2}{n_1} \sin i > 1. \quad (8.111)$$

What does this mean? At the angle

$$\sin i = \frac{n_1}{n_2} \Rightarrow r = 90^\circ, \quad (8.112)$$

the light ray grazes along the interface:



For angles  $i$  greater than this

$$k_1 = \sqrt{\frac{\omega^2}{c^2} \epsilon_1 - k_{\perp}^2} \Rightarrow k_1 \rightarrow i\kappa, \kappa \text{ real.}$$

Then the form of the wave in region 1 becomes

$$e^{i\vec{k}_{\perp} \cdot \vec{x}_{\perp}} e^{-\kappa z} e^{-i\omega t},$$

which means the wave penetrates a distance  $\sim 1/\kappa$  but is damped out exponentially and there is no net flow of energy into region 1 (under steady state conditions). This is the "quasi-static solution" possibility mentioned above.

Of course, we don't have to use the above Green function formalism to get reflection and transmission coefficients. Can simply enforce the correct BC's (normal  $\vec{D}$ , normal  $\vec{B}$ , tangential  $\vec{E}$ , tangential  $\vec{H}$ ) at the interface on a given plane wave.

However, this treatment exhibits more closely the connection

with the previous Green functions for free space and the conducting plane, and emphasizes that all the relevant information is in these functions.

### Simple model for constitutive relations

Let us now discuss some simple models for constitutive relations in metals, dielectrics and plasmas. These models lead to a qualitative understanding of the electromagnetic properties, especially the response to external fields, in these materials. There is a lot of physics here, but we must keep in the back of our minds that the following is only a phenomenological description.

In our development of the macroscopic Maxwell equations, we made a distinction between bound and free charges and currents. Of course in real materials, these charges and currents are almost always electrons. Consider such an electron, either bound or free, moving under the influence of an external electric field, but being slowed by a resistive - like force. We write

$$m[\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x}] = e \vec{E}(t). \quad (8.113)$$

This equation, and its solution, should be familiar to you (at least in one spatial dimension) from classical dynamics.  $\omega_0$  is a natural frequency of the electron (due to energy levels), and  $\omega_0^2 \vec{x}$  represents a restoring force for a bound electron ( $\omega_0 = 0$ ) for a free or conductive (valence) electron).  $\gamma$  is a damping constant primarily due to radiation for bound electrons, whereas for free electrons the origin is in collisions with other electrons, lattice imperfections, impurities, etc. [For bound electrons usually  $\gamma \ll \omega_0$ .] You will study the Green function solutions to this equation in a HW problem. I will not stop here to discuss the well-known aspects of this linear

differential equation, but will point out the relevant properties of its solutions that we will need.

First, let us discuss the model for conductors. We take  $\omega_0 = 0$  and write

$$m(\ddot{\vec{x}} + \gamma \dot{\vec{x}}) = e \vec{E}(t). \quad (8.114)$$

We define  $\vec{v}(t) = \dot{\vec{x}}$  and write this as

$$\frac{d}{dt} (e^{\gamma t} \vec{v}(t)) = \frac{e}{m} e^{\gamma t} \vec{E}(t). \quad (8.115)$$

Integrate from  $t' = -\infty$  (or from where the initial conditions are given):

$$\begin{aligned} \int_{-\infty}^t \frac{d}{dt'} (e^{\gamma t'} \vec{v}(t')) &= \frac{e}{m} \int_{-\infty}^t e^{\gamma t'} \vec{E}(t') dt', \\ \Rightarrow \vec{v}(t) &= \frac{e}{m} \int_{-\infty}^t e^{-\gamma(t-t')} \vec{E}(t') dt'. \end{aligned} \quad (8.116)$$

Notice in this model the response is non-local in time. That is,  $\vec{v}(t)$  depends upon  $\vec{E}(t')$  at earlier times. The main contribution to the integral comes from time differences of order  $\gamma^{-1}$ .

Assuming a constant density of conduction electrons,  $n_c$ , the current  $\vec{J}$  is given by

$$\vec{J} = n_c e \vec{v} = \frac{n_c e^2}{m} \int_{-\infty}^t e^{-\gamma(t-t')} \vec{E}(t') dt'. \quad (8.117)$$

For the specific case of a constant electric field, this becomes

$$\vec{J} = \frac{n_c e^2}{m\gamma} \vec{E} \equiv \sigma \vec{E}, \quad (8.118)$$

giving Ohm's law with  $\sigma$  being the static conductivity. Notice: inversely proportional to  $\gamma$  (makes sense).

A more general situation is where the electric field exhibits harmonic time variation, as in the case of em waves. Take ( $\vec{\mathcal{E}}(\omega)$  can in general be complex to describe different polarizations)

$$\vec{E}(t) = \text{Re}(\vec{\mathcal{E}}(\omega)e^{-i\omega t}). \quad (8.119)$$

Then, the current density becomes

$$\begin{aligned} \vec{J}(t) &= \text{Re} \left( \frac{n_c e^2}{m} \vec{\mathcal{E}}(\omega) \int_{-\infty}^t dt' e^{-\gamma(t-t')} e^{-i\omega t'} \right), \\ &= \text{Re} \left( \frac{n_c e^2}{m} \frac{1}{\gamma - i\omega} \vec{\mathcal{E}}(\omega) e^{-i\omega t} \right), \\ &= \text{Re} (\sigma(\omega) \vec{\mathcal{E}}(\omega) e^{-i\omega t}). \end{aligned} \quad (8.120)$$

This gives us a generalized Ohm's law with

$$\sigma(\omega) = \frac{n_c e^2}{m} \frac{1}{\gamma - i\omega}, \quad (8.121)$$

and which coincides with the static value above for  $\omega = 0$ .

Notice that the real and imaginary parts of  $\sigma(\omega)$ ,

$$\text{Re } \sigma(\omega) = \frac{n_c e^2}{m} \frac{\gamma}{\gamma^2 + \omega^2}, \quad (8.122)$$

$$\text{Im } \sigma(\omega) = \frac{n_c e^2}{m} \frac{\omega}{\gamma^2 + \omega^2}, \quad (8.123)$$

are even & odd functions of  $\omega$ , respectively. In addition, we get the amplitude and phase of  $\sigma(\omega)$  as

$$\sigma(\omega) = |\sigma(\omega)| e^{i\delta}, \quad (8.124)$$

$$|\sigma(\omega)| = \frac{n_c e^2}{m} \frac{1}{\sqrt{\gamma^2 + \omega^2}}, \quad (8.125)$$

$$\tan \delta = \frac{\omega}{\gamma}. \quad (8.126)$$

What is the meaning of a complex conductivity?

$$\begin{aligned} \vec{J}(t) &= \text{Re} (\sigma(\omega) \vec{\mathcal{E}}(\omega) e^{-i\omega t}), \\ &= \frac{n_c e^2}{m} \frac{1}{\sqrt{\gamma^2 + \omega^2}} \text{Re} (\vec{\mathcal{E}}(\omega) e^{-i\omega t + \delta}). \end{aligned} \quad (8.127)$$

⇒ Current and field out of phase. For copper,  $\sigma$  is essentially real well beyond the microwave region ( $\lesssim 10^{11} \text{ sec}^{-1}$ ). Finally if the conductivity is integrated over all frequencies, we find

$$\int_{-\infty}^{\infty} d\omega \sigma(\omega) = \frac{n_c e^2}{m} \int_{-\infty}^{\infty} d\omega \frac{\gamma + i\omega}{\gamma^2 + \omega^2} = \frac{n_c e^2}{m} \int_{-\infty}^{\infty} \frac{d\omega}{\gamma} \frac{1}{1 + \omega^2/\gamma^2} = \frac{n_c e^2}{m} \pi, \quad (8.128)$$

which is real independent of  $\gamma$  and can be used to determine  $n_c$ .

