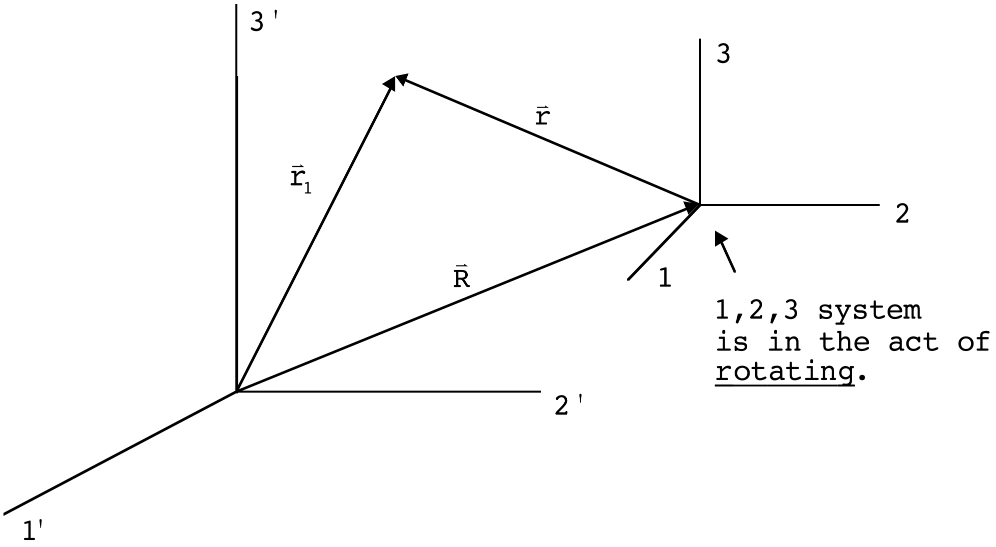
Chapter 10

Finite displacements and rotations

Need transformation connecting fixed and moving (body) noninertial frames. Picture:



Going to let the 1,2,3, system "evolve" an relation infinitesimal amount in time, dt

⇒

We are going to consider an infinitesimal of the $\vec{r}$ axes, but finite displacements, $\vec{R}$

(1,1′), (2,2′), (3,3′) axes coincide in direction at some instant of time, t. Clearly, the reason for this is to describe, for example, motions relative to the Earth, which is noninertial. This is all done for convenience, not any real physics reason. In fact, strictly speaking, this chapter has zero physics content! This does not mean, however, that these considerations are not useful or convenient.

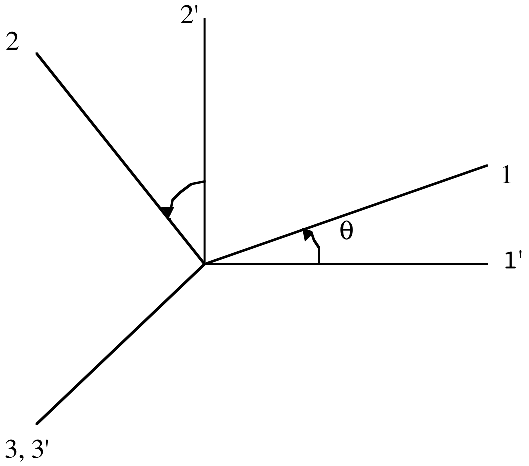
There are two types of transformations that will be involved:

1. Displacement: $\vec{r} = \vec{r}\,' - \vec{R}$. (see above figure)

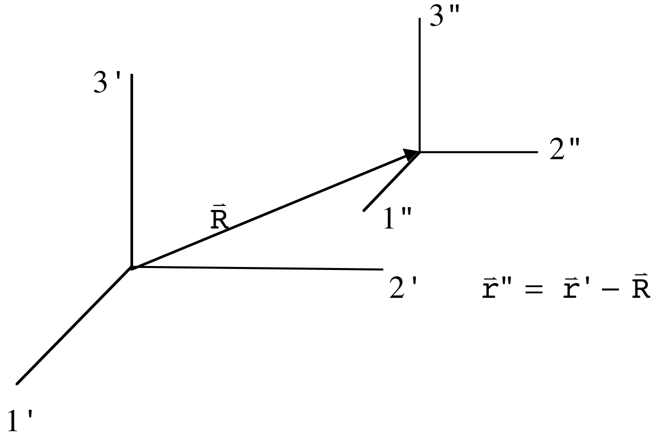
2. Rotation: (a “generic” passive rotation)

$$r_i = \sum_j \underset{\overset{\uparrow}{\text{passive}}}{\lambda_{ij}} r_j' .$$

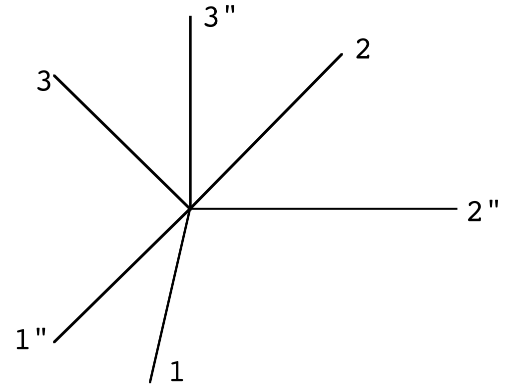
Picture: (specialized to rotation about 3,3′)



