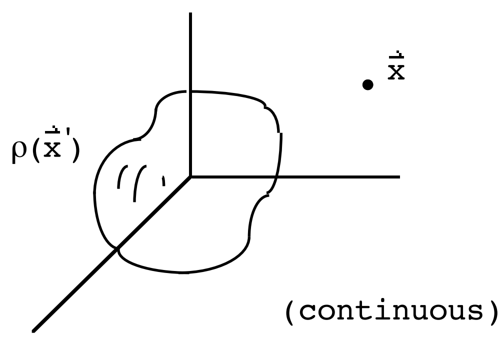
Chapter 4

Cartesian and spherical multipole expansions

Let's say we have:



The potential is given by

$$\Phi(\vec{x}) = \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{4.1}$$

Choose origin w/i charge distribution. If $r=|\vec{x}|$ is large compared to the characteristic dimensions of the distribution, we get

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{(\vec{x}'\cdot\vec{\nabla}')^{\ell}}{\ell!} \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) \Big|_{\vec{x}'=0}. \tag{4.2}$$

Notice

$$\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} = -\vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|}, \tag{4.3}$$

so that

$$\frac{(\vec{x}'\cdot\vec{\nabla}')^{\ell}}{\ell!}\ \frac{1}{|\vec{x}-\vec{x}'|}\Big|_{\vec{x}'=0} = (-1)^{\ell}\ \frac{(\vec{x}'\cdot\vec{\nabla}\ )^{\ell}}{\ell!}\ \frac{1}{r}, \qquad (4.4)$$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} - \vec{x}'\cdot\vec{\nabla}\frac{1}{r} + \frac{1}{2}\ (\vec{x}'\cdot\vec{\nabla})^2\ \frac{1}{r}\ - \cdots\ (r > r'). \qquad (4.5)$$

Work these out:

$$(\vec{x}'\cdot\vec{\nabla})\ \frac{1}{r} = \sum_i x_i'\nabla_i\ \frac{1}{\sqrt{x^2+y^2+z^2}} = -\sum_i \frac{x_i' x_i}{r^3} = -\ \frac{\vec{x}'\cdot\vec{x}}{r^3}, \qquad (4.6)$$

$$(\vec{x}'\cdot\vec{\nabla})^2\ \frac{1}{r} = \sum_{i,j} x_i' x_j' \nabla_i\nabla_j\ \frac{1}{\sqrt{x^2+y^2+z^2}} = -\sum_{i,j} x_i' x_j' \nabla_i\ \frac{x_j}{r^3}, \qquad (4.7)$$

or

$$(\vec{x}'\cdot\vec{\nabla})^2\ \frac{1}{r} = \sum_{i,j} x_i' x_j'\ \frac{(3x_i x_j\ -\ \delta_{ij} r^2)}{r^5}. \qquad (4.8)$$

Therefore

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}'\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j}\frac{x_i' x_j'}{r^5}\ (3x_i x_j\ -\ \delta_{ij} r^2) + \ldots \qquad (4.9)$$

Last term:

$$\sum_{i,j}\frac{x_i' x_j'}{r^5}\ (3x_i x_j\ -\ \delta_{ij} r^2) = \sum_{i,j}\ \frac{3x_i' x_j' x_i x_j\ -\ r'^2 r^2}{r^5}$$

$$= \sum_{i,j}\frac{x_i x_j}{r^5}\ (3x_i' x_j'\ -\ r'^2\delta_{ij}). \qquad (4.10)$$

$$\Rightarrow \Phi(\vec{x}) = \int d^3x'\rho(\vec{x}')\left\{\frac{1}{r} + \frac{\vec{x}'\cdot\vec{x}}{r^3} + \sum_{i,j}\frac{x_i x_j}{r^5}\ (3x_i' x_j'\ -\ r'^2\delta_{ij})\ + \ldots\right\},$$

(4.11)

$$\Phi(\vec{x}) = \frac{q}{r} + \frac{\vec{x}\cdot\vec{p}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \ldots \quad , \tag{4.12}$$

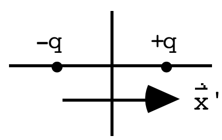
where

 # components

1 $\quad q = \int d^3x'\rho(\vec{x}')$ charge (scalar)

 # components

3 $\quad \vec{p} \equiv \int d^3x'\vec{x}'\rho(\vec{x}')$ el. dipole (vector)



$, \quad \vec{p} = q\,\vec{x}' \ .$

5 $\quad Q_{ij} \equiv \int d^3x'(3x_i'x_j' - \delta_{ij}r'^2)\rho(\vec{x}')$ quadrupole (tensor)

⇑increases like $2\ell + 1$

Notice ("traceless")

$$\sum_i Q_{ii} = \int d^3x'(3r'^2 - 3r'^2)\rho(\vec{x}') = 0, \tag{4.13}$$

$$\Rightarrow Q_{11} + Q_{22} + Q_{33} = 0. \tag{4.14}$$

Fields: $(r \neq 0)$

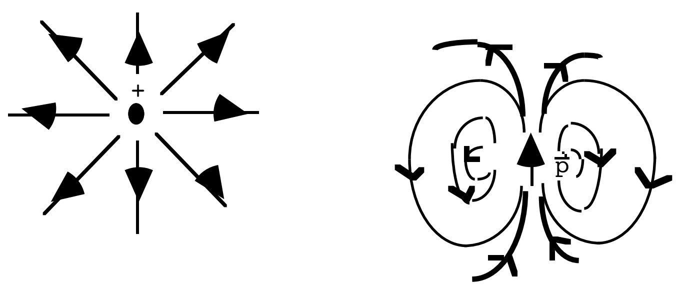
$$\vec{E} = -\vec{\nabla}\left(\frac{q}{r}\right) = \frac{q\hat{r}}{r^2} \quad \left(\text{point charge} \sim \frac{1}{r^2}\right), \tag{4.15}$$

$$\vec{E} = -\vec{\nabla}\left(\frac{\vec{x}\cdot\vec{p}}{r^3}\right) = \frac{3(\vec{x}\cdot\vec{p})\vec{x}-\vec{p}r^2}{r^5} \quad \left(\text{point dipole} \sim \frac{1}{r^3}\right), \tag{4.16}$$

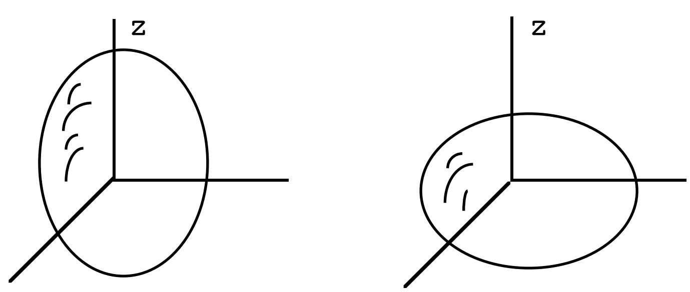
$$E_i = -\nabla_i \left(\frac{1}{2} \sum_{j,k} Q_{jk} \frac{x_j x_k}{r^5}\right), \text{ or}$$

$$E_i = \frac{1}{2} \left\{ \frac{5 \sum\limits_{j,k} Q_{jk} x_j x_j x_i - 2r^2 \sum\limits_{k} Q_{ik} x_k}{r^7} \right\} \text{ (point quadrupole} \sim \frac{1}{r^4}\text{)}. \tag{4.17}$$

Charge & dipole fields:



Two types of quadrupole distributions are: (rot. invariant about z-axis)



"prolate", $Q_{33} > 0$ "oblate", $Q_{33} < 0$

Obviously, both the location and orientation of our axes (in general) affect the moments.

The problem with the above is that the number of indices increases with increasing ℓ. Possible to avoid this by expanding in spherical harmonics. Of course

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{\ell,m} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi) \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m}(\theta',\phi'), \tag{4.18}$$

so

$$\Phi(\vec{x}) = \sum_{\ell,m} \frac{1}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi) \int d^3x' r'^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m}(\theta',\phi')\rho(\vec{x}'). \tag{4.19}$$

Introduce

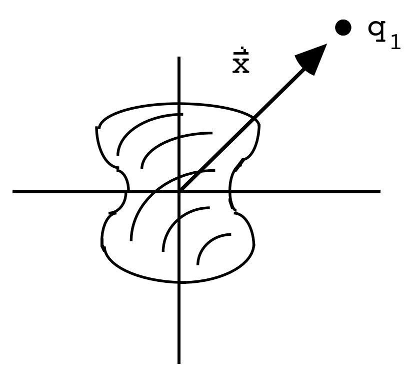
$$\rho_{\ell m} \equiv \int d^3x' r'^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m}\theta',\phi')\rho(\vec{x}'), \tag{4.20}$$

$$\Rightarrow \Phi(\vec{x}') = \sum_{\ell,m} \frac{1}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi)\rho_{\ell m}. \tag{4.21}$$

Multipole energy expansions

Energy of interaction. Point charge, q_1:

$$W = q_1\Phi(\vec{x}) = \frac{q_1 q}{r} + q_1 \frac{\vec{x}\cdot\vec{p}}{r^3} + \frac{q_1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \dots \,. \tag{4.22}$$



Introduce

$$\vec{E}^{(0)} = -\frac{q_1\vec{x}}{r^3} \Rightarrow \frac{\partial E_j^{(0)}}{\partial x_i} = q_1\left[\frac{3x_ix_j - \delta_{ij}r^2}{r^5}\right], \qquad (4.23)$$

(field of q_1 charge at origin). Then

$$W = q_1\phi - \vec{p}\cdot\vec{E}^{(0)} + \frac{q_1}{6}\sum_{ij} Q_{ij}\left[\frac{3x_ix_j - \delta_{ij}r^2}{r^5}\right], \qquad (4.24)$$

$$\Rightarrow W = q_1\phi - \vec{p}\cdot\vec{E}^{(0)} + \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j^{(0)}}{\partial x_i} + \cdots . \qquad (4.25)$$

(Notice difference in sign from book. Here, x_i is a distance from the <u>origin</u> to the <u>source</u>.) Can now imagine integrating over many q_1's: this gives us the energy of interaction of two arbitrary charge distributions. From this, we can read off various types of interaction:

Dipole - dipole:

$$W_{dd} = -\vec{p}_2\cdot\vec{E}_1^d = \frac{-3(\vec{x}\cdot\vec{p}_1)(\vec{x}\cdot\vec{p}_2)+(\vec{p}_1\cdot\vec{p}_2)r^2}{r^5}. \qquad (4.26)$$

⇑ ($\vec{x}\to-\vec{x}$ in previous expression which had the dipole at the origin)

Dipole - quadrupole:

$$W_{dQ} = -\vec{p}_2\cdot\vec{E}^Q = -\frac{1}{2}\sum_{i,j} Q_{ij}\left\{\frac{2r^2p_ix_j - 5x_ix_j(\vec{p}\cdot\vec{x})}{r^7}\right\}. \qquad (4.27)$$

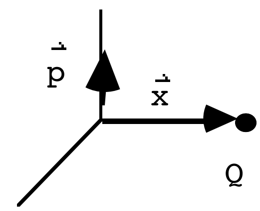
⇑ ($\vec{x}\to-\vec{x}$)

Same as quadrupole - dipole?

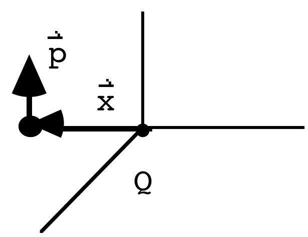
$$W_{Qd} = \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_j^{(0)}}{\partial x_i} = \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial}{\partial x_i}\left\{\frac{3(\vec{x}\cdot\vec{p})x_j - p_jr^2}{r^5}\right\},$$

$$W_{Qd} = \frac{1}{2}\sum_{i,j} Q_{ij}\left\{\frac{2r^2p_ix_j - 5x_ix_j(\vec{p}\cdot\vec{x})}{r^7}\right\} = -W_{dQ} \qquad (4.28)$$

Difference in sign occurs because in the first case we have



while in the second



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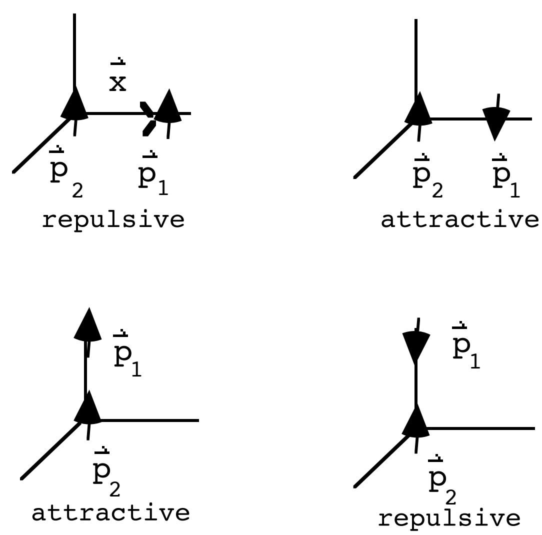
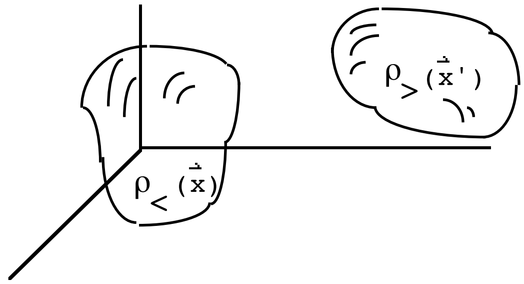
What we find in the dipole - dipole case is:



Explains a wealth of data having to do with forces between atoms in solids.