If we are dealing with bound electrons, we must include the  $\omega_0^2 \vec{x}$  term in the differential eq<sup>n</sup>. Let us examine again the situation of harmonic time dependence. Consider

$$m[\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x}] = e \text{Re} [\vec{\mathcal{E}}(\omega) e^{-i\omega t}], \quad (8.129)$$

The steady state solution to this is given by assuming

$$\vec{x}(t) = \text{Re} [\vec{x}(\omega) e^{-i\omega t}], \quad (8.130)$$

from which we find

$$\vec{x}(t) = \frac{e}{m} \text{Re} \left[ \frac{\vec{\mathcal{E}}(\omega) e^{-i\omega t}}{-\omega^2 + \omega_0^2 - i\gamma\omega} \right]. \quad (8.131)$$

Within this model we are trying to calculate the effective macroscopic properties (constitutive relations). If the electrons are in their "equilibrium" positions", then we assume that the polarization,  $\vec{P}$ , of the sample vanishes. However, for an electron displaced to  $\vec{x}$ , we assume

$$\vec{P}(t) = \frac{e}{V} \sum_{i=\text{atoms}} \vec{x}_i(t). \quad (8.132)$$

If  $\vec{x}_i = \vec{x} \forall i$ , then

$$\vec{P}(t) = n_b e \vec{x}(t) \equiv \text{Re} [\chi(\omega) \vec{\mathcal{E}}(\omega) e^{-i\omega t}], \quad (8.133)$$

↑ "bound" electrons

where the  $\omega$ -dependent susceptibility is given by

$$\chi(\omega) = \frac{n_b e^2}{m} \frac{1}{-\omega^2 + \omega_0^2 - i\omega\gamma}. \quad (8.134)$$

In the static case

$$\chi(0) = \frac{n_b e^2}{m\omega_0^2} > 0, \quad (8.135)$$

$$\Rightarrow \epsilon(0) \equiv 1 + 4\pi\chi(0) > 1 \text{ as expected.}$$

For arbitrary  $\omega$ , we can define a frequency dependent dielectric constant by

$$\vec{D}(t) \equiv \text{Re} (\epsilon(\omega) \vec{\mathcal{E}}(\omega) e^{-i\omega t}), \quad (8.136)$$

so that

$$\epsilon(\omega) = 1 + 4\pi\chi(\omega) = 1 + \frac{4\pi n_b e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}, \quad (8.137)$$

$$\Rightarrow \operatorname{Re} \varepsilon(\omega) = 1 + \frac{4\pi n_b e^2}{m} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}, \quad (8.138)$$

$$\operatorname{Im} \varepsilon(\omega) = 4\pi \frac{n_b e^2}{m} \frac{\omega \gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}. \quad (8.139)$$

The dependence of  $\varepsilon$  on  $\omega$  is called dispersion, which we'll study in more detail later.

There is an integral rule for  $\varepsilon(\omega)$  as well. Consider

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega \operatorname{Im} \varepsilon(\omega) &= 4 \frac{n_b e^2}{m} \int_{-\infty}^{\infty} d\omega \frac{\omega \gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}, \\ &= \frac{4\pi n_b e^2}{m}. \end{aligned} \quad (8.140)$$

This makes sense from the total current involved. We have

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}, \quad (8.141)$$

"conduction current"  $\uparrow$   $\uparrow$  "displacement current"

where

$$\vec{D} = \operatorname{Re}(\varepsilon(\omega) \vec{E} e^{-i\omega t}), \quad (8.142)$$

$$\vec{J} = \operatorname{Re}(\sigma(\omega) \vec{E} e^{-i\omega t}), \quad (8.143)$$

so

$$\frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \operatorname{Re} \left[ \left( \frac{4\pi}{c} \sigma(\omega) - \frac{i\omega}{c} \varepsilon(\omega) \right) \vec{E} e^{-i\omega t} \right]. \quad (8.144)$$

Write

$$\sigma_{\text{eff}} \equiv \sigma(\omega) - \frac{i\omega}{4\pi} \varepsilon(\omega). \quad (8.145)$$

Then

$$\begin{aligned}
\int_{-\infty}^{\infty} d\omega \sigma_{\text{eff}} &= \int_{-\infty}^{\infty} d\omega \left[ \text{Re } \sigma(\omega) + i \frac{4\pi}{c} \text{Im } \sigma(\omega) \right] \\
&+ \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \omega \left[ \text{Im } \epsilon(\omega) - i \frac{4\pi}{c} \text{Re } \epsilon(\omega) \right] \\
&= \frac{e^2 \pi}{m} (n_c + n_d). \tag{8.146}
\end{aligned}$$

conduction ↑
↑ displacement

We can also take the opposite point of view and define an effective dielectric constant:

$$\epsilon_{\text{eff}} \equiv \epsilon(\omega) + i \frac{4\pi\sigma(\omega)}{\omega}. \tag{8.147}$$

Can then write

$$k^2 = \frac{\mu\omega^2}{c^2} \left( \epsilon(\omega) + i \frac{4\pi\sigma(\omega)}{\omega} \right), \tag{8.148}$$

$$k \equiv \alpha + i\beta. \tag{8.149}$$

One can solve explicitly for  $\alpha$  and  $\beta$ ; see Prob. 17 at the end of the chapter for normally incident waves on a surface. The  $\beta$  coefficient causes damping and energy loss to the medium. The next chapter will deal with the forms induced for  $\beta$  when the waves we are dealing with propagate in a metallic waveguide.

### Model applications to plasmas, metals and dielectrics

Now let us discuss various applications of these formulas in matter. Using our model for  $\epsilon(\omega)$  (which we applied to bound electrons), we may replace our former  $\epsilon(\omega)$  by

$$\epsilon(\omega) = 1 + \frac{4\pi e^2}{m} \sum_i \frac{n_b^i}{\omega_0^{i2} - \omega^2 - i\omega\gamma^i}, \tag{8.150}$$

if we have many resonances. We define

$$\sum_{\mathbf{i}} n_{\mathbf{b}}^{\mathbf{i}} = N_{\mathbf{b}}, \quad (8.151)$$

which gives the total density of all bound electrons. Imagine there is a resonance at  $\omega_0 = 0$  in this material. Then we may write

$$\varepsilon(\omega) = \varepsilon_0(\omega) + i \frac{4\pi}{\omega} \left( \frac{e^2 n_{\mathbf{b}}^0}{m} \frac{1}{\gamma - i\omega} \right). \quad (8.152)$$

all other resonances  $\uparrow$   $\uparrow \omega_0 = 0$  part

But the quantity  $(\dots)$  is just of the form of  $\sigma(\omega)$  for a conductor (if we recognize that  $n_{\mathbf{b}}^0 \rightarrow n_{\mathbf{c}}$ ) and we have

$$\varepsilon(\omega) = \varepsilon_0(\omega) + i \frac{4\pi\sigma(\omega)}{\omega}, \quad (8.153)$$

again, showing that we may just think of conductors as materials with a resonance frequency  $\omega_0 = 0$ . Assuming we have a conductor, at low ( $\omega \ll \gamma$ ) frequencies ( $\varepsilon_0(\omega) \sim \text{const.}$ ), we get

$$\varepsilon(\omega) \approx \varepsilon_0 + i \frac{4\pi\sigma(0)}{\omega}. \leftarrow \text{free electron piece} \quad (8.154)$$

$\uparrow$   
usual static piece

Notice  $\varepsilon(\omega) \rightarrow +\infty$  as  $\omega \rightarrow 0^+$ . In our earlier discussions of dielectric Green functions, we recovered perfect conductor Green functions by taking the  $\rightarrow +\infty$  limit. Now we know why this works, although we see it is more appropriate to take this limit through positive imaginary numbers.

If we are working with a very good conductor, then from (low frequencies)

$$k^2 = \frac{\omega^2}{c^2} \mu \epsilon \approx i \frac{4\pi\mu\sigma(0)\omega}{c^2}, \quad (8.155)$$

$$\Rightarrow k \approx (1 + i) \frac{\sqrt{2\pi\omega\mu\sigma(0)}}{c}. \quad (8.156)$$

Thus, a plane wave solution inside the conductor is damped,

$$e^{i\vec{k}\cdot\vec{x}} \rightarrow e^{-\hat{k}\cdot\vec{x}/\delta} e^{i\hat{k}\cdot\vec{x}/\delta}, \quad (8.157)$$

where  $\delta$ , the "skin depth", is given by

$$\delta = \frac{c}{\sqrt{2\pi\omega\sigma(0)}}. \quad (8.158)$$

Notice that at high frequencies for either dielectrics ( $\omega \gg \omega_0^i$ , all  $i$ ) or conductors ( $\omega \gg \gamma$ ), we have

$$\epsilon(\omega) \approx 1 - \frac{4\pi e^2 N}{m\omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2}, \quad (8.159)$$

↓ "plasma frequency"

$$N = n_b + n_c.$$

(Actually, we also need  $\omega \gg \omega_p$  to apply this to dielectrics.)