Put them together. Step 1: displacement. At the end of step 1:



Now wish to rotate. Rotate the $\vec{r}''$ axes:



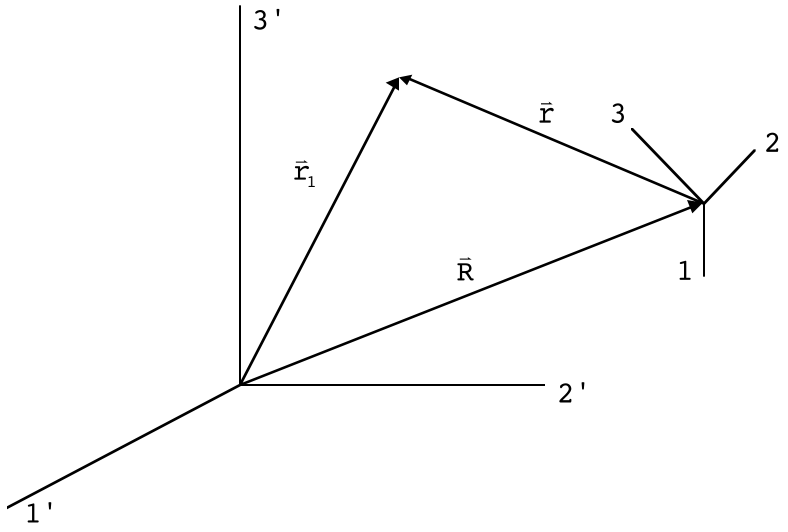
← A rotation about an arbitrary axis

$$r_i = \sum \lambda_{ij} r_j'' . \tag{10.1}$$

But $r''_j = r'_j - R_j$,so

$$r_i = \sum_j \lambda_{ij} \left(r_j' - R_j \right) . \tag{10.2}$$

Represents:



In matrix notation, this is

$$r = \lambda(r' - R). \qquad (10.3)$$

It's inverse is

$$r' - R = \lambda^{-1}r = \lambda^{T}r. \qquad (10.4)$$

We really need only the relationships between $\vec{r}$ and $\vec{r}\,'$ for an <u>infinitesimal</u> rotation. What is λ for such a situation? Remember

$$\lambda^{-1}\lambda = 1 \quad , \quad \lambda^{-1} = \lambda^{T},$$

$$\sum_{i} \lambda_{ij}\,\lambda_{ik} = \delta_{jk}\,. \qquad (10.5)$$

Assume

$$\lambda_{ij} = \delta_{ij} + \underset{\substack{\uparrow \\ \text{change in } \lambda \text{ necessary} \\ \text{to represent an infinitesimal} \\ \text{passive rotation}}}{\delta\lambda_{ij}}\,. \qquad (10.6)$$

Substitute above:

$$\sum \left(\delta_{ij} + \delta\lambda_{ij}\right)\left(\delta_{ik} + \delta_{ik}\right) = \delta_{jk}\,. \qquad (10.7)$$

$0^{\underline{th}}$ order: $\sum_{i} \delta_{ij}\delta_{ik} = \delta_{ik}\,. \quad \surd$

$1^{\underline{st}}$ order: $\sum_{i} \left(\delta_{ij}\,\delta\lambda_{ik} + \delta\lambda_{ij}\delta_{ik}\right) = 0\,,$

$$\Rightarrow \delta\lambda_{ij} + \delta\lambda_{kj} = 0\,.$$

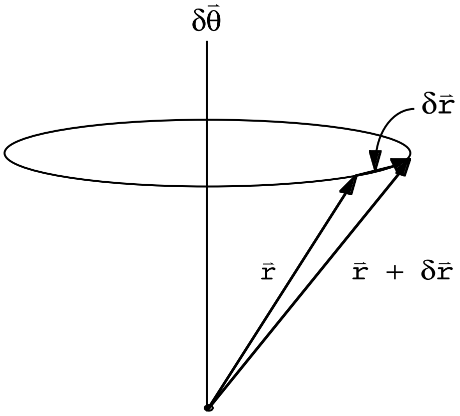
$$\text{or}\ \ \delta\lambda_{jk} = -\delta\lambda_{kj}\,,\ \ \underline{\text{antisymmetric}} \qquad (10.8)$$

Also implies there are only 3 <u>independent</u> elements:

$$\delta\lambda_{ij} = \begin{pmatrix} 0 & (1) & (2) \\ -(1) & 0 & (3) \\ -(2) & -(3) & 0 \end{pmatrix} \qquad (1),(2),(3)\ \text{arbitrary elements}$$

Instantaneous relations for velocity, acceleration

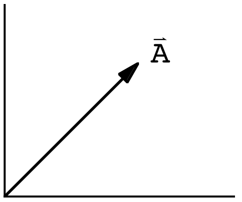
Now go back to Ch.1. Representation of an infinitesimal <u>active</u> rotation on a vector:



$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}\,,$$

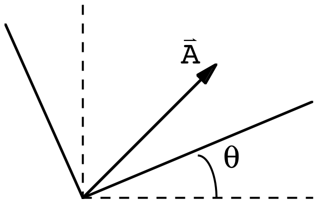
$$\text{or}\ \ \delta r_i = \sum_{j,k} \varepsilon_{ijk}\,\delta\theta_j\, r_k\,. \qquad (10.9)$$

Now, any active rotation is given by a passive rotation in the opposite direction. Start:

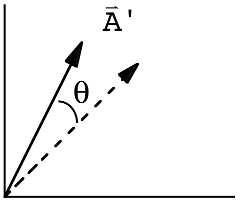


Passive, θ

(λ_{ij})



Active, θ:



↑

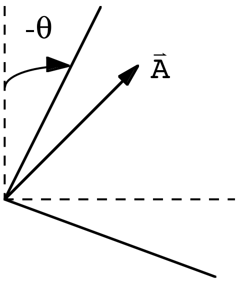
Passive, -θ:

$\left(\lambda_{ij}^{-1} = \lambda_{ij}^{T}\right)$

have the same

<u>components</u>

↓



This gives us two ways of specifying the effects of an active rotation:

<u>Way 1</u>: $\delta r_i = \sum_{j,k} \varepsilon_{ijk} \delta\theta_j r_k$. (active rotation)

<u>Way 2</u>: $\delta r_i = \underset{\text{(final)}}{\sum_j \lambda^T_{ij} r_j} - \underset{\text{(initial)}}{r_i}$, (passive inverse rotation)

$$\text{or} \quad \delta r_i = \sum_j \delta\lambda^T_{ij} r_j . \tag{10.10}$$

Must be the same:

$$\sum_k \delta\lambda^T_{ik} r_k = \sum_{j,k} \varepsilon_{ijk} \delta\theta_j r_k ,$$

$$\Rightarrow \delta\lambda^T_{ik} = \sum_j \varepsilon_{ijk} \delta\theta_j ,$$

$$\text{or} \quad \Rightarrow \delta\lambda_{ki} = \sum_j \varepsilon_{kij} \delta\theta_j . \tag{10.11}$$

Notice, as expected, $\delta\lambda_{ki}$ is antisymmetric in k,i.

Our relationship between the primed and unprimed coordinates are again,

$$r'_i - R_i = \sum_j \lambda^T_{ij} r_j . \tag{10.12}$$

Consider an infinitesimal change on both sides:

$$\delta r'_i - \delta R_i = \underset{①}{\sum_j \delta\lambda^T_{ij} r_j} + \underset{②}{\sum_j \lambda^T_{ij} \delta r_j} . \tag{10.13}$$

Why 2 terms on right side? ① arises from the rotation of the noninertial axes while ② is due to the independent motion of the particle relative to the $\bar{r}$ axes. Found earlier,

$$\delta\lambda^T_{ij} = \sum_k \varepsilon_{ikj}\, \delta\theta_k \, .$$

Of course also

$$\lambda^T_{ij} = \delta_{ij} + \delta\lambda^T_{ij}, \tag{10.14}$$

so

$$\delta r_i' - \delta R_i = \sum_{k,j} \varepsilon_{ikj}\delta\theta_k r_j + \sum_j \Big(\delta_{ij} + \underset{\substack{\uparrow \\ \text{can drop (2}^{\text{nd}}\text{ order} \\ \text{in small quantities)}}}{\delta\lambda^T_{ij}}\Big)\, \delta r_j \, . \tag{10.15}$$

Thus

$$\delta r_i' - \delta R_i = \sum_{k,j} \varepsilon_{ikj}\, \delta\theta_k r_j + \delta r_i \, . \tag{10.16}$$

Divide by δt $\left(\dfrac{\delta Q}{\delta t} \equiv \dfrac{dQ}{dt}\right)$:

$$\Rightarrow \quad \frac{dr_i'}{dt} - \frac{dR_i}{dt} = \frac{dr_i}{dt} + \sum_{k,j} \varepsilon_{ikj}\, \frac{d\theta_k}{dt}\, r_j \, . \tag{10.17}$$