Above becomes more complicated for higher order interactions. Expansion in spherical harmonics helps here also. In the case of non-overlapping charge densities, as in



we have

$$W = \int d^3x d^3x' \frac{\rho_<(\vec{x})\rho_>(\vec{x}')}{|\vec{x}-\vec{x}'|}. \qquad (4.29)$$

But

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+m}} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi), \qquad (4.30)$$

(Above picture: r>=r' r<=r)

$$\Rightarrow W = \sum_{\ell,m} \int d^3x' d^3x\, \rho_<(\vec{x}) \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi) \frac{r^{\ell}}{r'^{\ell+1}} Y^*_{\ell m}(\theta',\phi')\rho_>(\vec{x}'). \qquad (4.31)$$

Define

$$\rho_{\left(\substack{<\\>}\right)\ell m} \equiv \int d^3x \sqrt{\frac{4\pi}{2\ell+1}}\, r^{\left(\substack{\ell\\-\ell-1}\right)} Y_{\ell m}(\theta,\phi)\rho_{\substack{<\\>}}(\vec{x}). \qquad (4.32)$$

Then very simply:

$$W = \sum_{\ell,m} \rho_{<\ell m}\, \rho^*_{>\ell m}\, . \qquad (4.33)$$

Can establish relationships between the different sets of expansion coefficients. For example (see p.99 of J. for Y_{11}, Y_{1-1},Y_{10})

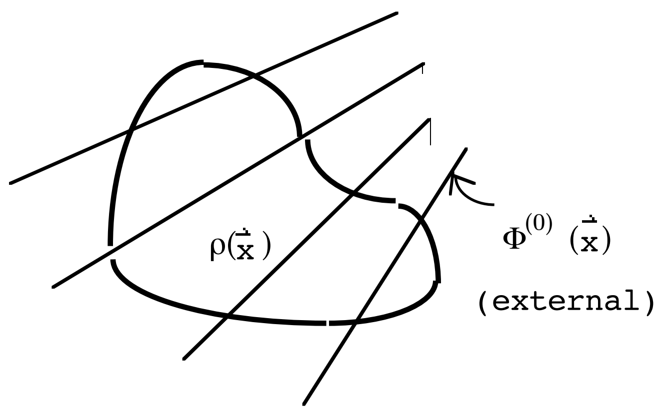
$$\vec{p} = \int d^3x' \rho(\vec{x}')r' \left[\sin\theta'\cos\phi\hat{i} + \sin\theta'\sin\phi'\hat{j} + \cos\theta'\hat{k} \right], \qquad (4.34)$$

$$\vec{p} = \sqrt{\frac{8\pi}{3}} \int d^3x' \rho(\vec{x}') r' \left[\frac{1}{2} [-Y_{11} + Y_{1-1}] \hat{i} + \frac{i}{2} [Y_{11} + Y_{1-1}] \hat{j} + \frac{1}{\sqrt{2}} Y_{10} \hat{k} \right], \tag{4.35}$$

$$\vec{p} = \frac{1}{\sqrt{2}} (-\rho^<_{11} + \rho^<_{1-1}) \hat{i} + \frac{i}{\sqrt{2}} (\rho^<_{11} + \rho^<_{1-1}) \hat{j} + \rho^<_{10} \hat{k}. \tag{4.36}$$

Can also see that if q=0, $\vec{p}$ is independent of origin. (True for higher moments as well as if all the lower ones vanish.)

Set up prob. 4.5(b) of Jackson by doing 4.5(a):



$$d\vec{F}(\vec{x}) = dq\, \vec{E}^{(0)}(\vec{x}), \tag{4.37}$$

$$\Rightarrow \vec{F} = \int dq\, \vec{E}^{(0)}(\vec{x}) = \int d^3x \rho(\vec{x})\, \vec{E}^{(0)}(\vec{x}) \tag{4.38}$$

Trick: introduce two <u>independent</u> sets of coordinates x_i and x_i', measured from the <u>same</u> origin. Look at each component:

$$\vec{E}^{(0)} = \vec{E}^{(0)}(0) + (\vec{x} \cdot \vec{\nabla}')\vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} + \frac{1}{2} (\vec{x} \cdot \vec{\nabla}')^2 \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} + \ldots . \tag{4.39}$$

3rd term really means:

$$(\vec{x} \cdot \vec{\nabla}')^2 \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} = \sum_{i,j} x_i \frac{\partial}{\partial x_i'} x_j \frac{\partial}{\partial x_j'} \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0}$$

$$= \sum_{i,j} x_i x_j \frac{\partial^2 \vec{E}^{(0)}(\vec{x}\,')}{\partial x_i' \partial x_j'} \bigg|_{x'=0}. \qquad (4.40)$$

A useful vector identity is (front of book)

$$\vec{\nabla}'(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla}')\vec{b} + (\vec{b}\cdot\vec{\nabla}')\vec{a} + \vec{a}\times(\vec{\nabla}'\times\vec{b}) + \vec{b}\times(\vec{\nabla}'\times\vec{a}). \qquad (4.41)$$

Therefore

$$\vec{\nabla}'\big(\vec{x}\cdot\vec{E}^{(0)}(\vec{x}\,')\big) = (\vec{x}\cdot\vec{\nabla}')\vec{E}^{(0)}(\vec{x}\,'), \quad \begin{pmatrix}\text{use electrostatics:} \\ \vec{\nabla}'\times\vec{E}^{(0)} = 0\end{pmatrix} \qquad (4.42)$$

$$\Rightarrow \vec{\nabla}'(\vec{x}\cdot(\vec{x}\cdot\vec{\nabla}')\vec{E}^{(0)}(\vec{x}\,')) = (\vec{x}\cdot\vec{\nabla}')(\vec{x}\cdot\vec{\nabla}')\vec{E}^{(0)}(\vec{x}\,'). \qquad (4.43)$$

[In the above we have to use:

$$\vec{\nabla}'\times\left[(\vec{x}\cdot\vec{\nabla}')\vec{E}^{(0)}(\vec{x}\,')\right] = \vec{\nabla}'\times\left[\vec{\nabla}'(\vec{x}\cdot\vec{E}^{(0)}(\vec{x}\,'))\right] = 0. \;]$$

Thus

$$\vec{E}^{(0)}(\vec{x}) = \vec{E}^{(0)}(0) + \vec{\nabla}'(\vec{x}\cdot\vec{E}^{(0)}(\vec{x}\,'))\Big|_{x'=0} + \frac{1}{2}\vec{\nabla}'\left\{\vec{x}\cdot[(\vec{x}\cdot\vec{\nabla}')\vec{E}^{(0)}(\vec{x}\,')]\right\}\Big|_{x'=0} + \ldots \,. \qquad (4.44)$$

More explicity on the last term:

$$\vec{E}^{(0)}(\vec{x}) = \vec{E}^{(0)}(0) + \vec{\nabla}'\big(\vec{x}\cdot\vec{E}^{(0)}(\vec{x}\,')\big)\Big|_{x'=0} + \frac{1}{2}\vec{\nabla}' \sum_{i,j} x_i x_j \frac{\partial}{\partial x_j'} E_i^{(0)}(\vec{x}\,')\Big|_{x'=0} + \ldots \,. \qquad (4.45)$$

$\vec{\nabla}'\cdot\vec{E}^{(0)} = 0$ for the external field in the region of interest, so we may add

$$-\frac{1}{6} r^2 \vec{\nabla}'\cdot\vec{E}^{(0)}(\vec{x}\,')\Big|_{x'=0} = -\frac{1}{6}\sum_i r^2 \frac{\partial \vec{E}_i^{(0)}}{\partial x_i'}\Big|_{x'=0}$$

$$= -\frac{1}{6}\sum_{i,j} r^2\,\delta_{ij}\frac{\partial \vec{E}_i^{(0)}}{\partial x_j'}\Bigg|_{x'=0}. \qquad (4.46)$$

We now find that

$$\vec{E}^{(0)}(\vec{x}) = \vec{E}^{(0)}(0) + \vec{\nabla}'(\vec{x}\cdot\vec{E}^{(0)}(\vec{x}'))\Big|_{x'=0}$$

$$+ \frac{1}{6}\vec{\nabla}'\sum_{i,j}(3x_ix_j - r^2\delta_{ij})\frac{\partial \vec{E}_i^{(0)}}{\partial x_j'}\Bigg|_{x'=0} + \dots\ . \qquad (4.47)$$

Thus, for the force, $\vec{F}$, from (4.38):

$$\vec{F} = \int d^3x\ \rho(\vec{x})\ \Big[\vec{E}^{(0)}(0) + \vec{\nabla}'(\vec{x}\cdot\vec{E}^{(0)}(\vec{x}')\Big|_{x'=0}$$

$$+ \frac{1}{6}\vec{\nabla}'\sum_{i,j}(3x_ix_j - r^2\delta_{ij})\frac{\partial E_i^{(0)}}{\partial x_i'}\Bigg|_{x'=0} + \dots\Big]. \qquad (4.48)$$

Using our def$^{\underline{n}}$s of q, $\vec{p}$ and Q_{ij}, this then shows that ($\vec{x}'\to\vec{x}$ now)

$$\vec{F} = q\ \vec{E}^{(0)}(0) + \Big\{\vec{\nabla}(\vec{p}\cdot\vec{E}^{(0)}(\vec{x})\Big\}\Big|_{x=0}$$

$$+ \Big\{\vec{\nabla}\ [\frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_i^{(0)}}{\partial x_j}\ ]\Big\}\Bigg|_{x=0} + \dots\ . \qquad (4.49)$$

An alternate form is given by

$$\vec{F} = q\ \vec{E}^{(0)}(0) + (\vec{p}\cdot\vec{\nabla})\vec{E}^{(0)}(\vec{x})\Big|_{x=0}$$

$$+ \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial^2\ \vec{E}^{(0)}}{\partial x_i\partial x_j}\Bigg|_{x=0} + \dots\ . \qquad (4.50)$$

Introduction of the "electric polarization" and "displacement field"

We can think of the above as applying to an atom. We want to deal with a macroscopic description instead of dealing with individual atoms. Let us therefore integrate over the density, $n(\vec{x})$, of such atoms. (q = 0 for neutral atoms)

$$\vec{F}_{bulk} \equiv \sum_{i=atoms} \vec{F}_i = \sum_i (\vec{p}_i \cdot \vec{\nabla}) \vec{E}^{(0)} \Big|_{x_i}, \qquad (4.51)$$

$$\vec{F}_{bulk} \approx \int d^3x \; n(\vec{x}) \left(\vec{p}(\vec{x}) \cdot \vec{\nabla}\right) \vec{E}^{(0)}(\vec{x}). \qquad (4.52)$$

⇑possible space dependence

Let us define

$$\vec{P}(\vec{x}) = n(\vec{x})\vec{p}(\vec{x}) \quad \left[\text{Units: } \frac{\text{dipole strength}}{\text{volume}} \sim \frac{q\ell}{\ell^3}\right] \qquad (4.53)$$

(same units as $\vec{E}$) as the "electric polarization". Clear that $\vec{P}$ can have $\vec{x}$ dependence either from the density or some intrinsic change in $\vec{P}$ from atom to atom. Now, integrate by parts (original integral over entire sample):

$$\vec{F}_{bulk} \approx \int_V d^3x (-\vec{\nabla}\cdot\vec{P}(\vec{x}))\vec{E}^{(0)}(\vec{x}) + \int_S da(\vec{P}\cdot\hat{n})\vec{E}^{(0)}(\vec{x}). \qquad (4.54)$$

"V" and "S" now refer to the idealized sample volume, surface. (In this form the surface, S, is not in the volume, V.) Compare with $\int d^3x \; \rho_{eff}(\vec{x})\vec{E}^{(0)}(\vec{x})$ to identify

$$\left.\begin{aligned} \rho^d_{eff}(\vec{x}) &= -\vec{\nabla}\cdot\vec{P}, \\ \sigma^d_{eff}(\vec{x}) &= \vec{P}\cdot\hat{n}. \end{aligned}\right\} \qquad (4.55)$$

[Can also show (prob.)

$$\rho^q_{eff}(\vec{x}) = \frac{1}{6}\sum_{i,j} \frac{\partial^2 q_{ij}(\vec{x})}{\partial x_i \partial x_j}, \qquad (4.56)$$

where $q_{ij}(\vec{x}) = n(\vec{x})Q_{ij}(\vec{x})$, which can also be written as a contribution to $\vec{P}$.]