Then, using the above relation between  $k$  and  $\epsilon$  gives

$$k = \frac{i}{c} \sqrt{\omega_p^2 - \omega^2}, \quad \omega^2 < \omega_p^2 \quad (\text{damping}) \quad (8.160)$$

or

$$k = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}, \quad \omega^2 > \omega_p^2 \quad (\text{oscillating}). \quad (8.161)$$

Such relations can be used to describe the response of plasmas, metals or dielectrics to em waves. For plasmas,  $\gamma \approx 0$  and  $n_b = 0$ , and the response is damped, almost down to  $\omega = 0$ . For dielectrics the above, as I said, only applies for  $\omega \gg \omega_p$ . For metals, this leads to "ultraviolet transparency" as one

increases  $\omega$  through  $\omega_p$ . The same thing happens for plasmas (except at microwave frequencies.)

We can get a different point of view on this by looking at the time behavior of fields rather than the space behavior (i.e.,  $k$  values) discussed above. First, let us consider the motion of free charges in the simplest possible case; a medium for which Ohm's law holds but where we can ignore the frequency dependence of the conductivity (highly resistive materials or possibly semiconductors):

$$\vec{J} = \sigma(0)\vec{E} = \frac{\sigma(0)}{\epsilon_0} \vec{D}. \quad (8.162)$$

$\uparrow$   
 const.

Combining this with charge conservation

$$\frac{\partial \rho(\vec{x}, t)}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{x}, t) = 0, \quad (8.163)$$

we get

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left( \frac{\sigma(0)}{\epsilon_0} \vec{D} \right) = 0. \quad (8.164)$$

But because

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho, \quad (8.165)$$

we find

$$\left[ \frac{\partial}{\partial t} + \frac{4\pi\sigma(0)}{\epsilon_0} \right] \rho(\vec{x}, t) = 0. \quad (8.166)$$

Then, given an initial charge density,  $\rho(\vec{x}, 0)$ , the solution to this is

$$\rho(\vec{x}, t) = \rho(\vec{x}, 0) \exp\left(-\frac{4\pi\sigma}{\epsilon_0} t\right). \quad (8.167)$$

The physical picture is: the free charges (and thus the fields associated with them) are all expelled from the interior to the surface. This is certainly the general picture also for conductors, but we will not be able to ignore the frequency dependence there.

In the case of conductors  $\sigma(\omega)$  becomes complex when  $\omega \sim \gamma$  (Assume  $\epsilon_0$  still slowly changing in  $\omega$ .) Introducing the Fourier transform of  $\rho(\vec{x}, t)$  through

$$\rho(\vec{x}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \rho(\vec{x}, \omega), \quad (8.168)$$

the above equation becomes

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ -i\omega + \frac{4\pi\sigma(0)}{\epsilon_0} \right] \rho(\vec{x}, \omega) = 0. \quad (8.169)$$

↓ now replace  $\sigma(0) \rightarrow \sigma(\omega)$

Also, take another derivative w.r.t. time:

$$\left( \frac{\partial}{\partial t} + \gamma \right) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ -i\omega + \frac{4\pi n_c e^2}{m\epsilon_0} \frac{1}{\gamma - i\omega} \right] \rho(\vec{x}, \omega) = 0,$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[ -i\omega^2 - i\gamma\omega + \frac{4\pi n_c e^2}{m\epsilon_0} \right] \rho(\vec{x}, \omega) = 0, \quad (8.170)$$

which is equivalent to the differential equation

$$\left[ \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} + \frac{\omega_p^2}{\epsilon_0} \right] \rho(\vec{x}, t) = 0, \quad (8.171)$$

where  $\omega_p$  now appears as a resonance frequency ( $\gamma \ll \omega_p$ ) and  $\gamma$  as the time underdamped time constant. Thus, the picture in resistors, (8.167), or metals or plasmas, (8.171), is: at low frequencies, the fields are damped in space and expelled from the interior\*; at higher frequencies ( $\omega \gg \gamma$ ) the time behavior becomes oscillating. Thus, even though the fields may be damped in space for  $\omega < \omega_p$ , they are not necessarily expelled from the interior, but continue to oscillate at the plasma frequency. Then, as  $\omega \sim \omega_p$  the system resonates  $\Rightarrow$  ultraviolet transparency.

### Kramers-Kronig relations

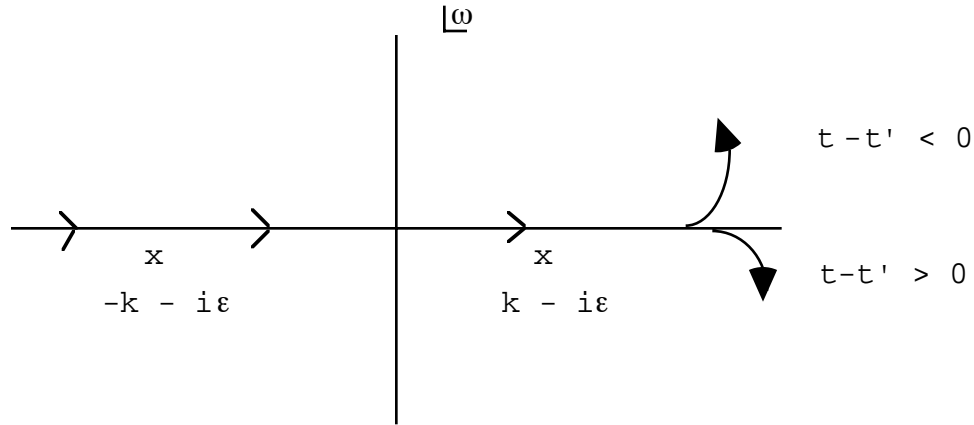
Kramers Kronig relations now. Let's go back to the evaluation of  $G^{\text{Ret}}$  by considering the  $\omega$ -integral first:

$$G^{\text{Ret}} = 4\pi \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{k^2 - \left(\frac{\omega}{c} + i\varepsilon\right)^2}. \quad (8.172)$$

Pole structure:

---

\* The time behavior is given by  $\rho(t) \sim e^{-(\gamma/2)t}$  for the underdamped case, corresponding to conductors, by (8.171). The interpretation is that the charge is moving to the surface. The time dependence in (8.167) for highly resistive materials in fact just corresponds to the severely overdamped case ( $\gamma^2 \gg \omega_p^2/\varepsilon_0 = 4\pi\sigma(0)/\varepsilon_0$ ) for large enough times ( $t \gg 2/\gamma$ ).



$$G^{\text{Ret}} = 4\pi c H(t-t') \int \frac{d^3k}{(2\pi)^3} \frac{1}{k} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \sin [kc(t-t')]. \quad (8.173)$$

This causal form followed from the absence of poles in the upper half  $\omega$ -plane. Switching gears back to dispersion relations, we have

$$\vec{D}(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{D}(\vec{x}, \omega). \quad (8.174)$$

Assume  $\vec{D}(\vec{x}, \omega) = \varepsilon(\omega) \vec{E}(\vec{x}, \omega)$ . Go backwards on  $\vec{E}(\vec{x}, \omega)$ :

$$\vec{E}(\vec{x}, \omega) = \int_{-\infty}^{\infty} dt' e^{i\omega t'} \vec{E}(\vec{x}, t'). \quad (8.175)$$

Put it together (do  $d\omega$  integral before  $dt'$ ):

$$\vec{D}(\vec{x}, t) = \int_{-\infty}^{\infty} dt' \vec{E}(\vec{x}, t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \varepsilon(\omega). \quad (8.176)$$

Let  $\tau \equiv t-t'$ , then can write this as

$$\vec{D}(\vec{x}, t) = \vec{E}(\vec{x}, t) + \int_{-\infty}^{\infty} d\tau \vec{E}(\vec{x}, t-\tau) G(\tau), \quad (8.177)$$

where

$$G(\tau) \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} [\varepsilon(\omega) - 1]. \quad \left( \int_{0-}^{\infty} d\tau G(\tau) e^{i\omega\tau} = \varepsilon(\omega) - 1 \right) \text{ "response function"}$$

(8.178)

Analyticity & absence of poles in the u.h.p. for  $\varepsilon(\omega)$  will now guarantee  $G(\tau) = 0, \tau < 0$ , just like  $G^{\text{Ret}}$ . Alternatively, given:

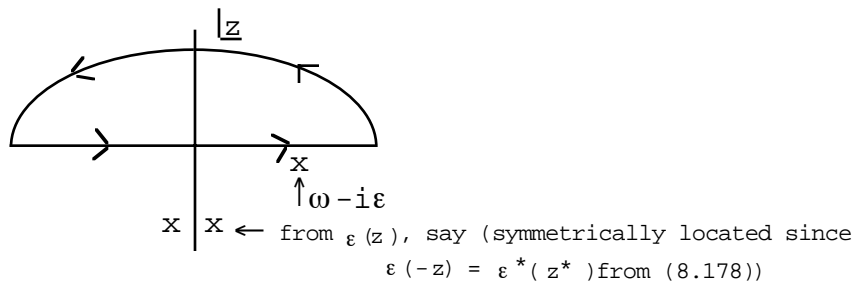
- |                                    |   |
|------------------------------------|---|
| 1. $G(\tau) = 0, \tau < 0$         | 3. $G(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ (nonconductors) |
| 2. $G(\tau)$ finite $\forall \tau$ | 4. $G$ cont. at $\tau = 0$  |
|                                    | ↑ (excludes <u>static</u> conductor<br>at all $\omega$ )                |

then  $\varepsilon(\omega)$  has no poles in the u.h.p. (#1) including the real  $\omega$ -axis (this is the reason for condition #3 above) and  $\varepsilon(\omega) - 1$  behaves at most like  $\omega^{-2}$  at high frequencies (this is a result of #4). In this model, the relation between  $\vec{D}$  and  $\vec{E}$  is nonlocal in time, like our simple model connecting  $\vec{J}$  and  $\vec{E}$ .