But

$$\omega_k \equiv \frac{d\theta_k}{dt}, \tag{10.18}$$

so

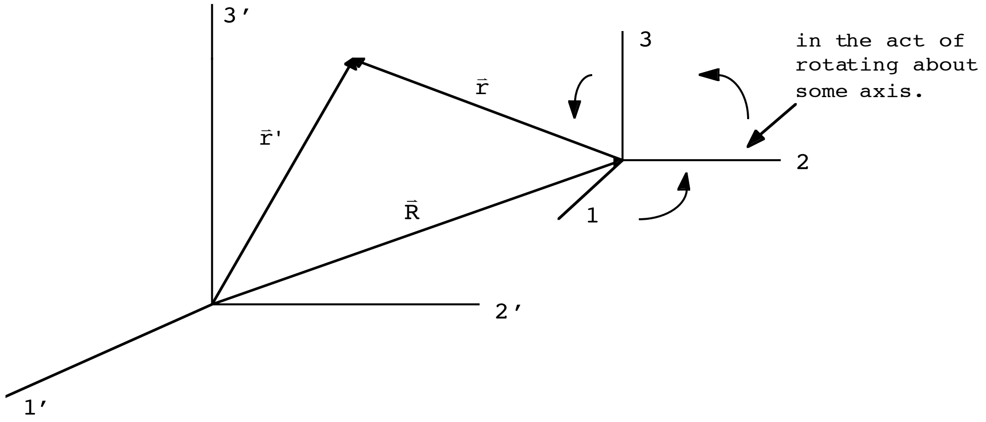
$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{R}}{dt} + \vec{\omega} \times \vec{r}\,, \qquad (10.19)$$

or

$$\vec{v}_f = \vec{v}_r + \vec{V} + \vec{\omega} \times \vec{r}\,. \qquad (10.20)$$

$\vec{v}_f \left(= \frac{d\vec{r}\,'}{dt}\right)$ is the velocity of the particle relative to the fixed frame. (It is the velocity as measured by an observer at rest in $\vec{r}\,'$) Must be a constant in magnitude and direction if the particle has no <u>rea</u>l forces acting on it. $\vec{v}_r$ is the velocity relative to the moving frame, whose axes coincide with the fixed aces at the given instant in time.

Picture:



Now do the second variation:

$$\delta r_i' - \delta R_i = \sum_j \delta\, \lambda^T_{ij}\, r_j + \sum_j \lambda^T_{ij}\, \delta r_j\,,$$

$$\Rightarrow \delta^2 r_i' - \delta^2 R_i = \sum \delta^2 \lambda^T_{ij} r_j + 2\sum_j \delta\lambda^T_{ij} \delta r_j + \sum_j \underset{\substack{\uparrow \\ \text{can replace by } \delta_{ij}}}{\lambda^T_{ij}} \delta^2 r_j . \qquad (10.21)$$

Before, we compared

$$\delta r_i = \varepsilon_{ijk}, \qquad (10.22)$$

to

$$\delta r_i = \sum_k \delta\lambda^T_{ik} r_k , \qquad (10.23)$$

and got

$$\delta\lambda^T_{ik} = \sum_j \varepsilon_{ijk} \delta\theta_j . \qquad (10.24)$$

Now

$$\delta^2 r_i = \sum_{j,k} \varepsilon_{ijk} \, \delta\left(\delta\theta_j r_k\right)$$

$$= \sum_{j,k} \varepsilon_{ijk} \left(\delta^2\theta_j r_k + \delta\theta_j \, r_k\right) . \qquad (10.25)$$

But

$$\delta r_k = \sum_{\ell,m} \varepsilon_{k\ell m} \, \delta\theta_\ell r_m , \qquad (10.26)$$

so

$$\delta^2 r_i = \sum_{j,k} \varepsilon_{ijk} \left(\delta^2\theta_j r_k + \underset{\text{(let } m\to k)}{\sum_{\ell,m} \varepsilon_{k\ell m} \delta\theta_j \delta\theta_\ell r_m}\right),$$

$$= \sum_{j,k} \varepsilon_{ijk}\, \delta^2\theta_j r_k + \sum_{\substack{j,k,\\ \ell,m}} \varepsilon_{ijm}\, \varepsilon_{m\ell k} \delta\theta_j\, \delta\theta_\ell r_k \,. \qquad (10.27)$$

On the other hand, compare this to

$$\delta^2 r_i = \sum_k \delta^2\lambda^T_{ik}\, r_k \,. \qquad (10.28)$$

Identify:

$$\delta^2\lambda^T_{ik} = \sum_j \varepsilon_{ijk}\, \delta^2\theta_j + \sum_{j,\ell,m} \varepsilon_{ijm}\varepsilon_{m\ell k}\, \delta\theta_j \delta\theta_\ell \,. \qquad (10.29)$$

$$\uparrow \qquad \uparrow\;\uparrow\uparrow \qquad \uparrow\;\;\uparrow \qquad \uparrow \qquad \uparrow\;\uparrow$$

$$(j) \quad (k)(k)(j)\;\;(k)\;\;(k) \quad\;\; (k) \quad (j)(k)$$

Put it all back together:

$$\delta^2 r'_i - \delta^2 R_i = \sum_{j,k} \varepsilon_{ikj}\delta^2\theta_k r_j + \sum_{\substack{j,k,\\ \ell,m}} \varepsilon_{ikm}\varepsilon_{m\ell j}\delta\theta_k\delta\theta_\ell r_j$$

$$+ 2\sum_{j,k} \varepsilon_{ikj}\delta\theta_k\delta r_j + \delta^2 r_i \,. \qquad (10.30)$$