What is the meaning of ρ_{eff}? It is the effective charge density contributed by all charges bound in the atoms. Effectively, for bulk material

$$\vec{\nabla}\cdot\vec{E} = 4\pi\,[\rho_{free} + \rho_{bound}]. \tag{4.57}$$

If we identify

$$\rho_{bound} = \rho^{d}_{eff} = -\vec{\nabla}\cdot\vec{P}, \tag{4.58}$$

then

$$\vec{\nabla}\cdot\vec{E} = 4\pi\,[\rho_{free} - \vec{\nabla}\cdot\vec{P}]. \tag{4.59}$$

Define ("displacement field")

$$\vec{D} = \vec{E} + 4\pi\,\vec{P}, \tag{4.60}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{D} = 4\pi\,\rho_{free}. \tag{4.61}$$

Will usually asume ("linear & isotropic")

$$\vec{P} = \chi(\vec{x})\,\vec{E}, \tag{4.62}$$

$$\Rightarrow \vec{D} = \varepsilon(\vec{x})\,\vec{E},\ \varepsilon(\vec{x}) = 1 + 4\pi\,\chi(\vec{x}). \tag{4.63}$$

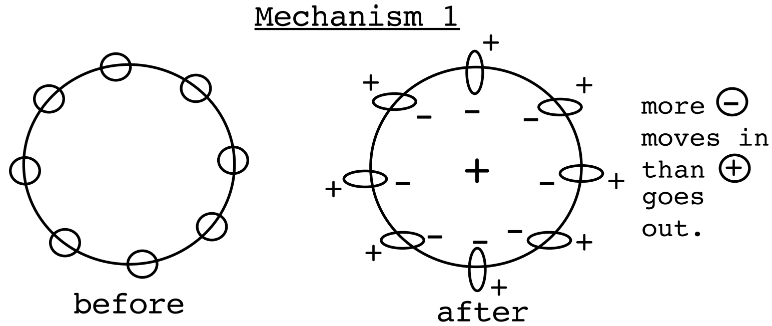
⇑dielectric constant

If ε = constant in the material, then

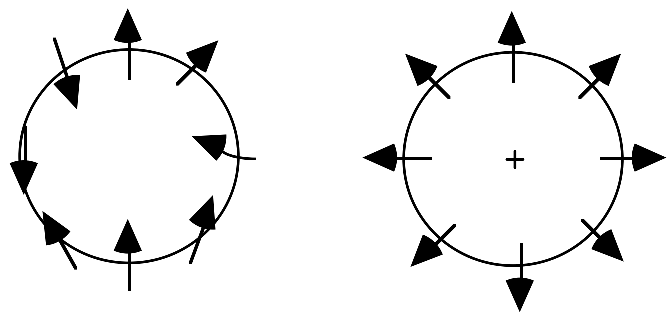
$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi\rho_{free}}{\varepsilon}. \tag{4.64}$$

One expects that electric fields are <u>reduced</u> so that $\varepsilon > 1$. This is understandable in that dipoles shield charge.

<u>Mechanism 1</u>



<u>Mechanism 2</u> $(W = -\vec{p}\cdot\vec{E}_{ext})$



This mechanism is temperature dependent (see section 4.6).

Mechanism 1: induced polarization
Mechanism 2: orientation polarization ("polar" substances)

Green functions in the presence of linear dielectrics

Go back to how we derived Green functions. Now have that

$$\vec{\nabla}\cdot[\varepsilon(\vec{x})\vec{\nabla}\Phi(\vec{x})\ ] = -4\pi\,\rho(\vec{x}). \tag{4.65}$$

(ρ understood to be free charge.) Now let us assume the Green function solves

$$\vec{\nabla}\cdot[\varepsilon(\vec{x})\vec{\nabla}\ G(\vec{x}',\vec{x})\ ] = -4\pi\,\delta(\vec{x}-\vec{x}'). \tag{4.66}$$

Still represents the electric field of a + unit charge. Go through old song and dance:

$$\int d^3x'\left\{G(\vec{x},\vec{x}')\vec{\nabla}'\cdot[\varepsilon(\vec{x}')\vec{\nabla}'\Phi(\vec{x}')]-\Phi(\vec{x}')\vec{\nabla}'\cdot[\varepsilon(\vec{x}')\vec{\nabla}'G(\vec{x},\vec{x}')]\right\}$$

$$= -4\pi\int d^3x'\left\{G(\vec{x},\vec{x}')\rho(\vec{x}') - \Phi(\vec{x}')\delta(\vec{x}-\vec{x}')\right\}. \tag{4.67}$$

$$\text{RHS} = 4\pi\,\Phi(\vec{x}) - 4\pi\int d^3x'G(\vec{x},\vec{x}')\rho(\vec{x}'). \tag{4.68}$$

$$\text{LHS} = \int d^3x'\vec{\nabla}'\cdot\left[G(\vec{x},\vec{x}')\varepsilon(\vec{x}')\vec{\nabla}'\Phi(\vec{x}') - \vec{\nabla}'G(\vec{x},\vec{x}')\varepsilon(\vec{x}')\Phi(\vec{x}')\right]$$

$$= \oint da' \left[\varepsilon(\vec{x}')G(\vec{x},\vec{x}') \frac{\partial \Phi}{\partial n'} - \varepsilon(\vec{x}')\Phi(\vec{x}') \frac{\partial G}{\partial n'} (\vec{x},\vec{x}') \right]. \qquad (4.69)$$

Choice of BC on $G(\vec{x},\vec{x}')$ now. Choose

$$G_D(\vec{x},\vec{x}') \Big|_{x' \text{on } s} = 0,$$

$$\Rightarrow \Phi(\vec{x}) = \int d^3x' G_D(\vec{x},\vec{x}')\rho(\vec{x}') - \frac{1}{4\pi} \oint da' \varepsilon(\vec{x}')\Phi(\vec{x}') \frac{\partial G_D}{\partial n'} . \qquad (4.70)$$

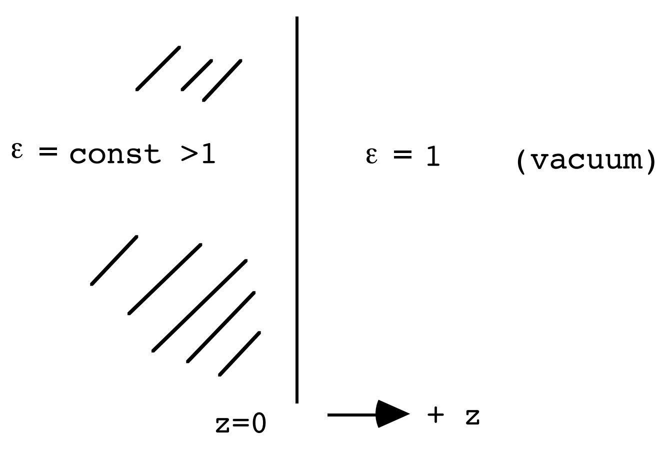
Can show that

$$G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x}), \qquad (4.71)$$

as before.

Green function for the dielectric slab

Now apply our knowlege to:



Get $G_D(\vec{x},\vec{x}')$ for $z' > 0$. Must solve

$$z > 0: \quad \nabla^2 G(\vec{x},\vec{x}') = -4\pi\, \delta(\vec{x}-\vec{x}'), \qquad (4.72)$$

$$z < 0: \quad \nabla^2 G(\vec{x},\vec{x}') = 0. \tag{4.73}$$

As usual, use

$$4\pi\,\delta(\vec{x} - \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, \delta(\vec{z} - \vec{z}'), \tag{4.74}$$

$$G(\vec{x},\vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}.(\vec{x}-\vec{x}')}\, g(z,z'), \tag{4.75}$$

$$- \nabla^2 G(\vec{x},\vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}.(\vec{x}-\vec{x}')} \left[k^2 - \frac{\partial^2}{\partial z^2} \right] g(z,z'). \tag{4.76}$$

So we get $(z' > 0)$

$$z > 0: \quad \left[- \frac{\partial^2}{\partial z^2} + k^2 \right] g(z,z') = \delta(z-z'), \tag{4.77}$$

$$z < 0: \quad \left[- \frac{\partial^2}{\partial z^2} + k^2 \right] g(z,z') = 0. \tag{4.78}$$

Our B.C.'s are

$$g \Big|_{0-}^{0+} = 0 \,, \quad (\Phi \text{ is continuous} \rightarrow \vec{E}_{||} \text{ is cont.}) \tag{4.79}$$

$$\varepsilon \frac{\partial}{\partial z}\, g \Big|_{0-} = \frac{\partial}{\partial z}\, g \Big|_{0+} . \qquad (D_n \text{ is cont.}) \tag{4.80}$$

The solutions in the various regions are:

$$z < 0: \quad g = Ae^{kz}, \quad (\text{finite as } z \rightarrow -\infty) \tag{4.81}$$

$$0 < z < z': \quad g = Be^{kz} + Ce^{-kz}, \tag{4.82}$$

$$z' < z: \quad g = De^{-kz}. \quad (\text{finite as } z \rightarrow +\infty) \tag{4.83}$$

The above BC's now require that

$$(4.79) \Rightarrow A = B+C, \tag{4.84}$$

$$(4.80) \;\Rightarrow\; \varepsilon k\, A = k(B-C), \tag{4.85}$$

from which we find

$$B = \frac{\varepsilon+1}{2} A, \quad C = \frac{1-\varepsilon}{2} A. \tag{4.86}$$

As z approaches z', we have

$$g \Big|_{z'-}^{z'+} = 0, \tag{4.87}$$

$$- \frac{\partial}{\partial z} g \Big|_{z'-}^{z'+} = 1, \tag{4.88}$$

which imply

$$De^{-kz'} = Be^{kz'} + Ce^{-kz'}, \tag{4.89}$$

$$kDe^{-kz'} + k(Be^{kz'} - Ce^{-kz'}) = 1. \tag{4.90}$$

Just give the solution. (Can check it for yourselves):

$$A = \frac{2}{\varepsilon + 1} \frac{1}{2k} e^{-kz'}, \tag{4.91}$$

$$B = \frac{1}{2k} e^{-kz'}, \tag{4.92}$$

$$C = - \frac{\varepsilon - 1}{\varepsilon + 1} \frac{1}{2k} e^{-kz'}, \tag{4.93}$$

$$D = - \frac{\varepsilon - 1}{\varepsilon + 1} \frac{1}{2k} e^{-kz'} + \frac{1}{2k} e^{kz'}. \tag{4.94}$$

Putting these back, we find that

$$z < 0: \quad g = \frac{2}{\varepsilon + 1} \frac{1}{2k} e^{-k(z' - z)} \left(= \frac{2}{\varepsilon + 1} \frac{1}{2k} e^{-k|z' - z|} \right), \tag{4.95}$$

$$0 < z < z': \quad g = \frac{1}{2k} \left[e^{-k(z' - z)} - \frac{\varepsilon - 1}{\varepsilon + 1} e^{-k(z + z')} \right], \tag{4.96}$$

$$z' < z: \quad g = \frac{1}{2k} \left[e^{-k(z - z')} - \frac{\varepsilon - 1}{\varepsilon + 1} e^{-k(z + z')} \right]. \tag{4.97}$$

Notice the last two combine as

$$z > 0: \quad g = \frac{1}{2k}\left[e^{-k|z-z'|} - \frac{\varepsilon-1}{\varepsilon+1}e^{-k(z+z')}\right]. \qquad (4.98)$$

Old result:

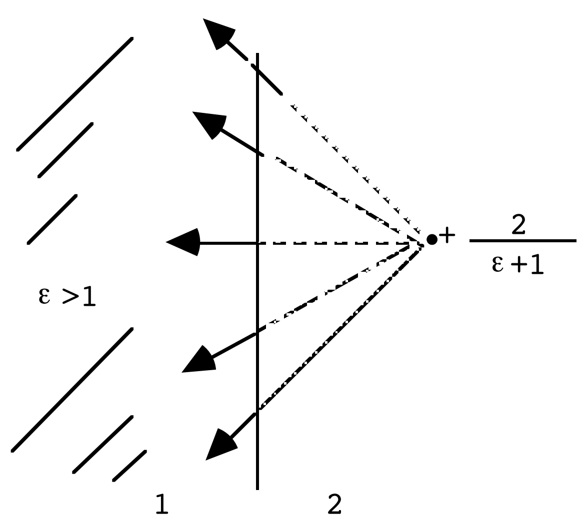
$$4\pi\int\frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')_\perp}\frac{1}{2k}e^{-k|z-z'|} = \frac{1}{|\vec{x}-\vec{x}'|}. \qquad (4.99)$$