Complete the argument:

$$0 = \frac{1}{2\pi} \oint dz \frac{\varepsilon(z) - 1}{z - \omega + i\varepsilon}, \quad (8.179)$$

where



We have that  $\lim_{\omega \rightarrow \infty} (\varepsilon(\omega) - 1) \sim \omega^{-2}$ , as follows:

↓ use  $\omega \rightarrow \omega + i\varepsilon$  rule.

$$\varepsilon(\omega) = 1 + \int_{0^-}^{\infty} d\tau G(\tau) e^{+i\omega\tau}, \quad (8.180)$$

$$G(\tau) = G(0^+) + \tau G'(0^+) + \dots, \quad (8.181)$$

$$\Rightarrow \varepsilon(\omega) = 1 + i \frac{G(0^+)}{\omega} - \frac{G'(0^+)}{\omega^2} + \dots \quad (8.182)$$

$$\text{Know } G(0^-) = 0 \quad \Rightarrow G(0^+) = 0. \quad (8.183)$$

continuity (#4)

$$\Rightarrow \lim_{\omega \rightarrow \infty} (\varepsilon(\omega) - 1) \sim \omega^{-2}. \quad (\text{Saw this in our simple model}) \quad (8.184)$$

Integral around the semi-circular part of C now vanishes.

$$\Rightarrow 0 = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\varepsilon(\omega') - 1}{\omega' - \omega + i\varepsilon}. \quad (8.185)$$

But

$$\frac{1}{\omega' - \omega + i\varepsilon} = P \frac{1}{\omega' - \omega} - i\pi \delta(\omega' - \omega), \quad (8.186)$$

or

$$-\frac{i}{2} (\varepsilon(\omega') - 1) + P \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\varepsilon(\omega') - 1}{\omega' - \omega} = 0, \quad (8.187)$$

$$\Rightarrow \varepsilon(\omega) - 1 = -\frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon(\omega') - 1}{\omega' - \omega}. \quad (8.188)$$

Take real and imaginary parts:

$$\operatorname{Re} \varepsilon(\omega) = 1 + \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega' \frac{\operatorname{Im} \varepsilon(\omega')}{\omega' - \omega}, \quad (8.189)$$

$$\operatorname{Im} \varepsilon(\omega) = -\frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega' \frac{\operatorname{Re} \varepsilon(\omega') - 1}{\omega' - \omega}. \quad (8.190)$$

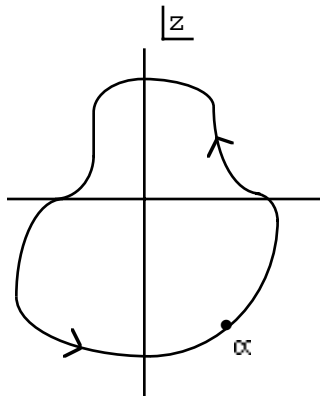
The "P" only reminds us what principle value means:

$$\Rightarrow \text{P} \frac{1}{\omega' - \omega} = \frac{1}{\omega' - \omega + i\varepsilon} + i\pi\delta(\omega' - \omega), \quad (8.191)$$

or

$$\text{P} \frac{1}{\omega' - \omega} = \frac{1}{\omega' - \omega - i\varepsilon} - i\pi\delta(\omega' - \omega). \quad (8.192)$$

So, for example, assume  $f(z)$  is nonsingular on  $C$ :



$$\begin{aligned} \text{P} \oint_C \frac{f(z) dz}{z - \alpha} &= 0 + i\pi f(\alpha) \\ &\text{or} \\ &= 2\pi i f(\alpha) - i\pi f(\alpha) \end{aligned}$$

The relation  $\varepsilon(-\omega) = \varepsilon^*(\omega^*)$  implies  $\text{Re } \varepsilon(\omega)$  is even and  $\text{Im } \varepsilon(\omega)$  is odd in  $\omega$  for  $\omega$  on the real axis. This means

$$\text{Re } \varepsilon(\omega) - 1 = \frac{2}{\pi} \text{P} \int_0^{\infty} d\omega' \frac{\omega' \text{Im } \varepsilon(\omega')}{\omega'^2 - \omega^2}, \quad (8.193)$$

$$\text{Im } \varepsilon(\omega) = \frac{2\omega}{\pi} \text{P} \int_0^{\infty} d\omega' \frac{\text{Re } \varepsilon(\omega') - 1}{\omega'^2 - \omega^2}. \quad (8.194)$$

Let's now go back and look at an aspect of the K-K dispersion relations we have already seen in our simple model: sum rules. We saw that  $\varepsilon(\omega) = 1 - \frac{G'(0^+)}{\omega^2} + \dots$ ,

$$\lim_{\omega \rightarrow \infty} \varepsilon(\omega) = 1 - \frac{G'(0)}{\omega^2}. \quad (8.195)$$

Therefore define

$$\omega_p^2 \equiv \lim_{\omega \rightarrow \infty} \omega^2 (1 - \varepsilon(\omega)). \quad (8.196)$$

Can then show from the K-K dispersion relations that this gives

$$\omega_p^2 = \frac{2}{\pi} \int_0^{\infty} d\omega \omega \text{Im } \varepsilon(\omega). \quad (8.197)$$

In our simple model with

$$\varepsilon(\omega) = 1 + \frac{4\pi e^2}{m} \sum_{\mathbf{i}} \frac{n_{\mathbf{b}}^{\mathbf{i}}}{\omega_0^{\mathbf{i}2} - \omega^2 - i\omega\gamma^{\mathbf{i}}}, \quad (8.198)$$

either one of the above statements give

$$\omega_p^2 = \frac{4\pi e^2}{m} \sum_{\mathbf{i}} n_{\mathbf{b}}^{\mathbf{i}} = \frac{4\pi^2 e^2}{m} N_{\mathbf{b}}, \quad (8.199)$$

which recovers the model form of the square of the plasma frequency.

There is another sum rule for the real part of  $\epsilon(\omega)$ . Go back to our model (one resonance)

$$\text{Re}(\epsilon(\omega)-1) = \frac{4\pi n_b e^2}{m} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}. \quad (8.200)$$

Let's integrate this:

$$\int_0^{\infty} d\omega \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} = \frac{\pi}{2\gamma} \quad (\text{Did this before}) \quad (8.201)$$

$$\int_0^{\infty} d\omega \frac{1}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} = \frac{\pi}{2\gamma \omega_0^2}. \quad (8.202)$$

$$\text{so model} \Rightarrow \int_0^{\infty} d\omega [\text{Re } \epsilon(\omega) - 1] = 0. \quad (8.203)$$

In fact, simply using the K-K dispersion relations (not the model), we can show

$$\int_0^N d\omega' [\text{Re } \epsilon(\omega') - 1] = \frac{\omega_p^2}{N} + \mathcal{O}\left(\frac{1}{N^3}\right). \quad (8.204)$$

How do these statements about Re, Im sums change if there is a pole at  $\omega = 0$ ? Can then show that

$$\epsilon_{\text{eff}}(\omega) = \epsilon(\omega) + \frac{i4\pi\sigma(\omega)}{\omega}, \quad (8.205)$$

↑ satisfies usual K-K relations

where

$$\sigma(\omega) = \frac{\gamma\sigma(0)}{\gamma - i\omega}, \quad (8.206)$$

give

$$\operatorname{Re}(\epsilon_{\text{eff}}(\omega) - 1) = \frac{2}{\pi} \text{P} \int_0^{\infty} d\omega' \frac{\omega' \operatorname{Im} \epsilon_{\text{eff}}(\omega')}{\omega'^2 - \omega^2}, \quad (8.207)$$

$$\operatorname{Im} \epsilon_{\text{eff}}(\omega) = \frac{4\pi\sigma(0)}{\omega} - \frac{2\omega}{\pi} \text{P} \int_0^{\infty} d\omega' \frac{\operatorname{Re} \epsilon_{\text{eff}}(\omega') - 1}{\omega'^2 - \omega^2}, \quad (8.208)$$

and

$$\omega_{\text{peff}}^2 = \omega_p^2 + 4\pi\gamma\sigma(0), \quad (8.209)$$

$$\int_0^N d\omega' [\operatorname{Re} \epsilon_{\text{eff}}(\omega') - 1] = \frac{\omega_{\text{peff}}^2}{N} - 2\pi^2\sigma(0) + \mathcal{O}\left(\frac{1}{N^3}\right). \quad (8.210)$$

HW probs. 6, 7, and 8 fill in some of the gaps in the above derivations.