Divide by dt^2 $\left(\dfrac{\delta^2 Q}{\delta t^2} = \dfrac{d^2 Q}{dt^2}\right)$:

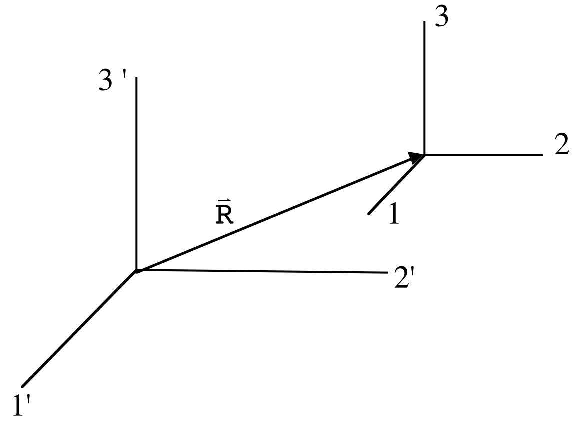
$$\Rightarrow \frac{d^2 r'_i}{dt^2} - \frac{d^2 R_i}{dt^2} = \sum_{k,j} \varepsilon_{ikj} \frac{d^2 Q_k}{dt^2} + \frac{d^2 r_i}{dt^2}$$

$$+ \sum_{k,m} \varepsilon_{ikm} \frac{d\theta_k}{dt} \left(\sum_{\ell,j} \varepsilon_{m\ell j} \frac{d\theta_\ell}{dt} r_j \right) + 2\sum_{j,k} \varepsilon_{ikj} \frac{d\theta_k}{dt} \frac{dr_j}{dt} \,. \qquad (10.31)$$

Identify $\omega_i = \dfrac{d\theta_i}{dt}$, $\dot{\omega}_i = \dfrac{d^2\theta_i}{dt^2}$ and write in vector notation:

$$\ddot{\bar{r}}' = \ddot{\bar{R}} + \ddot{\bar{r}} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + 2\bar{\omega} \times \dot{\bar{r}} \,. \qquad (10.32)$$

$\ddot{\vec{R}}$ represents the acceleration of the <u>origin</u> of the $\vec{r}$ coordinate system relative to the $\vec{r}\,'$ origin. Will be zero if we consider uniform motion. Picture:



Also have $\dot{\vec{\omega}} = 0$ in the case of constant angular velocity (magnitude and direction).

Useful Earth coordinate choices

Write in the <u>inertial system</u>

$$\vec{F} = m\ddot{\vec{r}}, \tag{10.33}$$

Write in the <u>noninertial system</u>

$$\vec{F}_{eff} = m\ddot{\vec{r}}\,. \tag{10.34}$$

which gives

$$\vec{F}_{eff} = m\left(\ddot{\vec{r}}\,' - \ddot{\vec{R}} - \dot{\vec{\omega}} \times \vec{r} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \dot{\vec{r}}\right),$$

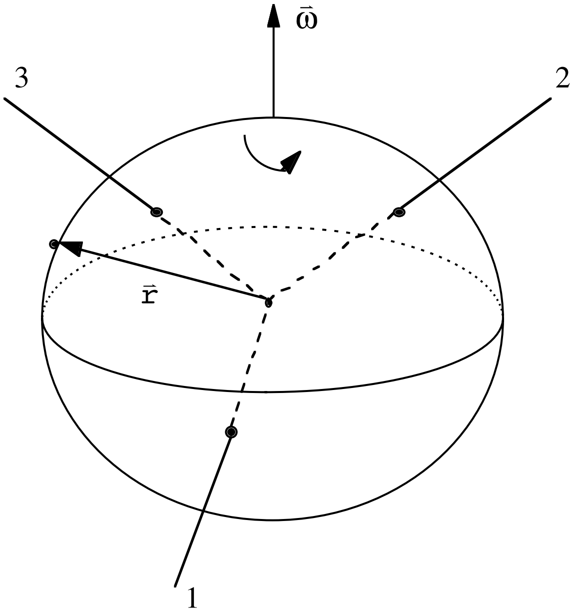
or

$$\vec{F}_{eff} = \vec{F} - m\left(\ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} - \underset{\substack{\uparrow \\ \text{“centifugal”}}}{\vec{\omega} \times (\vec{\omega} \times \vec{r})} + \underset{\substack{\uparrow \\ \text{“Coriolis”}}}{2\vec{\omega} \times \dot{\vec{r}}}\right). \quad (10.35)$$

<u>Remember</u>:

Deflection is to the right in northern hemishpere and to the left in the southern (relative to the initial direction) → (leads to deflection of air masses in a counter-clockwise direction in the northern hemisphere.)