Therefore ($z' > 0$)

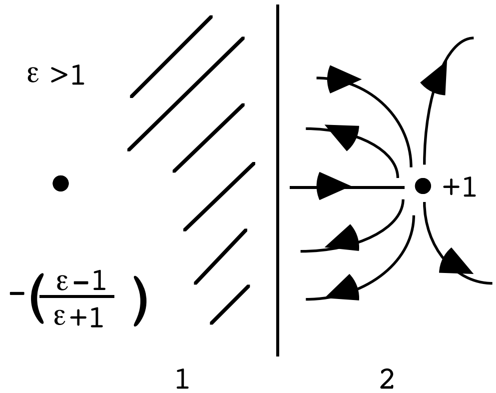
$$z < 0: \quad G(\vec{x},\vec{x}') = \frac{1}{\varepsilon}\frac{2\varepsilon}{\varepsilon+1}\frac{1}{|\vec{x}-\vec{x}'|}, \qquad (4.100)$$

$$z > 0: \quad G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{\varepsilon-1}{\varepsilon+1}\frac{1}{|\vec{x}-\vec{x}''|}. \qquad (4.101)$$

where $\vec{x}'' = (x',y',-z')$. Also gives $z'<0$, $z>0$ solution from symmetry of G: $G(\vec{x},\vec{x}')=G(\vec{x}',\vec{x})$. Interpretation: ($z < 0$)



$z > 0$:



Put them both together for final solution.

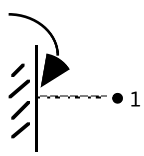
Charge on the interface?

$$-\vec{\nabla}\cdot\vec{P} = \rho_{\text{bound}}, \tag{4.102}$$

$$\Rightarrow -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n}_{21} = \sigma_{\text{bound}}. \tag{4.103}$$

$$\vec{P}_1 = \frac{\varepsilon-1}{4\pi}\vec{E}_1,\ \vec{P}_2 = 0, \tag{4.104}$$

$$\Rightarrow \sigma_{\text{bound}} = \frac{-1}{2\pi}\ \frac{\varepsilon-1}{\varepsilon+1}\ \frac{z'}{(\rho^2+z'^2)^{3/2}}\ .\ \text{(4.47 of J. with } q=1,\ \varepsilon_1=1) \tag{4.105}$$

($\rho^2=x^2+y^2$, as measured from 

.) For what it's worth, there is a surface delta function here, as we have seen before for a conductor (see p.54).

$$\lim_{z'\to 0^+} \frac{-2z'}{[\rho^2 + z'^2]^{3/2}} \to -4\pi\delta(\vec{x}_\perp-\vec{x}_\perp'), \tag{4.106}$$

$$\Rightarrow \sigma_{\text{bound}} \to -\frac{\varepsilon-1}{\varepsilon+1}\,\delta(\vec{x}_\perp - \vec{x}_\perp'). \tag{4.107}$$

Look at special cases: $\varepsilon\to\infty$ (perfect conductor)

$$z < 0: \quad G(\vec{x},\vec{x}') = 0. \text{ (no } \vec{E} \text{ field in conductor)} \tag{4.108}$$

$$z > 0: \quad G(\vec{x},\vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|}. \tag{4.109}$$

Neumann B.C. given (for one side) as $\varepsilon \to 0$.

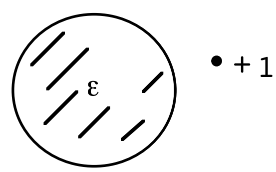
Trivial case, $\varepsilon \to 1$:

$$\text{all } z: \quad G(\vec{x},\vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}.$$

(Can do all this also by method of images, as in the book.)

Green function for the dielectric sphere

Next problem: (dielectric sphere)



Need to solve:

$$r > a: \quad -\nabla^2 G(\vec{x},\vec{x}') = 4\pi\delta(\vec{x} - \vec{x}'), \tag{4.110}$$

$$r < a: \quad -\vec{\nabla}\left[\varepsilon\vec{\nabla}\, G(\vec{x},\vec{x}')\right] = 0. \tag{4.111}$$

Assume

$$G(\vec{x},\vec{x}') = 4\pi\sum_{\ell,m} Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)\; g_\ell(r,r'). \tag{4.112}$$

As usual, get

$$-\frac{\partial}{\partial r}\left(r^2\frac{dg_\ell}{\partial r}\right) + \ell(\ell + 1)g_\ell = \delta(r - r') \ , \ r > a \qquad (4.113)$$

$$\varepsilon\left[-\frac{d}{dr}\left(r^2\frac{dg_\ell}{dr}\right) + \ell(\ell + 1)g_\ell\right] = 0, \ r < a \ . \qquad (4.114)$$

BC are

$$g_\ell \Big|_{a-}^{a+} = 0. \qquad (\Phi \text{ is cont.} \Rightarrow \vec{E}_{||} \text{ cont.}) \qquad (4.115)$$

$$\varepsilon\frac{\partial g_\ell}{\partial r}\Big|_{a-} = \frac{\partial g_\ell}{\partial r}\Big|_{a+} . \quad (D_n \text{ is continuous}) \qquad (4.116)$$

The solutions are

$$r < a: \quad g_\ell = A_\ell\, r^\ell \ , \qquad (4.117)$$

$$a < r < r': \quad g_\ell = B_\ell\, r^\ell + C_\ell\, r^{-\ell-1}, \qquad (4.118)$$

$$r' < r: \quad g_\ell = D_\ell\, r^{-\ell-1}. \qquad (4.119)$$

The above BC requires

$$(4.115) \Rightarrow A_\ell a = B_\ell\, a + C_\ell a^{-\ell-1}, \qquad (4.120)$$

$$(4.116) \Rightarrow \varepsilon A_\ell\, \ell a^{\ell-1} = \ell B_\ell\, a^{\ell-1} - (\ell+1)C_\ell a^{-\ell-2}. \quad (4.121)$$

from which we find (leave it to you again)

$$B_\ell = \frac{\ell(1+\varepsilon)+1}{2\ell+1}\, A_\ell, \qquad (4.122)$$

$$C_\ell = \frac{\ell(1-\varepsilon)+1}{2\ell+1}\, a^{2\ell+1}\, A_\ell. \qquad (4.123)$$

Other conditions at r = r' are:

$$g_\ell \Big|_{r'-}^{r'+} = 0, \qquad (4.124)$$

$$-r'^2 \frac{\partial}{\partial r} g_\ell \Big|_{r'_-}^{r'_+} = 1. \tag{4.125}$$

which give

$$B_\ell r'^\ell + C_\ell \; r'^{-\ell-1} = D_\ell \; r'^{-\ell-1}, \tag{4.126}$$

$$-r'^2 \Big[-(\ell+1) D_\ell \; r'^{-\ell-2} - (\ell B_\ell r'^{\ell-1} - (\ell+1) \; C_\ell r'^{-\ell-2}) \Big] = 1. \tag{4.127}$$

Again, I'll just give the solution:

$$A_\ell = \frac{1}{\ell(1+\varepsilon)+1} \; \frac{1}{r'^{\ell+1}}, \tag{4.128}$$

$$B_\ell = \frac{1}{2\ell+1} \; \frac{1}{r'^{\ell+1}}, \tag{4.129}$$

$$C_\ell = \frac{-(\varepsilon-1)\ell}{\ell(1+\varepsilon)+1} \; \frac{1}{2\ell+1} \; \frac{a^{2\ell+1}}{r'^{\ell+1}}, \tag{4.130}$$

$$D_\ell = C_\ell + \frac{r'^\ell}{2\ell+1}. \tag{4.131}$$

The Green function is now given by: (r' > a)

$$r < a: \quad G(\vec{x},\vec{x}') = \sum_{\ell,m} Y^*_{\ell m}(\theta',\phi') \; Y_{\ell m}(\theta,\phi) \; \frac{4\pi}{\ell(1+\varepsilon)+1} \; \frac{r^\ell}{r'^{\ell+1}}, \tag{4.132}$$

$$r > a: \quad G(\vec{x},\vec{x}') = \sum_{\ell,m} Y^*_{\ell m}(\theta',\phi') \; Y_{\ell m}(\theta,\phi) \; \frac{4\pi}{2\ell+1} \; \frac{r_<^\ell}{r_>^{\ell+1}}$$

$$- \sum_{\ell,m} Y^*_{\ell m}(\theta',\phi') \; Y_{\ell m}(\theta,\phi) \; \frac{4\pi}{2\ell+1} \; \frac{\ell(\varepsilon-1)}{\ell(1+\varepsilon)+1} \; \frac{a^{2\ell+1}}{(rr')^{\ell+1}}. \tag{4.133}$$

Can write as $\left(P_\ell = \sum_m \frac{4\pi}{2\ell+1} \; Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) \right)$

$$r < a: \quad G(\vec{x},\vec{x}') = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{\ell(1+\varepsilon)+1} \; \frac{r^\ell}{r'^{\ell+1}} \; P_\ell(\cos\gamma),$$ (also gives r'<a, r>a form)

(4.134)

$$r > a: \quad G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \sum_{\ell=1}^{\infty} \frac{(\varepsilon-1)\ell}{\ell(1+\varepsilon)+1} \frac{a^{2\ell+1}}{(rr')^{\ell+1}} P_\ell(\cos\gamma). \quad (4.135)$$

No more image charge interpretation (that I know of) except when $\varepsilon \to \infty$.

Look at r>a solution when r'>> a. We have

$$G(\vec{x},\vec{x}') \approx \frac{1}{|\vec{x}-\vec{x}'|} - \frac{\varepsilon-1}{\varepsilon+2} \frac{a^3}{r^2 r'^2} \cos\gamma, \quad (4.136)$$

$$\left(\cos\gamma = \frac{\vec{x}\cdot\vec{x}'}{rr'}\right)$$

$$\Rightarrow \quad G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \frac{\vec{x}\cdot\vec{p}}{r^3}, \quad \vec{p} = \frac{\varepsilon-1}{\varepsilon+2} a^3 \left(-\frac{\vec{x}'}{r'^3}\right). \quad (4.137)$$

$\Uparrow$ $\vec{E}'(0)$

$\vec{p}$ is the induced dipole moment of the sphere due to the positive charge. Says that the dipole moment induced in a sphere of radius a by a <u>uniform</u> electric field is

$$\vec{p} = \frac{\varepsilon-1}{\varepsilon+2} a^3 \vec{E}_{const}. \quad (4.138)$$

Other case (r < a):

$$G(\vec{x},\vec{x}') \approx \frac{1}{r'} + \frac{3}{\varepsilon+2} \frac{r}{r'^2} \cos\gamma, \quad (4.139)$$

$$G(\vec{x},\vec{x}') = \frac{1}{r'} + \frac{3}{\varepsilon+2} \frac{\vec{x}\cdot\vec{x}'}{r'^2}, \quad (4.140)$$

$$G(\vec{x},\vec{x}') = \frac{1}{r'} - \frac{3}{\varepsilon+2} \vec{x}\cdot\vec{E}'(0). \quad (4.141)$$

$\Uparrow$point charge's electric field at origin

Says, for the positive charge very far away, the electric field in the sphere is approximately

$$\vec{E} = -\vec{\nabla}G(\vec{x},\vec{x}') = \frac{3}{\varepsilon+2}\vec{E}'(0), \tag{4.142}$$

which is less than $\vec{E}'(0)$ ($\varepsilon > 1$).