### Dispersion in one-dimension: theory and example

Let's discuss some more consequences of dispersion relations. Limit the discussion to some background and an example which, however, exhibits the dispersive character of materials.

First, let's notice something. Wave eq<sup>n</sup> for  $\epsilon = \text{const.}$  was

$$\left( \nabla^2 - \frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = 0. \quad (8.211)$$

Phase velocity was

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{\mu\epsilon}}. \quad (8.212)$$

In regions where  $\epsilon > 1$ ,  $v_p < c$ , but for  $\epsilon < 1$ , for example in

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad (8.213)$$

we have  $v_p > c$ . Special relativity says that energy transport must take place with a speed  $\leq c$ . Is this a contradiction?

Some background on the equations we will try to understand:

$$\vec{\nabla} \times \vec{H} = \frac{1}{c} \dot{\vec{D}} + \frac{4\pi}{c} \vec{J}, \quad (8.214)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \dot{\vec{B}}. \quad (8.215)$$

Drop  $\vec{J}$  (we thereby neglect conductors)

$$\vec{D}(\vec{x}, t) = \vec{E}(\vec{x}, t) + \int_{0^-}^{\infty} d\tau \vec{E}(\vec{x}, t-\tau) \underbrace{\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega'\tau} [\epsilon(\omega')-1]}_{G(\tau)}, \quad (8.216)$$

↑ includes point  
at  $\tau = 0$   
(important later)

$G(\tau)$

so ( $\mu=1$ )

$$\begin{cases} \vec{\nabla} \times \vec{B} = \frac{1}{c} \dot{\vec{E}} + \frac{1}{c} \int_{0^-}^{\infty} d\tau \frac{\partial}{\partial t} \vec{E}(\vec{x}, t-\tau) G(\tau), \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \dot{\vec{B}}. \end{cases} \quad (8.217)$$

$$\Rightarrow \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{c^2} \int_{0^-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} \vec{E}(\vec{x}, t-\tau) G(\tau). \quad (8.218)$$

This is our new, non local (in time) wave equation. Let  $\vec{E} \ni u$  (1 dimension) for simplicity. We want to solve

$$\frac{\partial^2 u(\mathbf{x}, t)}{\partial \mathbf{x}^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2} \int_{0^-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} u(\mathbf{x}, t-\tau) G(\tau). \quad (8.219)$$

Try solution (we will make sure it's real later)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} [A(\omega)e^{+ik(\omega)x} + B(\omega)e^{-ik(\omega)x}]. \quad (8.220)$$

Why include 2 terms? Wave with a given  $\omega$  can go in + or - directions. Compute:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} (-k^2) [\dots], \quad (8.221)$$

$$\frac{1}{c} \frac{\partial^2 u}{\partial t^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{-\omega^2}{c^2} [\dots], \quad (8.222)$$

$$\frac{1}{c^2} \int_{0^-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} u(x,t-\tau)G(\tau) = \int_{0^-}^{\infty} d\tau G(\tau) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-\tau)} \left( \frac{-\omega^2}{c^2} \right) [\dots]. \quad (8.223)$$

By def<sup>n</sup>  $\int_{0^-}^{\infty} d\tau G(\tau)e^{i\omega\tau} = \varepsilon(\omega)-1$ . Therefore

$$\frac{1}{c^2} \int_{0^-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} u(x,t-\tau)G(\tau) = -\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left( -\varepsilon(\omega) \frac{\omega^2}{c^2} \right) [\dots].$$

↑ cancels with above (8.224)

Therefore, this is a solution if

$$k^2 = \varepsilon(\omega) \frac{\omega^2}{c^2}, \quad (8.225)$$

$$k = \pm n(\omega) \frac{\omega}{c}. \quad (k = k(\omega)). \quad (8.226)$$

(Choose + sign in (8.226), thus defining  $n(\omega)$ .) Then need A, B in terms of initial conditions on  $u(x,t)$ :

$$u(0,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} [A(\omega) + B(\omega)],$$

(8.227)

$$\frac{\partial u(0,t)}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega (ik) e^{-i\omega t} [A(\omega) - B(\omega)].$$
 (8.228)

Can invert and solve for A, B:

$$A(\omega) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{+i\omega t} \left[ u(0,t) + \frac{1}{ik(\omega)} \frac{\partial u(0,t)}{\partial x} \right],$$
 (8.229)

$$B(\omega) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{+i\omega t} \left[ u(0,t) - \frac{1}{ik(\omega)} \frac{\partial u(0,t)}{\partial x} \right].$$
 (8.230)

(8.229), (8.230) show that  $A^*(-\omega) = A(\omega)$ ,  $B^*(-\omega) = B(\omega)$  if  $k^*(-\omega) = -k(\omega)$ , which renders  $u(x,t)$  real. One can show that this makes  $u(x,t)$  real and restricts the integration in (8.220) to positive  $\omega$  values. There are therefore only two independent real functions. Then, from (8.226)

$$n^*(-\omega) = n(\omega).$$
 (8.231)

(Same property as for  $\epsilon(\omega)$ .)

On the other hand, assume (real part understood)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{+i\omega(k)t} [A(k)e^{-ikx}].$$
 (8.232)

Now (leave off c.c. term again)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{-i\omega t} (-k^2) A(k)e^{-ikx},$$
 (8.233)

$$\frac{1}{c} \frac{\partial^2 u}{\partial t^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{-i\omega t} \left( \frac{-\omega^2}{c^2} \right) A(k) e^{-ikx}, \quad (8.234)$$

$$\frac{1}{c^2} \int_{0^-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} u(x, t-\tau) G(\tau) = -\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{-i\omega t} \left( -\varepsilon \frac{\omega^2}{c^2} \right) A(k) e^{-ikx}. \quad (8.235)$$

A solution if

$$k^2 = \varepsilon(k) \frac{\omega^2(k)}{c^2}. \quad (8.236)$$

Pick one root & stick with it. Then (now put in the c.c. term)

$$u(x, 0) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dk [Ae^{+ikx} + A^*e^{-ikx}], \quad (8.237)$$

$$\frac{\partial u(x, 0)}{\partial t} = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dk (-i\omega) [Ae^{+ikx} - A^*e^{-ikx}]. \quad (8.238)$$

Invert and solve for  $A(k)$ ,  $A^*(-k)$ :

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \left[ u(x, 0) + \frac{i}{\omega(k)} \frac{\partial u(x, 0)}{\partial t} \right], \quad (8.239)$$

$$A^*(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \left[ u(x, 0) - \frac{i}{\omega(k)} \frac{\partial u(x, 0)}{\partial t} \right]. \quad (8.240)$$

This shows we must pick our root consistent with

$$\omega^*(k) = \omega(-k). \quad (8.241)$$

Now we know something about the equation we are solving, let's take a special case.

Take the example (used in Jackson, Section 7.9):

$$\omega(k) = v \left( 1 + \frac{a^2 k^2}{2} \right), \quad (8.242)$$

$$\Rightarrow \varepsilon(\omega) = \frac{2c^2}{a^2 \omega} \left( \frac{1}{v} - \frac{1}{\omega} \right). \quad (8.243)$$

Notice:

$\omega > v$ ,  $k$  real, oscill. in space,  
 $\omega < v$ ,  $k$  imag., damped in space.