For the Earth, we often choose:



For this choice $\dot{\vec{\omega}} = \ddot{\vec{R}} = 0$,so

$$\vec{F}_{\text{eff}} = \vec{F} - m\left(\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \dot{\vec{r}}\right). \tag{10.36}$$

Sum up. Because instantaneously our axes coincide in direction, we have

$$\vec{r}' = \vec{r} + \vec{R}\,,$$

$$\dot{\vec{r}}' = \dot{\vec{r}} + \dot{\vec{R}} + \vec{\omega} \times \vec{r}\,,$$

$$\ddot{\vec{r}}' = \ddot{\vec{r}} + \dot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}).$$

How come we do not get $\dot{\vec{r}}'$ as simply the time derivative of the expression $\vec{r}'$ for example? Because $\vec{r}'$ and $\vec{r}$ are referred to different axes which are rotating as well as moving with velocity $\dot{\vec{R}}$ with respect to one another. There is another way of viewing this process more in line with the book's derivation. When a change in $\vec{r}$ is considered, calculated with respect to the moving axes, we have

$$\underset{\substack{\uparrow\\ \text{fixed}}}{(\delta\vec{r})_f} = \underset{\substack{\uparrow\\ \text{rotating}}}{(\delta\vec{r})_r} - (\delta\vec{r})_{\text{passive}}\,, \tag{10.37}$$

From before:

$$(\delta\vec{r})_{\text{passive}} = -(\delta\vec{r})_{\text{active}} = -\overset{\substack{\text{understood in fixed frame}\\ \downarrow}}{\delta\vec{\theta}} \times \vec{r}\,,$$

$$\Rightarrow (\delta\vec{r})_f = (\delta\vec{r})_r + \delta\vec{\theta} \times \vec{r}\,. \tag{10.38}$$

Thus

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}. \qquad \left(\vec{\omega} = \frac{d\vec{\theta}}{dt}\right) \tag{10.39}$$

This is true for any vector, not just $\vec{r}$, as long as the fixed and rotating axes instantaneously coincide in direction. Including the effect of translation of the coordinate origin, this now gives

$$\left(\frac{d\vec{r}\,'}{dt}\right)_f = \underset{\substack{\uparrow \\ \text{make} \\ \text{replacement}}}{\left(\frac{d\vec{r}}{dt}\right)_f} + \underset{\substack{\uparrow \\ \text{understood in fixed} \\ \text{frame}}}{\frac{d\vec{R}}{dt}}$$

$$\Rightarrow \dot{\vec{r}}\,' = \dot{\vec{r}} + \dot{\vec{R}} + \vec{\omega} \times \vec{r} \ \text{ as before.}$$

Apply the same reasoning to get $\ddot{\vec{r}}\,'$:

$$\left(\frac{d\dot{\vec{r}}\,'}{dt}\right)_f = \underline{\underline{\left(\frac{d\dot{\vec{r}}}{dt}\right)_f}} + \frac{d\dot{\vec{R}}}{dt} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \overset{\substack{\text{see above} \\ \downarrow}}{\underline{\underline{\left(\frac{d\vec{r}}{dt}\right)_f}}}\,. \tag{10.40}$$

However

$$\left(\frac{d\dot{\vec{r}}}{dt}\right)_f = \left(\frac{d\dot{\vec{r}}}{dt}\right)_r + \vec{\omega} \times \dot{\vec{r}}\,, \tag{10.41}$$

$$\Rightarrow \ddot{\vec{r}}\,' = \ddot{\vec{r}} + \vec{\omega} \times \dot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\dot{\vec{r}} + \vec{\omega} \times \vec{r}\right),$$

$$\Rightarrow \ddot{\vec{r}}\,' = \ddot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right).$$

This is also as before.

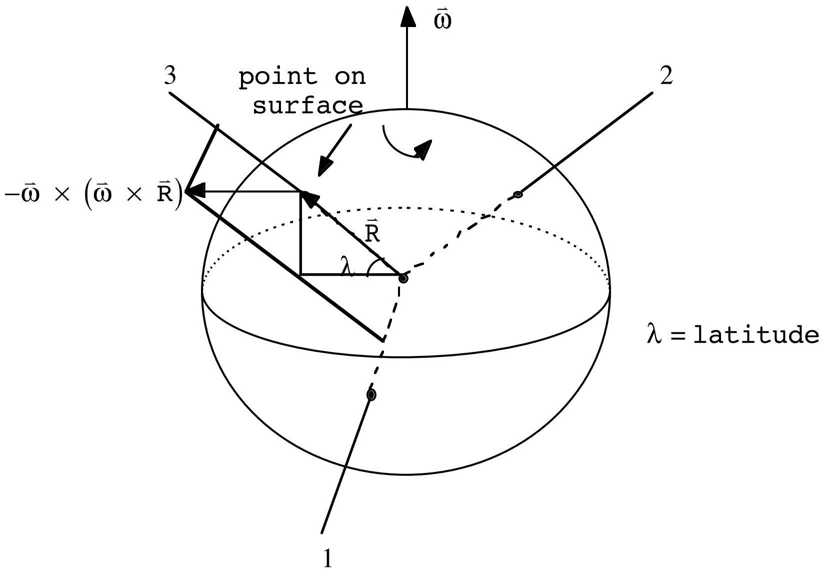
Get back to $\vec{F}_{\text{eff}}$ on Earth (static case, $\dot{\vec{r}} = 0$)

$$\vec{F}_{\text{eff}} = \vec{F} - m\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right). \tag{10.42}$$