Field energy and dielectrics

In the absence of any constitutive relation between $\vec{D}$ and $\vec{E}$, all we know for a given material is

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho, \tag{4.143}$$

$$\vec{\nabla} \times \vec{E} = 0. \tag{4.144}$$

The second eq$^{\underline{n}}$ implies we may still take

$$\vec{E} = -\vec{\nabla}\Phi. \tag{4.145}$$

We continue to require $\vec{F} = q\vec{E}$ so that Φ as usual has the meaning of potential energy. Energy to move an infinitismal charge:

$$\delta W_1 = \int_A^B (-\vec{F}_1)\cdot d\vec{\ell} = -\delta q_1 \int_A^B \vec{E}\cdot d\vec{\ell}, \tag{4.146}$$

⇑work <u>on</u> the charge

$$\Rightarrow \delta W_1 = \delta q_1(\Phi_B - \Phi_A). \tag{4.147}$$

Take A to be our reference point (can be at ∞ or any other point)

$$\Phi_A = 0, \quad \Phi_B \to \Phi(\vec{x}_1). \tag{4.148}$$

Move another charge from A to B':

$$\delta W_2 = \delta q_2\Phi(\vec{x}_2) + \mathcal{O}(\delta q_1\, \delta q_2). \tag{4.149}$$

Add them up:

$$\delta W = \sum_i \delta W_i = \sum_i \delta q_i \, \Phi(\vec{x}_i), \tag{4.150}$$

$$\delta q_i \to d^3x \, \delta\rho(\vec{x}), \tag{4.151}$$

$$\delta W = \int d^3x \, \delta\rho(\vec{x})\Phi(\vec{x}) \quad \underline{\text{always}} \text{ true.} \tag{4.152}$$

($W = \frac{1}{2}\int d^3x \, \rho(\vec{x})\Phi(\vec{x})$ is not, however, necessarily implied by this.) Now since

$$\rho = \frac{1}{4\pi} \vec{\nabla}\cdot\vec{D}, \tag{4.153}$$

$$\Rightarrow \delta\rho = \frac{1}{4\pi} \vec{\nabla}\cdot\delta\vec{D}, \tag{4.154}$$

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3x \; \vec{\nabla}\cdot\delta\vec{D} \;\; \Phi(\vec{x}), \tag{4.155}$$

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3x \left[\vec{\nabla}\cdot(\delta\vec{D}\Phi) - \delta\vec{D}\cdot\nabla\Phi \right], \tag{4.156}$$

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3x \; \vec{E}\cdot\delta\vec{D} + \frac{1}{4\pi}\int_S (\Phi\delta\vec{D})\cdot\hat{n}da. \tag{4.157}$$

Surface term vanishes for localized charge distribution:

$$\frac{1}{4\pi}\int_S (\delta\vec{D}\cdot\hat{n})\Phi da \underset{R\to\infty}{\Rightarrow} \frac{V}{4\pi}\int_S \delta\vec{D}\cdot\hat{n}da = V\delta q, \tag{4.158}$$

But at large distances $V \sim \frac{Q_{tot}}{R} \to 0$ as $R \to \infty$,

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3x \; \vec{E}\cdot\delta\vec{D}. \tag{4.159}$$

As far as we can go unless we can write this as a perfect differential. Assuming relations of the form

$$D_\alpha = \sum_\beta \varepsilon_{\alpha\beta} \, E_\beta, \tag{4.160}$$

we have ($\varepsilon_{\alpha\beta}$ are assumed independent of variations in the charge density)

$$\delta D_\alpha = \sum_\beta \varepsilon_{\alpha\beta}\, \delta E_\beta, \qquad (4.161)$$

$$\Rightarrow \sum_\alpha E_\alpha \delta D_\alpha = \sum_{\alpha,\beta} \varepsilon_{\alpha\beta}\, \delta E_\beta E_\alpha. \qquad (4.162)$$

Also

$$\sum_\alpha D_\alpha \delta E_\alpha = \sum_{\alpha,\beta} \varepsilon_{\alpha\beta}\, E_\beta \delta E_\alpha, \qquad (4.163)$$

$$\Rightarrow \vec{D}\cdot\delta\vec{E} = \vec{E}\cdot\delta\vec{D} \quad \text{if } \varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}, \qquad (4.164)$$

$$\Rightarrow \vec{E}\cdot\delta\vec{D} = \frac{1}{2}\,\delta(\vec{E}\cdot\vec{D}). \qquad (4.165)$$

[The above certainly includes the case where $\vec{D} = \varepsilon\vec{E}$, $\varepsilon = \varepsilon(\vec{x})$. It excludes, however, for example, a situation where

$$\vec{D} = \varepsilon(\vec{E}^2)\vec{E}$$

for then

$$\vec{E}\cdot\delta\vec{D} = \vec{D}\cdot\delta\vec{E} + \delta\varepsilon(\vec{E}^2)\vec{E}^2.]$$

So, for a certain class of constitutive relations, we get

$$W = \frac{1}{8\pi}\int d^3x\ \vec{E}\cdot\vec{D} \underset{\substack{\uparrow \\ \text{isotropic}}}{=} \frac{1}{8\pi}\int d^3x\ \varepsilon(\vec{x})\vec{E}^2(\vec{x}). \qquad (4.166)$$

Other expressions for W can be developed. In particular since we know all static properties are in $G(\vec{x},\vec{x}')$, should be able to relate it to W. Since $\vec{E} = -\vec{\nabla}\,\Phi$, above says of course (localized dist. again)

$$W = \frac{1}{2}\int d^3x\ \rho(\vec{x})\Phi(\vec{x}). \qquad (4.167)$$

⇑ tip off that self-energies included

Now remember (Dirichlet)

$$\Phi(\vec{x}) = \int d^3x'\rho(\vec{x}')G_D(\vec{x},\vec{x}') - \frac{1}{4\pi}\oint_S da'\Phi(\vec{x}')\varepsilon(\vec{x}')\frac{\partial G_D}{\partial n'}. \qquad (4.168)$$

The surface, S, being referred to in (4.168) are surfaces of the entire volume where fields are defined, not the surfaces of dilectircs. Let's say that $\Phi\big|_s = 0$ (Certainly true for free space). Then another expression for the energy is

$$W = \frac{1}{2}\int d^3x\rho(\vec{x})\Phi(\vec{x}) = \frac{1}{2}\int d^3x\ d^3x'\rho(\vec{x})G_D(\vec{x},\vec{x}')\rho(\vec{x}'). \qquad (4.169)$$

This is just a generalization of: (Ch.1)

$$W = \frac{1}{2}\int d^3x d^3x'\ \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \qquad (4.170)$$

Now instead of introducing charge, think of introducing a dielectric. Amount of energy to do this?

$$\Delta W \equiv W-W_0, \qquad (4.171)$$

$$\Rightarrow \quad \Delta W = \frac{1}{2}\int d^3x d^3x'\rho(\vec{x})\ [G_D(\vec{x},\vec{x}') - G_D^0(\vec{x},\vec{x}')]\rho(\vec{x}'), \qquad (4.172)$$

$$\begin{aligned}\Delta W &= \frac{1}{2}\int d^3x\rho(\vec{x})\ [\Phi(\vec{x}) - \Phi_0(\vec{x})] \\ &= -\frac{1}{8\pi}\int d^3x\ \vec{D}\cdot\vec{\nabla}[\Phi(\vec{x}) - \Phi_0(\vec{x})] \\ &= \frac{1}{8\pi}\int d^3x\ \vec{D}\cdot(\vec{E}-\vec{E}_0). \qquad (4.173)\end{aligned}$$

Now consider

$$\int d^3x\ \vec{E}\cdot\vec{D} = \int d^3x\ \vec{E}\cdot\vec{D}_0 + \int d^3x\ \vec{E}\cdot(\vec{D}-\vec{D}_0)\ \text{(by parts)}. \qquad (4.174)$$

$$\underbrace{\qquad\qquad}_{\int d^3x\ \Phi\ \vec{\nabla}\cdot(\vec{D}-\vec{D}_0)=0}$$

Therefore

$$\Delta W = -\frac{1}{8\pi}\int d^3x\ [\vec{E}\cdot\vec{D}_0-\vec{D}\cdot\vec{E}_0]. \tag{4.175}$$

If

some dielectrics already ⇓ present

$$\vec{D} = \varepsilon(x)\vec{E},\ \vec{D}_0 = \varepsilon_0\ \vec{E}_0 \tag{4.176}$$

(integration effectively ⇓ over volume of dielectric)

$$\Rightarrow \Delta W = -\frac{1}{8\pi}\int d^3x\ (\varepsilon_0-\varepsilon)\ \vec{E}\cdot\vec{E}_0, \tag{4.177}$$

or if $\varepsilon_0 = 1$

$$\Delta W = \int d^3x(-\frac{1}{2}\ \vec{P}\cdot\vec{E}_0). \tag{4.178}$$

⇑not a perma. dipole

Bulk forces on dielectrics: theory

By using (4.178) or other means of finding an appropriate energy expression, we may find the <u>total</u> force from

$$\vec{F}\cdot\delta\vec{x} = -\delta_Q W, \tag{4.179}$$

⇑fixed charges

$$\Rightarrow\ \vec{F}_Q = -\left(\frac{\partial W}{\partial\vec{x}}\right)_Q \tag{4.180}$$

On the other hand, consider the movement of a dielectric in the presence of conductors kept at <u>fixed voltage</u>. (Assume all free charges are on surface of conductors.) Now we expect

$$\vec{F}_V = -\frac{\delta}{\delta\vec{x}}(W+W_b)_V. \quad (4.181)$$

⇑ ⇑ battery energy
field energy

But

$$W = \frac{1}{2}\sum_i \int da\sigma_i(\vec{x})V_i \quad (4.182)$$

$$= \frac{1}{2}\sum_i Q_iV_i, \quad (4.183)$$

$$\Rightarrow \quad \delta_V W = \frac{1}{2}\sum_i \delta Q_i V_i. \quad (4.184)$$

On the other hand, the battery's change in energy is

$$\delta_V W_b = \sum_i \delta\overline{Q}_i V_i. \quad (4.185)$$

But

$$\delta\overline{Q}_i = -\delta Q_i, \quad (4.186)$$

$$\Rightarrow \delta_V W_b = -2\delta_V W, \quad (4.187)$$

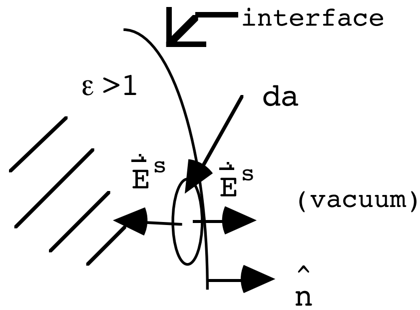
$$\Rightarrow \vec{F}_V = +\left(\frac{\partial W}{\partial\vec{x}}\right)_V. \quad (4.188)$$

It is important to realize that in a given static situation that we must have $\vec{F}_V = \vec{F}_Q$ in spite of the minus sign differences in (4.180) and (4.188).

Energy methods as discussed above are helpful, but they give you no idea of where the forces originate (although they are usually simpler). Go back to our $\vec{F}_{bulk}$ (from (4.54):

$$\vec{F}_{bulk} = \int_S da(\vec{P}\cdot\hat{n})\vec{E}^{(0)}(\vec{x}). \quad (4.189)$$

Consider a small surface element da:



Near the surface of each da:

$$\vec{E} = \vec{E}^{(0)} + \vec{E}^{r} + \vec{E}^{s}, \tag{4.190}$$

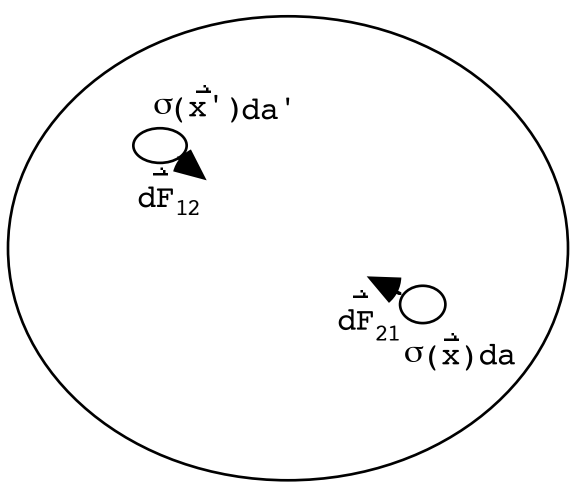
where $\vec{E}^s$ is the self-field ($\vec{E}^s = \pm 2\pi\sigma\hat{n}$), $\vec{E}^r$ is from the rest of the surface and $\vec{E}^{(0)}$ is external. Therefore the average field at the interface is

$$\frac{\vec{E}_1+\vec{E}_2}{2} = \vec{E}^{(0)} + \vec{E}^{r}. \tag{4.191}$$

From (4.189) and (4.191)

$$\vec{F}_{bulk} = \int_S da(\vec{P}\cdot\hat{n})\left[\frac{\vec{E}_1+\vec{E}_2}{2} - \vec{E}^r(\vec{x})\right]. \tag{4.192}$$

Let's consider:



Newton's third law tells us that $d\vec{F}_{12} = -\, d\vec{F}_{21}$. This implies that the second term in (4.210) is zero when the integration is over the entire surface. This means we can always use $\vec{E}^{(0)}$ or $\frac{\vec{E}_1+\vec{E}_2}{2}$ in such expressions. However, this is not to say that there are not self-forces or stresses w/i a given material; one only has to recall the outward pressure on the surfaces of a conductor, $2\pi\sigma^2$, we found in Chs.1,2. to realize this.