Our eq<sup>1</sup> is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2} \int_{0^-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} u(x, t-\tau) G(\tau), \quad (8.244)$$

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega [\varepsilon(\omega) - 1] e^{-i\omega\tau}. \quad (8.245)$$

Can transfer derivatives,

$$\frac{\partial}{\partial t} u \rightarrow - \frac{\partial}{\partial \tau} u, \quad (8.246)$$

and integrate by parts ( $G(\tau)$  is supposed to vanish at  $\tau=0^-, \infty$ )

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2} \int_{0^-}^{\infty} d\tau u(x, t-\tau) \frac{\partial^2 G(\tau)}{\partial \tau^2}. \quad (8.247)$$

Problem:  $\varepsilon \sim \frac{1}{\omega}$  for large  $\omega$ . Strictly speaking, our model does not apply. Implies a discontinuous response function. However, let's just plow ahead and see what we get.

Need 2<sup>nd</sup> derivative of response function:

$$\frac{\partial^2}{\partial \tau^2} G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega\tau} \left[ \frac{2c^2}{a^2} \left( 1 - \frac{i\omega}{iv} \right) + \omega^2 \right],$$

$$= \frac{2c^2}{a^2} \left[ \delta(\tau) + \frac{1}{iv} \delta'(\tau) \right] - \delta''(\tau). \quad (8.248)$$

Plug in above. Get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \int_{0^-}^{\infty} d\tau u(x, t-\tau) \left[ \frac{2c^2}{a^2} (\delta(\tau) - \frac{i}{v} \delta'(\tau)) - \delta''(\tau) \right] + \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \quad (8.249)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2}{a^2} \left[ u - \frac{i}{v} \frac{\partial u}{\partial t} \right]. \quad (8.250)$$

Let  $y(x, t) \equiv u(x, t)e^{ivt}$ . Then

$$- \frac{a^2 v}{2} \frac{\partial^2 y}{\partial x^2} = i \frac{\partial y}{\partial t}. \quad (8.251)$$

So, we are essentially solving the Schrödinger equation. The em wave packet in this case is behaving like a nonrelativistic particle of mass

$$m_{\text{eff}} = \frac{\hbar}{a^2 v}. \quad (8.252)$$

General solution:

$$y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{(ikx - \omega t)} g(k), \quad (8.253)$$

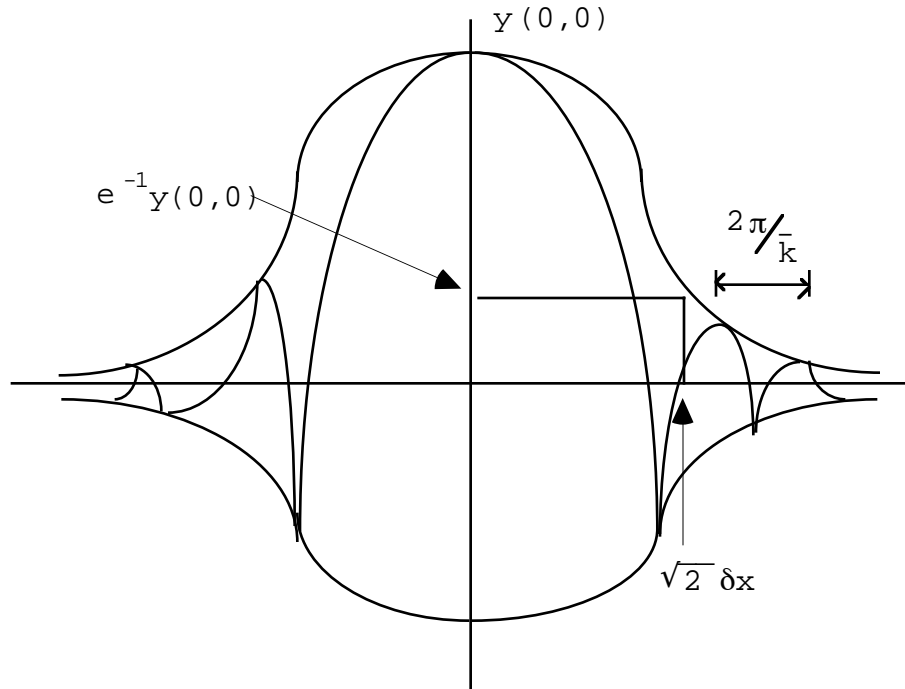
where

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} y(x, 0). \quad (8.254)$$

Let

$$y(x,0) = \frac{c}{\sqrt{\delta x}} \exp\left(i\bar{k}x - \frac{x^2}{2\delta x^2}\right). \quad (8.255)$$

where  $\delta x$ ,  $\bar{k}$  are constants. Notice,  $\langle k \rangle = \bar{k}$  ( $\langle \dots \rangle$  usual Q.M. meaning). Real part looks like:



Plug in and complete the squares. Result is:

$$u(x,t) = y(x,t)e^{-i\omega t}$$

$$= \frac{c}{\sqrt{\delta x + \frac{ia^2vt}{\delta x}}} \exp\left(\frac{-(x-a^2v\bar{k}t)^2}{2(\delta x^2+ia^2vt)}\right) \exp(i\bar{k}x - i\omega(\bar{k})t). \quad (8.256)$$

↓ envelope

↑ phase

$$\text{phase speed} = \frac{v}{\bar{k}} \left(1 + \frac{a^2 \bar{k}^2}{2}\right) = \frac{\omega(\bar{k})}{\bar{k}}, \quad (8.257)$$

$$\text{envelope speed} = a^2 v \bar{k} \left(= \frac{\hbar \bar{k}}{m_{\text{eff}}}\right) = \left. \frac{d\omega(k)}{dk} \right|_{k=\bar{k}}. \quad (8.258)$$

The quantity  $\left. \frac{d\omega(k)}{dk} \right|_{k=\bar{k}}$  is called the group velocity and is (for materials that are weakly dispersive) the approximate speed of energy transmission. This corresponds to making the "stationary phase approximation",

$$\frac{\partial(kz - \omega(|k|)t)}{\partial k} \approx 0,$$

in the argument of the exponent in (8.232).

### Problems

1. From the results derived in class for the reflection and transmission coefficients for  $E_{\perp}$  and  $B_{\perp}$ , find similar expressions for  $E_{\parallel}$  and  $B_{\parallel}$ . For example, show that

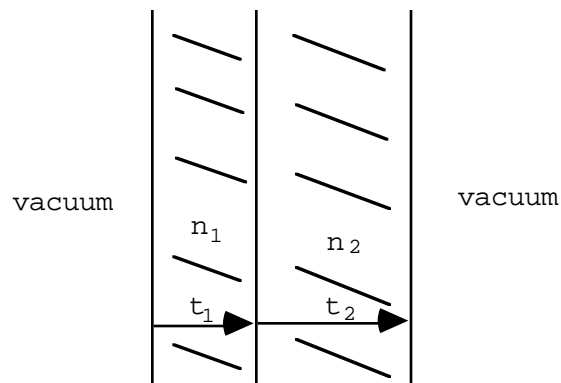
$$\frac{E'_{\parallel}}{E_{\parallel}} = \frac{2n_2 \cos r}{n_1 \cos i + n_2 \cos r} \quad , \quad (\text{transmission})$$

$$\frac{E''_{\parallel}}{E_{\parallel}} = \frac{n_2 \cos r - n_1 \cos i}{n_1 \cos i + n_2 \cos r} \quad . \quad (\text{reflection})$$

(Don't forget to present results for  $B_{\parallel}$  as well.)

2. Consider again the situation of plane waves incident on a flat interface. This time however, don't set  $\mu = 1$ . Derive the equations satisfied by  $E_{\perp}$  and  $H_{\perp}$  and show that their Green functions are interchanged under  $\epsilon(z) \leftrightarrow \mu(z)$ . Use this fact to write down the transmission and reflection coefficients for  $E_{\perp}$ . (You may check against Eq.(7.39) of Jackson.)

3. Jackson, problem 7.8. On the (c) part write out the transfer matrices needed to describe the layers:



4. The Green function for the linear DE

$$\left[ \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] \vec{x}(t) = \frac{e}{m} \vec{E}(t) ,$$

satisfies

$$\left[ \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] G(t, t') = \delta(t - t') .$$

Causality requires

$$G(t, t') = 0 , t < t' .$$

(a) Show that in general

$$G(t', t'') = G(-t'', -t') .$$

(b) Show that the particular solution is  $(\vec{x}, \dot{\vec{x}} = 0$  at  $t = -\infty)$

$$\vec{x}(t) = \frac{e}{m} \int_{-\infty}^{t^+} dt' G(t, t') \vec{E}(t') ,$$

where  $t^+ = t + \epsilon$ ,  $\epsilon > 0$ .