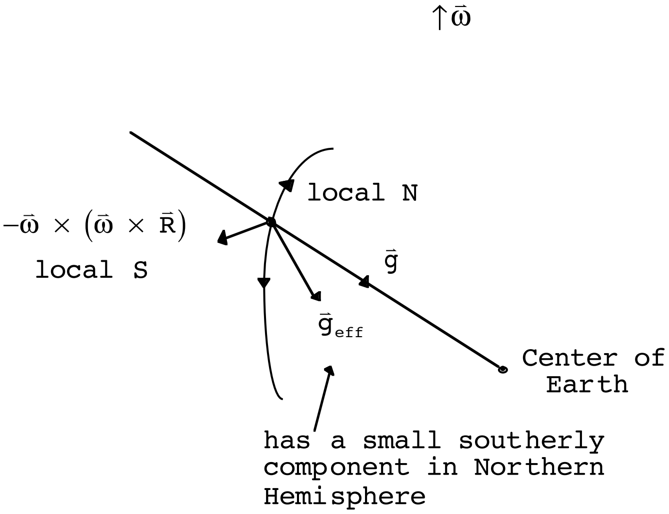
This means that the <u>effective</u> acceleration due to gravity and the Earth is given by

$$\vec{g}_{\text{eff}} = \underset{\substack{\uparrow \\ \text{gravity} \\ \text{only}}}{\vec{g}} - \vec{\omega} \times \left(\vec{\omega} \times \underset{\substack{\uparrow \\ \vec{R}\text{ points from center} \\ \text{of Earth to surface}}}{\vec{R}}\right). \tag{10.43}$$

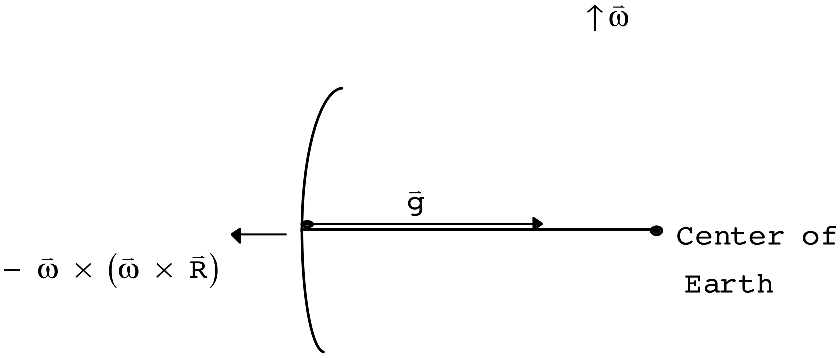
Picture:



As we can see, $-\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right)$ has components along 3 and 1 axes. Another view (corresponds to the 13 plane above):



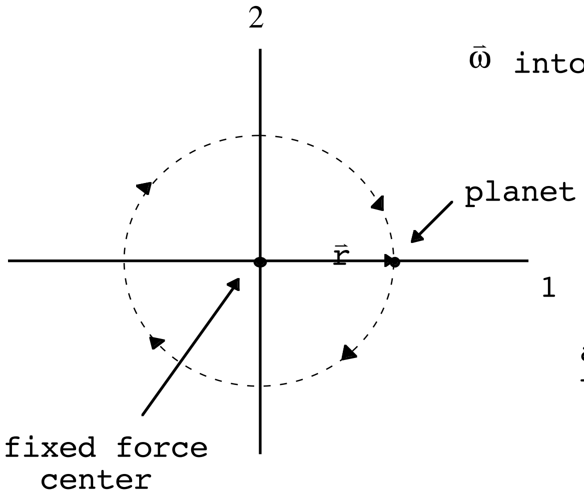
At equator, it’s particularly simple:



You will find the angular deviation of a plumb line from the true vertical caused by this effect in a problem.

<u>Example</u>: central force problem.

10.18



$\vec{\omega}$ into page ($\dot{\vec{\omega}} \neq 0$ in general)

1,2 axes rotate so that planet is always located along the "1" axis.

$$\vec{F} = -\frac{dU}{dr}\hat{e}_1 \ , \ \vec{R} = 0$$

$$m\ddot{\vec{r}} = \vec{F} - m\left(\cancelto{0}{\ddot{\vec{R}}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\omega \times \dot{\vec{r}}\right),$$

$$\Rightarrow \begin{cases} \vec{\omega} = -\omega\hat{e}_3 \ , \ \dot{\vec{\omega}} = -\dot{\omega}\hat{e}_3, \\ \vec{r} = r\hat{e}_1 \ , \dot{\vec{r}} = \dot{r}\hat{e}_1 \ , \quad \left(\dot{\hat{e}}_1 = 0 \text{ in rotating frame}\right) \\ \ddot{\vec{r}} = \ddot{r}\hat{e}_1, \\ \dot{\vec{\omega}} \times \vec{r} = -\dot{\omega}r\hat{e}_2, \\ 2\vec{\omega} \times \dot{\vec{r}} = -2\omega\dot{r}\hat{e}_2, \\ \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2 r\hat{e}_1. \end{cases}$$