Nonlinear dielectric example: a phenomenological quark confinement model

Before I go on, I want to develop one model where there is a nonlinear relation between $\vec{E}$ and $\vec{D}$. General expression:

$$\delta W = \frac{1}{4\pi}\int d^3x\; \vec{E}\cdot\delta\vec{D}. \tag{4.193}$$

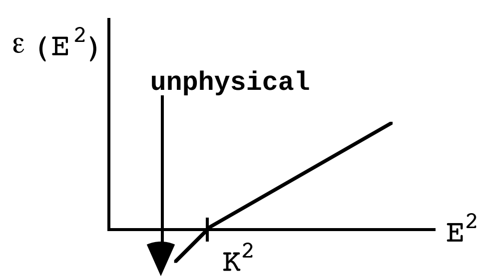
Model:

$$\vec{E} = -\vec{\nabla}\Phi, \tag{4.194}$$

$$\vec{D} = \varepsilon\vec{E}, \tag{4.195}$$

$$\text{but } \varepsilon = 2\alpha\,\ell n\left(\frac{E^2}{K^2}\right) \quad (E^2 = \vec{E}\cdot\vec{E}, \alpha > 0). \tag{4.196}$$

Extremely nonlinear. Picture:



Fix this up by saying that D=0 (or ε=0) outside the region where $E^2 > K^2$. Our only hope for getting an expression for W is if we can write the above δW as a perfect differential. Consider

$$\delta\left[\frac{1}{2}\vec{E}\cdot\vec{D} + \alpha\vec{E}^2\right] = \frac{1}{2}\vec{D}\cdot\delta\vec{E} + \frac{1}{2}\vec{E}\cdot\delta\vec{D} + 2\alpha\vec{E}\cdot\delta\vec{E}. \qquad (4.197)$$

Now

$$\delta\vec{D} = \delta(\varepsilon\vec{E}) = \left(2\alpha\,\ell n\frac{E^2}{K^2}\right)\delta\vec{E} + \frac{4\alpha}{E^2}(\vec{E}\cdot\delta\vec{E})\vec{E}, \qquad (4.198)$$

$$\Rightarrow \vec{E}\cdot\delta\vec{D} = \left(2\alpha\,\ell n\frac{E^2}{K^2}\vec{E}\right)\cdot\delta\vec{E} + 4\alpha\vec{E}\cdot\delta\vec{E}, \qquad (4.199)$$

$$= \vec{D}\cdot\delta\vec{E} + 4\alpha\vec{E}\cdot\delta\vec{E}, \qquad (4.200)$$

$$\Rightarrow \delta\left[\frac{1}{2}\vec{E}\cdot\vec{D} + \alpha\vec{E}^2\right] = \frac{1}{2}\vec{E}\cdot\delta\vec{D} - 2\alpha\vec{E}\cdot\delta\vec{E} + 2\alpha\vec{E}\cdot\delta\vec{E} + \frac{1}{2}\vec{E}\cdot\delta\vec{D}, \qquad (4.201)$$

$$\delta\left[\frac{1}{2}\vec{E}\cdot\vec{D} + \alpha\vec{E}^2\right] = \vec{E}\cdot\delta\vec{D}\ ! \qquad (4.202)$$

Therefore

$$W = \frac{1}{4\pi}\int d^3x\left[\frac{1}{2}\vec{E}\cdot\vec{D} + \alpha\vec{E}^2\right]. \qquad (4.203)$$

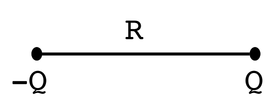
If we write

$$E = |\vec{E}| = Kf(D), \qquad (4.204)$$

then from the above we have that

$$f(D) \geq 1 \qquad (4.205)$$

whenever $E^2 \geq K^2$. This can be used to give a lower bound on the energy of certain charge configurations. Given



then

$$W > \frac{1}{8\pi}\int d^3x\,\vec{E}\cdot\vec{D} = \frac{K}{8\pi}\int d^3x f(D)D, \qquad (4.206)$$

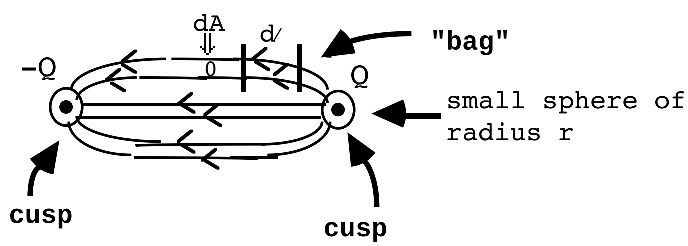
$$\underbrace{\qquad}_{ED}$$

$$\Rightarrow W > \frac{K}{8\pi}\int d^3x\ D. \quad \text{(a pos. def. number)}. \tag{4.207}$$

Choose

$$d^3x = d\ell dA. \tag{4.208}$$

Picture:

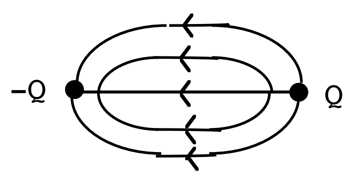


$$W > \frac{K}{8\pi}\int d\ell dA\ D > \frac{K\ell_{min}}{8\pi}\underbrace{\int d\vec{A}\cdot\vec{D}}_{4\pi|Q|}, \tag{4.209}$$

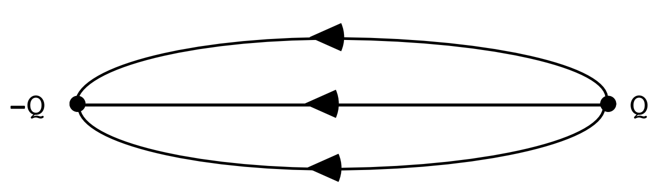
$(\ell_{min} = R-2r)$

$$\Rightarrow W > \frac{1}{2}K(R-2r)|Q|. \quad \text{(only for } R>>r) \tag{4.210}$$

=> Potential grows at least linearly at large R. By computer simulation, can show that it in fact saturates the lower limit for large R. What does this describe?



small separation

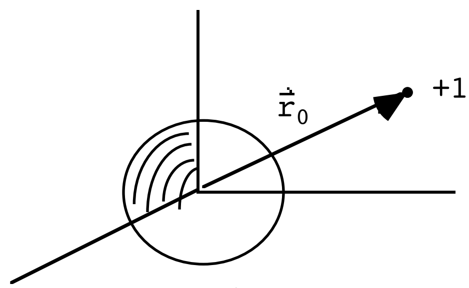


large separation

System becomes string-like and confined. This is a phenomenological quark model for mesons due to S.L. Adler.

Bulk forces on dielectrics: examples

Finish this up with two problems as examples of forces on dielectrics. First problem (similar to one assigned):



Get the force between the dielectric and the positive unit charge. Do it 3 ways. (I must be mad.) First way (easiest):

$$W = \frac{1}{2}\int d^3x d^3x' \rho(\vec{x})\left[G_D(\vec{x},\vec{x}') - G_D^0(\vec{x},\vec{x}')\right]\rho(\vec{x}'). \qquad (4.211)$$

Use

$$\rho(\vec{x}) = \delta(\vec{x}-\vec{r}_0). \qquad (4.212)$$

Then $\left(\text{use } r>a \text{ form of } G_D(\vec{x},\vec{x}')\right)$

$$W = \frac{1}{2}\left[G_D(\vec{x},\vec{x}') - G_D^0(\vec{x},\vec{x}')\right]\Big|_{\underbrace{\vec{x},\vec{x}'=\vec{r}_0}_{\text{be careful!}}}. \qquad (4.213)$$

(must be done as a limit since $G_D(\vec{x},\vec{x})$, $G_D^0(\vec{x},\vec{x}) = \infty$.) This gives

$$W = -\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(\varepsilon-1)\ell}{\ell(\varepsilon+1)+1}\frac{a^{2\ell+1}}{r_0^{2\ell+2}}\underbrace{P_\ell(1)}_{=1}, \qquad (4.214)$$

$$\Rightarrow \quad W = -\frac{\varepsilon-1}{2r_0}\sum_{\ell=1}^{\infty}\frac{\ell}{\ell(1+\varepsilon)+1}\left(\frac{a}{r_0}\right)^{2\ell+1}. \qquad (4.215)$$

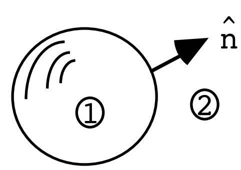
At large r_0:

$$W \approx -\left(\frac{\varepsilon-1}{\varepsilon+2}\right)\frac{a^3}{2r_0^4} \quad \Rightarrow F_r = -\left(\frac{\varepsilon-1}{\varepsilon+2}\right)\frac{2a^3}{r_0^5}. \qquad (4.216)$$

⇑force on <u>charge</u> (take origin on sphere)

Always pulled toward charge. Force is different from conducting sphere which is inverse cube at large r_0. Same problem using explicit force expression: (hardest way)

$$F_z = \int da\ (\vec{P}\cdot\hat{n})\left(\frac{E_{2z}+E_{1z}}{2}\right). \qquad (4.217)$$



$$(\vec{P}_1\cdot\hat{n}) = \frac{1}{4\pi}\left(E_{2r}\big|_a - E_{1r}\big|_a\right), \qquad (4.218)$$

$$G_1 \approx \frac{1}{r'} + \frac{3}{2+\varepsilon}\frac{r}{r'^2}\underbrace{\cos\theta}_{P_1} + \frac{5}{3+2\varepsilon}\frac{r^2}{r'^3}\underbrace{\frac{1}{2}(3\cos^2\theta-1)}_{P_2}, \qquad (4.219)$$

(taking z-axis along $\vec{r}\,' = \vec{r}_0$)

$$G_2 \approx \frac{1}{r'} + \frac{r}{r'^2}\cos\theta + \frac{r^2}{r'^3}\frac{1}{2}(3\cos^2\theta-1) - \frac{(\varepsilon-1)}{2+\varepsilon}\frac{a^3}{r^2r'^2}\cos\theta$$

$$- \frac{2(\varepsilon-1)}{3+2\varepsilon} \frac{a^5}{r^3 r'^3} \frac{1}{2}(3\cos^2\theta - 1) + \ldots \cdot \qquad (4.220)$$

$$E_{1r} = -\frac{\partial}{\partial r} G_1 \ , \ E_{2r} = -\frac{\partial}{\partial r} G_2, \qquad (4.221)$$

$$\Rightarrow E_{1r}\Big|_a \approx -\frac{3}{2+\varepsilon} \frac{1}{r'^2} \cos\theta - \frac{5}{3+2\varepsilon} \frac{a}{r'^3} (3\cos^2\theta - 1), \qquad (4.222)$$

$$E_{2r}\Big|_a \approx -\frac{3\varepsilon}{2+\varepsilon} \frac{1}{r'^2} \cos\theta - \frac{5\varepsilon}{3+2\varepsilon} \frac{a}{r'^3} (3\cos^2\theta - 1), \qquad (4.223)$$

$$\Rightarrow \vec{P}\cdot\hat{n}\Big|_a \approx \frac{1}{4\pi} \Big[\frac{3(1-\varepsilon)}{2+\varepsilon} \frac{1}{r'^2} \cos\theta + \frac{5(1-\varepsilon)}{3+2\varepsilon} \frac{a}{r'^3} (3\cos^2\theta - 1) \Big]. \qquad (4.224)$$

Likewise

$$E_{1z} = -\frac{\partial}{\partial z} G_1, \qquad (4.225)$$

$$E_{1z} = -\frac{\partial}{\partial z} \Big(\frac{3}{2+\varepsilon} \frac{z}{r'^2} + \frac{5\varepsilon}{3+2\varepsilon} \frac{1}{r'^3} \frac{1}{2}(3z^2 - r^2) + \ldots\Big).$$

$$(4.226)$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$E_{1z}\Big|_a = -\frac{3}{2+\varepsilon} \frac{1}{r'^2} - \frac{10}{3+2\varepsilon} \frac{a}{r'^3} \cos\theta, \qquad (4.227)$$

$$E_{2z}\Big|_a = -\frac{3}{2+\varepsilon} \frac{1}{r'^2} - \frac{3(\varepsilon-1)}{2+\varepsilon} \frac{1}{r'^2} \cos^2\theta - \frac{2a}{r'^3} \cos\theta$$
$$+ \frac{(\varepsilon-1)}{3+2\varepsilon} \frac{a}{r'^3} [-15 \cos^3\theta + 9\cos\theta]. \qquad (4.228)$$

$$\Rightarrow \frac{E_{2z}+E_{1z}}{2}\Big|_a = -\frac{3}{2+\varepsilon} \frac{1}{r'^2} - \frac{3}{2} \frac{(\varepsilon-1)}{2+\varepsilon} \frac{1}{r'^2} \cos^2\theta$$
$$-\Big(\frac{8+2\varepsilon}{3+2\varepsilon}\Big) \frac{a}{r'^3} \cos\theta + \frac{(\varepsilon-1)}{3+2\varepsilon} \frac{a}{r'^3} \Big[-\frac{15}{2}\cos^3\theta + \frac{9}{2}\cos\theta \Big]. \qquad (4.229)$$