(c) Show explicitly that  $H(t - t')$  is the step function.)

$$G(t, t') = \begin{cases} \frac{H(t - t')}{\sqrt{\omega_0^2 - \gamma^2/4}} e^{-\gamma/2(t - t')} \sin[\sqrt{\omega_0^2 - \gamma^2/4}(t - t')] , & \omega_0 > \frac{\gamma}{2} , \\ \frac{H(t - t')}{\sqrt{\gamma^2/4 - \omega_0^2}} e^{-\gamma/2(t - t')} \sinh[\sqrt{\gamma^2/4 - \omega_0^2}(t - t')] , & \omega_0 < \frac{\gamma}{2} . \end{cases}$$

(d) Set  $\omega_0 = 0$  and show that we recover the result

$$\vec{v}(t) = \frac{e}{m} \int_{-\infty}^t dt' e^{-\gamma(t-t')} \vec{E}(t') .$$

5. Given the time equation for a field  $\psi(t')$ ,

$$\left(\frac{d^2}{dt'^2} + \gamma \frac{d}{dt'} + \omega_0^2\right)\psi(t') = f(t'),$$

and its Green function,

$$\left(\frac{d^2}{dt'^2} + \gamma \frac{d}{dt'} + \omega_0^2\right)G(t',t) = \delta(t-t'),$$

where  $G(t',t)=G(-t,-t')$  (time reversal) and  $G(t,t')=0$  for  $t < t'$  (causality), show that the complete solution (particular plus homogeneous) for  $\psi(t)$  is given by  $(G \equiv G(t,t'))$  everywhere;  $t_i$  is the given initial time)

$$\psi(t) = \int_{t_i}^{\infty} dt' G f(t') + \left(\psi(t') \frac{dG}{dt'} - G \frac{d\psi(t')}{dt'} - \gamma \psi(t') G\right) \Big|_{t'=t_i}.$$

6. Show that (complex integration or otherwise)

$$\int_{-\infty}^{\infty} d\omega \frac{\omega^2 \gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2 \gamma^2} = \pi.$$

7.(a) Using the definition

$$\omega_p^2 \equiv \lim_{\omega \rightarrow \infty} \omega^2 (1 - \epsilon(\omega))$$

and the K.K dispersion relations, show that

$$\omega_p^2 = \frac{2}{\pi} \int_0^{\infty} d\omega' \omega' \text{Im } \epsilon(\omega').$$

(b) Likewise, show that

$$\int_0^N d\omega' [\operatorname{Re} \varepsilon(\omega') - 1] = \frac{\omega_p^2}{N} + O\left(\frac{1}{N^3}\right).$$

8.(a) Assume an effective (dynamic) dielectric constant, which models a conductor, given by

$$\varepsilon_{\text{eff}}(\omega) = \varepsilon(\omega) + \frac{4\pi i \sigma(\omega)}{\omega}$$

where

$$\sigma(\omega) = \frac{\gamma \sigma(0)}{\gamma - i\omega}$$

and where  $\varepsilon(\omega)$  satisfies the usual K.K. dispersion relations. Show that  $\varepsilon_{\text{eff}}(\omega)$  satisfies the modified relations:

$$\operatorname{Re} \varepsilon_{\text{eff}}(\omega) - 1 = \frac{2}{\pi} P \int_0^{\infty} d\omega' \frac{\omega' \operatorname{Im} \varepsilon_{\text{eff}}(\omega')}{\omega'^2 - \omega^2},$$

$$\operatorname{Im} \varepsilon_{\text{eff}}(\omega) = \frac{4\pi\sigma(0)}{\omega} - \frac{2\omega}{\pi} P \int_0^{\infty} d\omega' \frac{(\operatorname{Re} \varepsilon_{\text{eff}}(\omega') - 1)}{\omega'^2 - \omega^2}$$

(b) In this situation, show that the modified forms of the results in 7(a) and (b) are

$$\begin{aligned} \omega_{\text{eff}}^2 &\equiv \lim_{\omega \rightarrow \infty} \omega^2 (1 - \varepsilon_{\text{eff}}(\omega)) \\ &= \omega_p^2 + 4\pi\gamma\sigma(0) \end{aligned}$$

and

$$\int_0^N d\omega' [\operatorname{Re} \varepsilon_{\text{eff}}(\omega') - 1] = \frac{(\omega_p)_{\text{eff}}^2}{N} - 2\pi^2\sigma(0) + O\left(\frac{1}{N^3}\right).$$

9. Use the constraint

$$\int_{-\infty}^{\infty} dt \frac{\partial \vec{P}(\vec{x}, t)}{\partial t} \cdot \vec{E}(\vec{x}, t) > 0,$$

where  $\vec{E}(\vec{x}, t)$  is an arbitrary electric field to show that  $\text{Im} \epsilon(\omega) > 0$  for real  $\omega > 0$ . Assume that

$$\begin{aligned} \vec{E}(\vec{x}, \omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \vec{E}(\vec{x}, t), \quad \epsilon(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \epsilon(t), \\ \vec{P}(\vec{x}, \omega) &= \frac{(\epsilon(\omega) - 1)}{4\pi} \vec{E}(\vec{x}, \omega). \end{aligned}$$

Comment: Note that ( $\mu=1$ )

$$\vec{J} = c \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}(\vec{x}, t)}{\partial t},$$

so the above is just a statement that the total work done by the field on the *bound* charges is positive.

10. We briefly discussed the behavior of the dielectric constant at high frequencies in metals or plasmas where it was given approximately by ( $\mu=1$ )

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

where  $\omega_p$  is the plasma frequency.

(a) Derive the explicit form of the response function,  $G(\tau)$ , for this  $\epsilon(\omega)$ . Find the group,  $v_g$ , and phase,  $v_p$ , velocities implied by this dielectric relation, and demonstrate that  $v_p > c > v_g$ .

(b) In one dimension, show that the electric field ( $\vec{E}(\vec{x}, t) \rightarrow u(x, t)$ ) satisfies the modified wave equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{\omega_p^2}{c^2} u.$$

(c) Solve this equation (called the Klein equation) for a "box" of length "a" and boundary conditions  $u|_{x=0,a} = 0$ . Show that the resonant frequencies are given by

$$\omega_n^2 = \omega_p^2 + \left(\frac{cn\pi}{a}\right)^2, \quad n=1,2,3,\dots$$

11. (Modified Jackson prob.7.28.) (a) Show that a wave traveling in the z-direction which has a finite extent in the transverse directions (large compared to it's wavelength) has an electric field given approximately by (real part understood)

$$\vec{E}(\vec{x}, t) \simeq \left( \vec{\epsilon}_0(\vec{x}_T) + \frac{i\hat{e}_3}{k} \vec{\nabla} \cdot \vec{\epsilon}_0(\vec{x}_T) \right) e^{i(kz-\omega t)},$$

where  $\vec{x}_T = (x, y)$ . Notice, the "transverse" wave has developed a z-component!

(b) Show that the associated magnetic field (to the same order) is given by

$$\vec{B}(\vec{x}, t) \simeq \frac{kc}{\omega} \left( \hat{e}_3 \times \vec{\epsilon}_0(\vec{x}_T) + \frac{i\hat{e}_3}{k} \vec{\nabla} \cdot (\hat{e}_3 \times \vec{\epsilon}_0(\vec{x}_T)) \right) e^{i(kz-\omega t)}.$$

Notice again,  $\vec{B}(\vec{x}, t)$  develops a small z-component. Also notice a dispersion relation between k and  $\omega$  has not been specified so that the above can be used in vacuum or nonvacuum regions.

12. (Modified Jackson prob.7.29.) (a) As an extension of the last problem, consider circularly polarized plane waves where

$$\vec{\epsilon}_0(\vec{x}_T) = E_0(\hat{e}_1 \pm i\hat{e}_2),$$

holds exactly ( $E_0$  is a complex constant in space). Show that all components of the time-averaged angular momentum (real part understood)

$$\vec{L} = \frac{1}{8\pi c} \int d^3x \vec{x} \times (\vec{E} \times \vec{B}^*),$$

vanish.

(b) Now, using the space dependent fields from problem 11 with

$$\vec{\epsilon}_0(\vec{x}_T) = E_0(\vec{x}_T)(\hat{e}_1 \pm i\hat{e}_2),$$

(assume  $E_0(\vec{x}_T)$  is an even function of  $x$  and  $y$ ) show that  $L_1$  and  $L_2$  still vanish, but that  $L_3$  is given by

$$L_3 \simeq \pm \frac{1}{4\pi\omega} \int d^3x |E_0|^2.$$

Also show to lowest order in slowly varying  $E_0(\vec{x}_T)$ ,

$$U = \frac{1}{16\pi} \int d^3x (|\vec{E}|^2 + |\vec{B}|^2),$$

$$\Rightarrow U \simeq \frac{1}{4\pi} \int d^3x |E_0|^2,$$

so that

$$\frac{L_3}{U} \simeq \pm \frac{1}{\omega},$$

where throughout the upper sign is for positive helicity, the lower sign is for negative helicity.