Notice that $\frac{1}{r'^4}$ terms go like $\int_{-1}^{1} d\cos\theta \begin{pmatrix}\cos\theta \\ \cos^3\theta\end{pmatrix} = 0$. Lowest order terms:

$$F_z = \frac{2\pi a^3}{4\pi r'^5} \int_{-1}^{1} dx \left\{ \left(\frac{3(1-\varepsilon)}{2+\varepsilon} \frac{(\varepsilon-1)}{3+2\varepsilon} \left[-\frac{15}{2} x^4 + \frac{9}{2} x^2 \right] - \frac{3(1-\varepsilon)}{2+\varepsilon} \left(\frac{8+2\varepsilon}{3+2\varepsilon} \right) x^2 \right. \right.$$
$$\left. + \frac{5(1-\varepsilon)}{3+2\varepsilon} (3x^2-1) \left[-\frac{3}{2+\varepsilon} - \frac{3}{2} \frac{(\varepsilon-1)}{2+\varepsilon} x^2 \right] \right\}, \qquad (4.230)$$

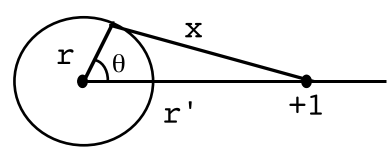
∘
∘ (much algebra)
∘

$$F_z = \frac{2a^3}{r'^5} \left(\frac{\varepsilon-1}{2+\varepsilon} \right). \qquad (4.231)$$

Can also calculate the force explicitly using the <u>external</u> field:

$$F_z = \int da (\vec{P} \cdot \hat{n})\, E_z^0 \,. \qquad (4.232)$$

⇑ <u>external</u> field



$$\Phi_0 = \frac{1}{x} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\theta), \qquad (4.233)$$

$$\Phi_0 \approx \frac{1}{r'} + \frac{r}{r'^2} \cos\theta + \frac{r^2}{r'^3} \frac{1}{2} (3\cos^2\theta - 1) + \cdots, \qquad (4.234)$$

$$\Rightarrow E_z^0 \Big|_a = -\frac{\partial}{\partial z} \Phi_0 \Big|_a \approx -\frac{1}{r'^2} - 2 \frac{a}{r'^3} \cos\theta. \qquad (4.235)$$

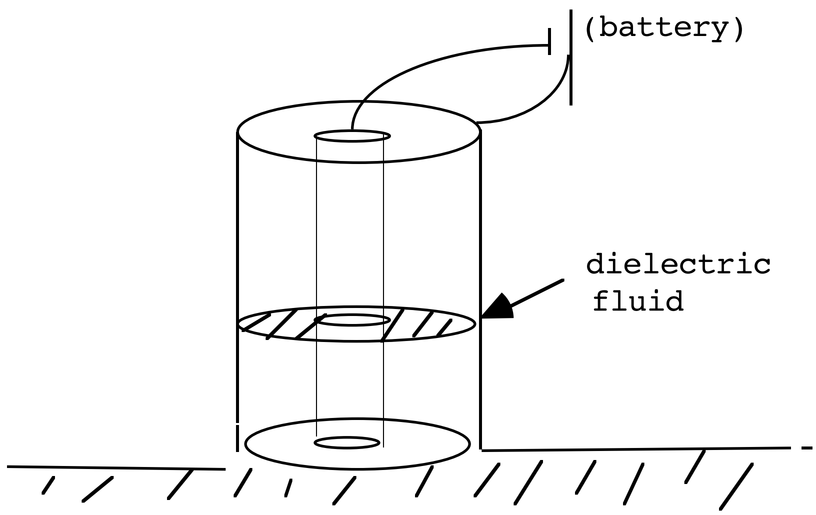
(Compare with complicated expression for $\frac{E_{2z} + E_{1z}}{2}$ above)

$$F_z \approx 2\pi a^2 \frac{1}{4\pi} \frac{a}{r'^5} \int_{-1}^{1} dx \left\{ -\frac{6(1-\varepsilon)}{2+\varepsilon} x^2 - \frac{5(1-\varepsilon)}{3+2\varepsilon} (3x^2 - 1) \right\}.$$

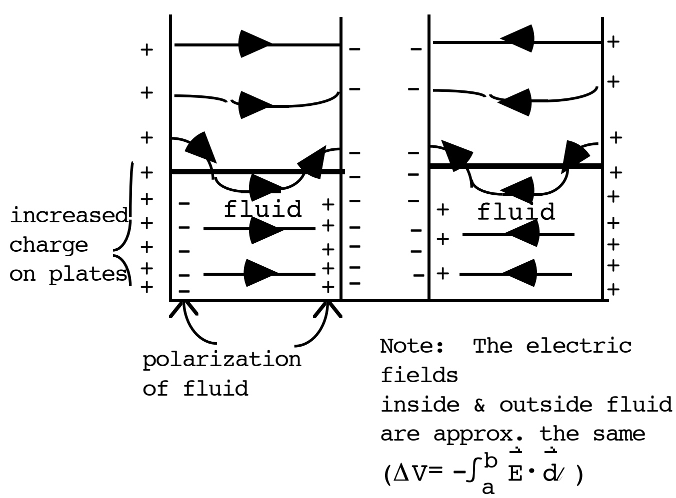
(4.236)

$$F_z \approx \frac{2a^3}{r'^5}\frac{\varepsilon - 1}{2 + \varepsilon}. \tag{4.237}$$

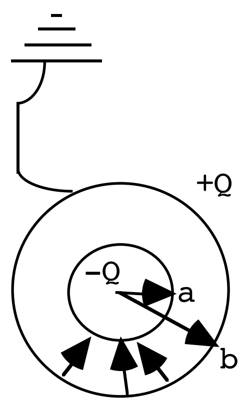
Another problem: 4.13 of Jackson



What's going on: (cross section of cylindrical tank)



Top view:



Use an energy method-easier. Inside or outside the fluid:

$$E_r \approx -\frac{V}{\ell n\,\frac{b}{a}}\frac{1}{r}. \tag{4.238}$$

⇑inward if V>0

$$4\pi\ Q_{plates} = \int \vec{D}\cdot\hat{n}\ da \Rightarrow \Delta Q_{plates} = \frac{V\Delta z}{2\ell n\ b/a}\ (\varepsilon-1), \tag{4.239}$$

$$= \frac{2\pi\ V\Delta z}{\ell n\ b/a}\ \chi. \tag{4.240}$$

battery supplies energy⇓

$$\Delta W^{battery} = V\Delta Q^{battery} = -\ V\Delta Q^{plates} = -\ \frac{2\pi\chi\Delta zV^2}{\ell n\ b/a}. \tag{4.241}$$

Add up energies:

$$\Delta W^{tot} = \Delta W^{gravity} + \Delta W^{field} + \Delta W^{battery} = 0, \tag{4.242}$$

$$\Delta W^{gravity} = \rho g(\Delta z)^2\pi(b^2-a^2), \tag{4.243}$$

$$\Delta W^{field} = \frac{1}{2}\ V\Delta Q^{plates}, \quad (= -\ \frac{1}{2}\ \Delta W^{battery}) \tag{4.244}$$

$$\Rightarrow 0 = \rho g(\Delta z)^2\ \pi(b^2-a^2) - \frac{\pi\chi\Delta zV^2}{\varepsilon\ell n\ b/a}, \tag{4.245}$$

$$\Rightarrow \chi \approx \frac{(b^2-a^2)\rho g\Delta z\ell n(b/a)}{V^2}. \tag{4.246}$$

(Many other ways of solving this.)

Problems

1. Theorem: It is always possible to find an origin such that the dipole moments, $\vec{p}$, vanish for a charge distribution whose total charge, q, is nonvanishing.

Either prove this theorem or give a counter example.

2. A charge distribution has multipole moments q, $\vec{p}$, Q_{ij} with respect to one set of coordinate axes, and moments q', $\vec{p}\,'$ and Q'_{ij} with respect to another set whose origin is located at the point $\vec{R} = (X,Y,Z)$ relative to the first. Determine explicitly the connections between the monopole, dipole and quadrupole moments in the two coordinate frames.

3. The center of a cubical volume with sides L is centered at the coordinate origin and is aligned with the x,y,z axes. Within the volume is a charge density $\rho(x,y,z) = Kx$, where K is a constant.

(a) Calculate the monopole, dipole and all the quadrupole moments of this charge distribution.

(b) What is the leading form of the electric field far away from the cube?

4.(a) Show that an alternate form for the force we found in (4.49) of the script is

$$\vec{F} = q\,\vec{E}^{(0)}(0) + (\vec{p}\cdot\vec{\nabla})\,\vec{E}^{(0)}(\vec{x})\Big|_{x=0} + \frac{1}{6}\sum_{i,j} Q_{ij}\,\frac{\partial^2\vec{E}^{(0)}(\vec{x})}{\partial x_i \partial x_j}\Big|_{x=0} + \cdots$$

(b) Use the third term on the right above to argue that atoms with a nonzero quadrupole moment density, $q_{ij}(\vec{x}) \equiv n(\vec{x})\,Q_{ij}(\vec{x})$, contribute a bulk effective charge density,

$$\rho^{Q}_{eff}(\vec{x}) = \frac{1}{6} \sum_{i,j} \frac{\partial^2 q_{ij}(\vec{x})}{\partial x_i \partial x_j} .$$

5. For a cylindrically symmetric quadupole ($Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$, all other Q_{ij}'s = 0) in an external potential, $\Phi(\vec{x})$, show that the energy of interaction between the field and the quadrupole is

$$W = \frac{1}{4} Q_{33} \frac{\partial^2 \Phi}{\partial z^2},$$

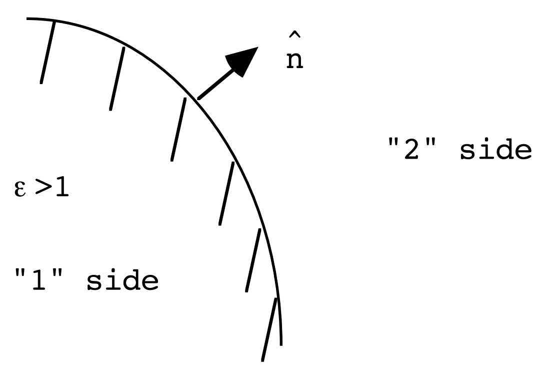
and the resulting force on the quadupole is

$$\vec{F} = \frac{1}{4} Q_{33} \frac{\partial^2 \vec{E}}{\partial z^2}.$$

6. Show that the normal force per unit area, $\vec{\mathfrak{F}} \cdot \hat{n}$, ($\hat{n}$ directed outward from the dielectric) on an arbitrary dielectric surface is given by

$$\vec{\mathfrak{F}} \cdot \hat{n} = \frac{1}{8\pi} (E^2_{2n} - E^2_{1n}),$$

where E_{2n}, E_{1n} are surface normal components of $\vec{E}_2$ and $\vec{E}_1$.



7.(a) Show that in the volume of a linear, isotropic material with dielectric constant ε, the free charge and bound charge densities are related by

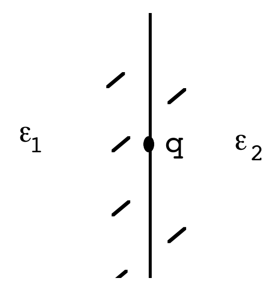
$$\rho_{bound} = \left(\frac{1-\varepsilon}{\varepsilon}\right) \rho_{free}.$$

(b) Show that (a) implies that the total bound surface charge for arbitrary geometry is given as

$$\int da\, \sigma_{bound} = \left(\frac{\varepsilon-1}{\varepsilon}\right) Q_{free},$$

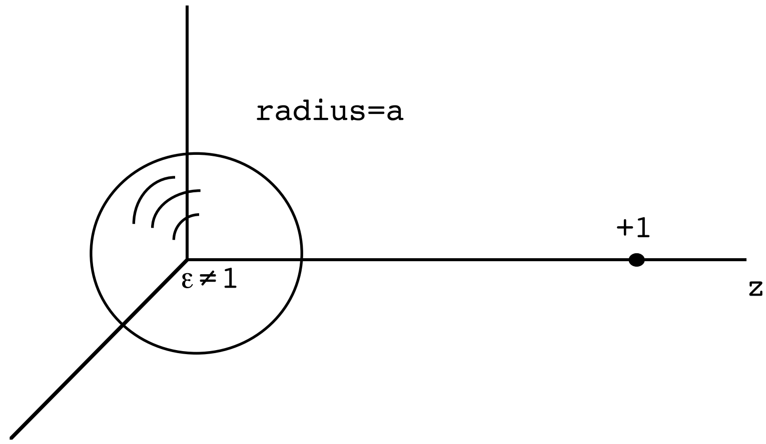
where Q_{free} is the total free charge in the volume.