(c) Explain the physics behind the result that the wave in (a) has  $L_3 = 0$ , while the (b) wave has  $L_3 \neq 0$ .

### Other Problems

13.(a) Given the representation ((7.115) of Jackson in cgs units) for  $\epsilon(z)$  in the upper half complex plane,

$$\epsilon(z) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\epsilon(\omega') - 1}{\omega' - z},$$

show that

$$\text{Im}(\epsilon(z)) = \frac{\text{Im}(z)}{2\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im}(\epsilon(\omega'))}{|\omega' - z|^2}.$$

(b) Using (a) or any other method, for

$$\epsilon(-z) = \epsilon^*(z^*),$$

and

$$\text{Im}(\epsilon(\omega)) > 0, \text{ for } \omega > 0$$

show that  $\text{Im}(\epsilon(z)) > 0$  everywhere in the first quarter complex plane (this means  $\text{Im}(z), \text{Re}(z) > 0$ ).

14. We may write the causal connection between  $\vec{D}$  and  $\vec{E}$ , Eq.(8.177) of the notes, as

$$\vec{D}(\vec{x}, t) = \vec{E}(\vec{x}, t) + \int_{0^-}^{\infty} d\tau H(\tau) g(\tau) \vec{E}(\vec{x}, t-\tau), \quad (1)$$

where  $H(\tau)$  is the Heaviside step function,

$$H(\tau) = i \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{e^{-iv\tau}}{v + i\epsilon}. \quad (2)$$

( $H(\tau) = 1, \tau > 0; H(\tau) = 0, \tau < 0; \epsilon > 0$ )  $g(\tau)$  has the Fourier transform

$$g(\tau) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega'\tau} \tilde{g}(\omega'), \quad (3)$$

and is a real function ( $\tau > 0$ ).

(a) Using (2) and (3) in (1), and given that  $\vec{D}(\omega) = \epsilon(\omega)\vec{E}(\omega)$ , show that one has that

$$\epsilon(\omega) - 1 = i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\tilde{g}(\omega')}{\omega - \omega' + i\epsilon}. \quad (4)$$

(b) Given that  $g(-\tau) = -g(\tau)$  (allowed from (1)), show that  $\tilde{g}(\omega')$  is odd in  $\omega'$  and purely imaginary. Then, show that (4) results in

$$\text{Re } \epsilon(\omega) - 1 = + \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im } \epsilon(\omega')}{\omega' - \omega}. \quad (5)$$

15. Given (as derived for example in the last problem)

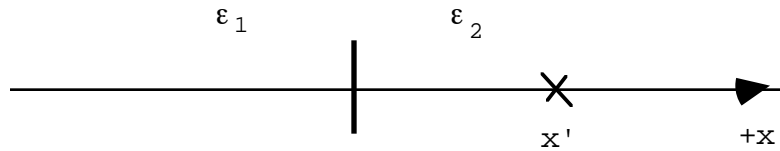
$$\text{Re } \epsilon(\omega) - 1 = + \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im } \epsilon(\omega')}{\omega' - \omega}, \quad (1)$$

show that the other K-K relation,

$$\text{Im } \epsilon(\omega) = - \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega' \frac{\text{Re } \epsilon(\omega') - 1}{\omega' - \omega}, \quad (2)$$

can be obtained from (1) by an appropriate contour integration. (The K-K relations are really the K-K relation.)

16. A one-dimensional interface has a discontinuity in dielectric constant as shown.



In other words,

$$\epsilon = \begin{cases} \epsilon_2, & x > 0, \\ \epsilon_1, & x < 0. \end{cases}$$

The origin of coordinates is taken at the interface and the source point has  $x' > 0$ . Solve the one-dimensional wave equation,

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right) G(x, t; x', t') = \delta(t-t') \delta(x-x'),$$

subject to the usual boundary conditions for tangential electric field components. Give an image source interpretation of your results, if possible. Be sure to solve for both the  $x > 0$  and  $x < 0$  regions. (This problem is very close to the reduced Green function problem solved in the notes, except the final step, which is the reconstruction of  $G(x, t; x', t')$  itself. See if you can give an image and new wave velocity interpretation to  $G(x, t; x', t')$  in the regions  $x > 0$  and  $x < 0$ .)

17.(a) Given Eq.(8.148) of the notes,

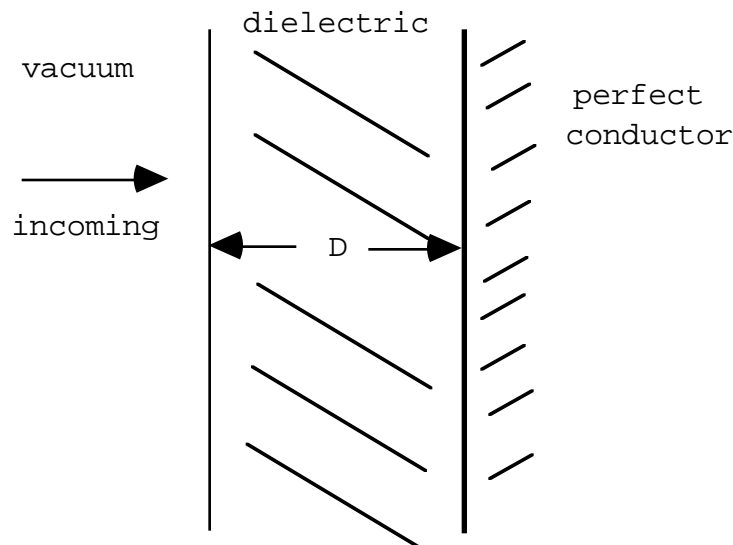
$$k^2 = \frac{\mu\omega^2}{c^2} \left( \epsilon + i \frac{4\pi\sigma}{\omega} \right),$$

write  $k = \alpha + i\beta$  and solve explicitly for  $\alpha$ ,  $\beta$ , (Consider  $\epsilon$ ,  $\sigma$  real and frequency independent.)

(b) A plane polarized electromagnetic wave of frequency  $\omega$  in free space is incident normally on a flat surface with dielectric constant  $\epsilon$ , permeability  $\mu$  and conductivity  $\sigma$ . Obtain an exact expression in terms of  $\alpha$  and  $\beta$  for the phase,  $\phi(\omega)$ , of the reflected wave relative to the incident wave. Find the high ( $\sigma/\omega \ll 1$ ) and low ( $\sigma/\omega \gg 1$ ) frequency limits of  $\phi(\omega)$  in terms of  $\epsilon$  and  $\sigma$ . Show that the phase is close to  $\pi$  in either the high and low frequency limits, increasing in one case, decreasing in the other, if  $\epsilon/\mu > 1$ .

(c) Find a simple formula for the critical frequency,  $\omega_{cr}$ , at which this phase is maximized. In addition, get an expression for this maximum phase,  $\phi_{max}$ . (I'm not sure how realistic this is since we have assumed  $\epsilon$  and  $\sigma$  frequency independent.)

18. Consider a plane electromagnetic wave of angular frequency  $\omega$  incident normally on a planar nonconducting material of depth  $D$  and index of refraction,  $n$ , from vacuum. The backing material is a perfect conductor. Assume the incoming wave has the initial form  $\text{Re}(\vec{E}_0 e^{i(kx - \omega t)})$ , where  $\vec{E}_0$  is linearly polarized.



Write down the boundary conditions on the electric, magnetic field components at the two interfaces. Solve for the fields

everywhere and show that the phase,  $\phi$ , of the reflected wave relative to the incoming is given by

$$\phi = \frac{2}{n \cot(k_1 D) - \left(\frac{1}{n}\right) \tan(k_1 D)},$$

where  $k_1 = \frac{\omega n}{c}$ .

19. (a) Consider a material with purely bound charge within the dispersion model considered in the notes. Using the model formula for displacements for a harmonic electric field, show that for a system with a single resonance,  $\omega_0$ , and a damping constant,  $\gamma$ , one obtains

$$\left[ \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} + \omega_0^2 + \frac{4\pi e^2 n_b}{m} \right] \rho_b(\vec{x}, t) = 0,$$

where  $n_b$  is the density of bound charges.