8. A point free charge, q, is located directly at the plane interface of two infinte dielectric slabs, as shown. The dielectric constant on the left is ε_1, on the right, ε_2.

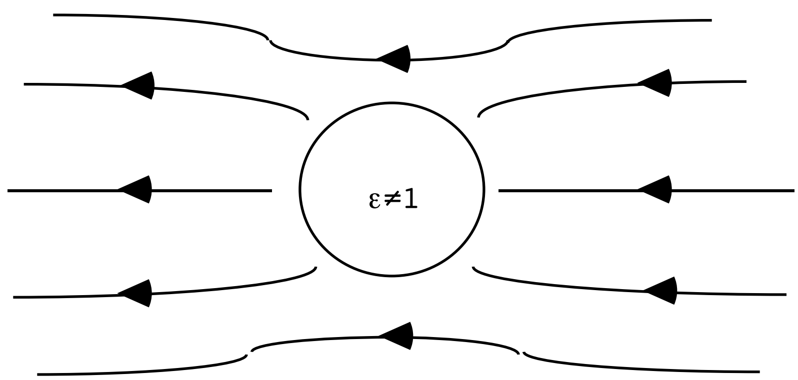


Starting from first principles, find the $\vec{E}$ and $\vec{D}$ fields everywhere.

9. From the Green function solution in the notes for a positive unit charge in the presence of a spherical dielectric,

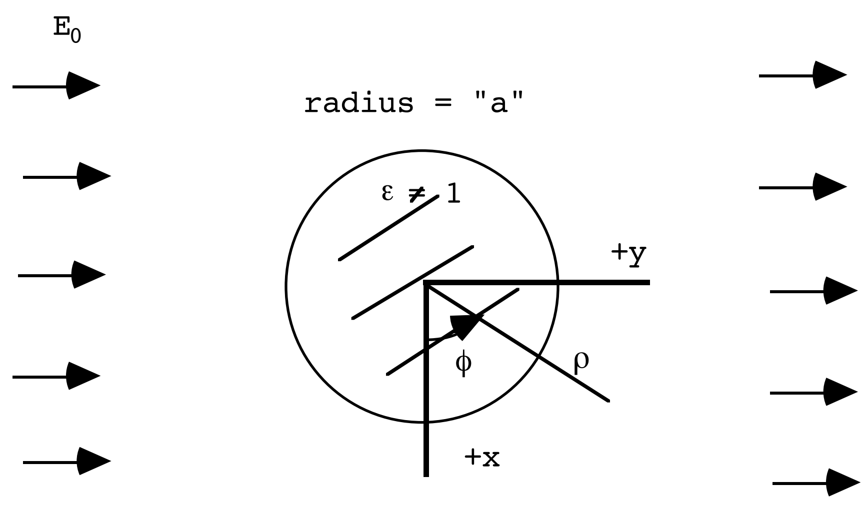


generate the potential, both inside and outside, for a spherical dielectric placed in an initially uniform electric field



10. Write down the Green function for a dielectric sphere when $r' < a$ (The unit charge is inside the sphere. Part of the solution can be gotten directly by using the symmetry $G_D(\vec{x}', \vec{x}) = G_D(\vec{x}, \vec{x}')$ and the solution $r' > a$ given in the notes.) In the limit $r \gg a$, identify an effective dipole moment of the system and evaluate the total polarization charge on the surface.

11. An infinitely long cylinder of dielectric material is placed in an initially uniform electric field of magnitude E_0 pointing in the +y direction as shown.

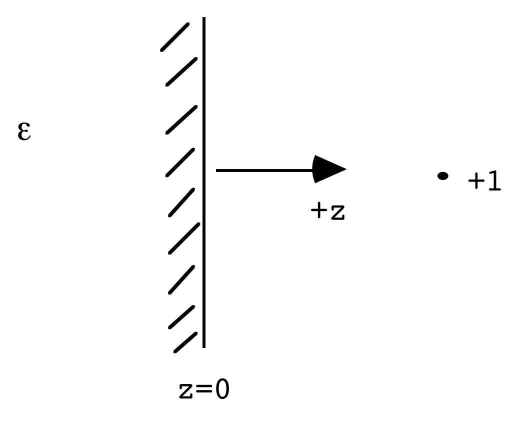


Show that the potentials inside and outside the cylinder are given by

$$\Phi_{in} = -\frac{2E_0}{\varepsilon+1}\rho \sin\phi,$$

$$\Phi_{out} = -E_0\rho\sin\phi + E_0\frac{\varepsilon-1}{\varepsilon+1}a^2\rho\sin\phi\ .$$

12. Calculate the force on the half-infinite dielectric plane,



due to the presence of the unit charge at z'>0. Do it:

(a) By evaluating the force between the unit charge and the image charge.

(b) By using an energy method explained in class:

$$\Delta W = \frac{1}{2}\int d^3x d^3x' \rho(\vec{x})[G_D(\vec{x},\vec{x}')-G_D^0(\vec{x},\vec{x}')]\rho(\vec{x}').$$

(c) By explicitly evaluating the force on the induced surface charge density. (Be careful <u>which</u> electric field you use, as there is a discontinuity in $\vec{E}$ at z=0.)

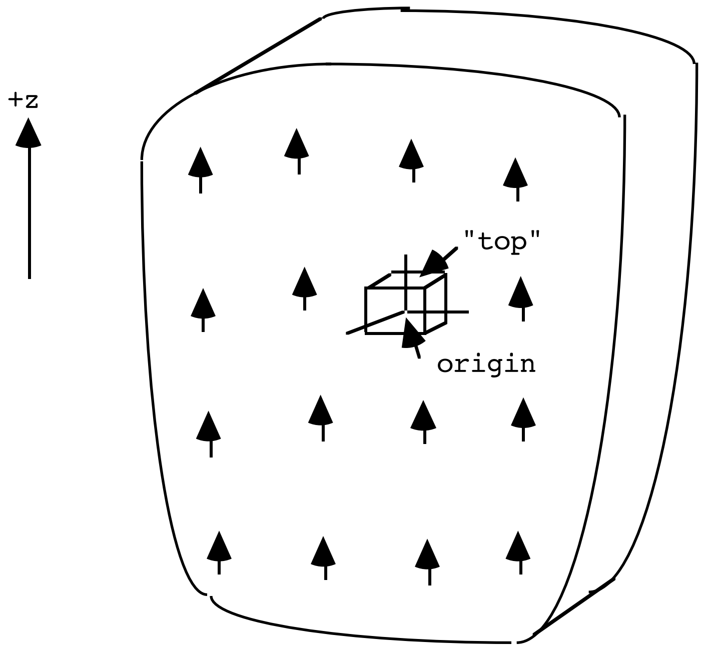
13. A piece of linear dielectric material is brought slowly into a region where an initial electric field, $\vec{E}_0(\vec{x})$, has already been established. The change in energy of the system is

$$W - W_0 = -\frac{1}{2}\int_V d^3x\ \vec{P}(\vec{x})\cdot\vec{E}_0(\vec{x}) ,$$

where $\vec{P}(\vec{x})$ is the polarization vector and the integration is only over the volume, V, of the introduced dielectric. Using the energy expression above and the Green's function for this situation, find the force between a dielectric sphere of radius "a" and a charge, Q, located a distance, d, from the center of the sphere when d >> R. [This is worked out three other ways in Ch.4!]

14. Finish the integration and algebra leading from (4.239) of the script to (4.240).

15. There is a cubic hole of vacuum within a piece of material which has a uniform electric polarization, $\vec{P} = P_0\hat{z}$:



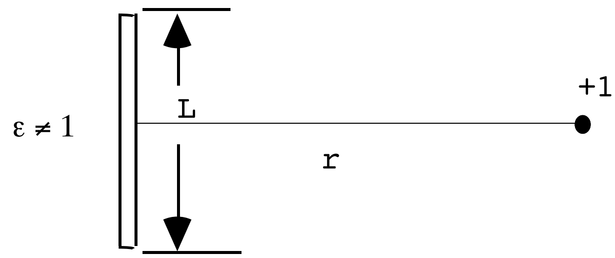
(a) Taking the origin of coordinates at the center of the cubic hole, show that the electric field at the origin can be expressed as

$$\vec{E}(0) = 2P_0 \int_{\text{"top"}} \frac{\vec{x}'da'}{r'^3},$$

where the integration is over the "top" of the cube.
(b) Using symmetry and the concept of solid angle, argue that this expression reduces to

$$\vec{E}(0) = \frac{4\pi}{3} P_0\hat{z}.$$

16. Find the approximate force (attraction, repulsion?) between a dielectric rod of length L and radius a (L>>a) and a positive unit point charge located a distance r>>L,a from the rod. The point charge is located perpendicular to the rod's axis, on the rod's midpoint plane:



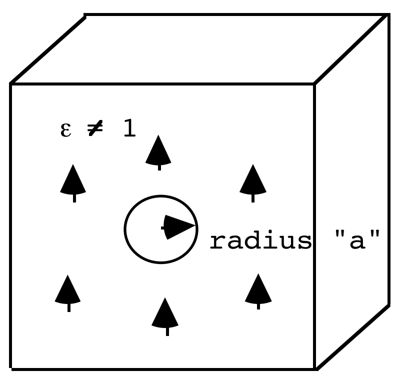
[Hint: The point charge's $\vec{E}$ field at the rod's position will be approximately uniform.]

17.(a) Given a material with a space dependent polarization, $\vec{P}$, show that the potential at an arbitrary point is given by

$$\Phi(\vec{x}) = \int ds' \frac{\vec{P}\cdot\hat{n}'}{|\vec{x}-\vec{x}'|} - \int d^3x' \frac{\vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}.$$

[Note: The volume V' is considered not to include the surface, S'.]

(b) Apply part (a) to a spherical bubble of vacuum with radius "a" enclosed in a semi-infinite dielectric slab.

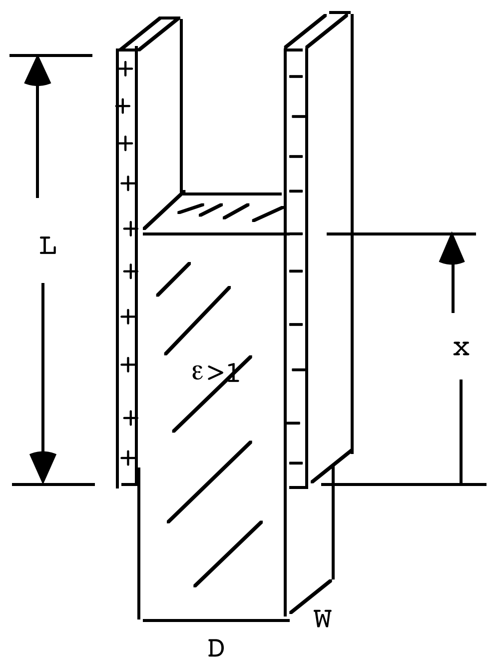


$P = P_0 \hat{k}$

Assume the slab has a uniform polarization, $\vec{P} = P_0\hat{z}$, outside the sphere. Show that the electric field everywhere inside the sphere is given by

$$\vec{E} = \frac{4\pi}{3} P_0\hat{z}.$$

18.(a) Two parallel conducting plates of a capacitor of length L and width W are separated by a distance D. The region between the plates is filled to a distance x with a dielectric material with constant ε. If the plates are maintained at a constant potential V by connection to a battery, calculate the force, F_x, on the dielectric block. Neglect edge effects. Is the block pulled in or pushed out of the capacitor?



(b) The dielectric block has been withdrawn and a fixed charge ± Q has been placed on the plates. The magnitude of this charge is given by $\frac{V}{D} = \frac{4\pi Q}{LW}$ so that the potential in (a) is established when the dielectric block is not yet inserted. The block is reinserted a distance x so that it partially fills the space between the plates. Again neglecting edge effects, calculate the force, F_x, on the block.

(c) Now imagine that the inserted dielectric material is wafer-thin (D<<W). Can you provide an approximate solution for this case as well?