"2" components:

$$0 = -m\,(-\dot{\omega}r - 2\omega\dot{r})\,.$$

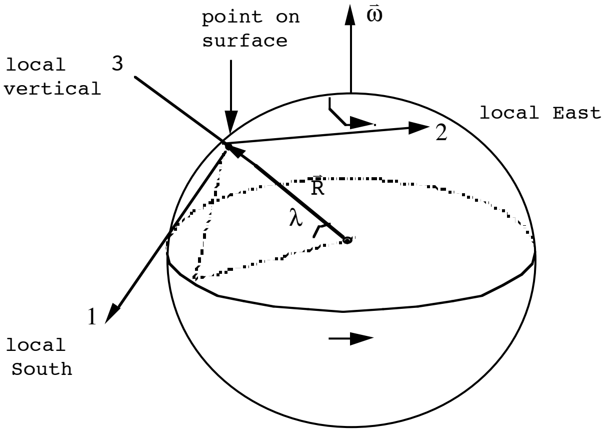
integrate ⇒ $r^2\omega = \frac{\ell}{m}$, $\omega = \frac{\ell}{mr^2}$, as before (conservation of angular momentum). "1" components:

$$m\ddot{r} = -\frac{dU}{dr} - m\left(-\omega^2 r\right),$$

$$\Rightarrow \ddot{r} - \omega^2 r = -\frac{dU}{dr}.$$

or $\ddot{r} - \dfrac{\ell^2}{m^2 r^3} = -\dfrac{dU}{dr}$, also as before.

There are other choices of noninertial coordinate systems which can simply motion problems near the Earth's surface. Consider the choice:



For this choice we have

$$\vec{F}_{\text{eff}} = \vec{F} - m\left(\ddot{\vec{R}} + \overset{0}{\cancel{\dot{\vec{\omega}}}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \dot{\vec{r}}\right). \quad (10.44)$$

We need to compute $\ddot{\vec{R}}$. Using the general formula

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A},$$

for any $\vec{A}$, we get

$$\left(\dot{\vec{R}}\right)_f = \vec{\omega} \times \vec{R} ,$$

since $\left(\dfrac{d\vec{R}}{dt}\right)_r = 0$. Again applying the above general equation, we get

$$\left(\frac{d^2\vec{R}}{dt^2}\right)_f = \vec{\omega} \times \left(\frac{d\vec{R}}{dt}\right)_f = \vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) .$$

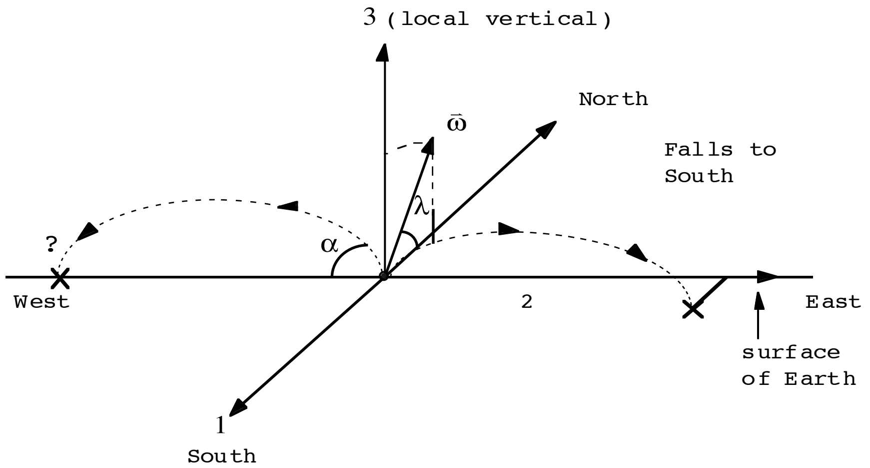
Notice that we now get

$$\vec{F}_{eff} = \vec{F} - m\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) - \overset{\substack{\text{usually small} \\ \downarrow}}{m\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)} - 2m\vec{\omega} \times \dot{\vec{r}} . \qquad (10.45)$$

* [The fact that $\ddot{\vec{R}} \neq 0$ and can not be neglected contradicts Marion's statement in the footnote on p.346 of the text.] *

Deflection of projectiles near Earth's surface

Using this $\vec{F}_{eff}$, we can now investigate the motion of projectiles near the Earth's surface. Qualitative situation:



$$\omega = 7.29 \times 10^{-5}\ \frac{\text{radians}}{\text{second}}$$

$$\vec{\omega} = -\omega \cos\lambda\, \hat{e}_1 + \omega \sin\lambda\, \hat{e}_3 .$$

When projected initially to the East, the particle trajectory will gain a Southerly component due to 1. centrigulal force and 2. Coriolis force. However, if initially projected West, the force will now deflect the particle to the North. The interesting question is: which deflection will be bigger when projected to the West?

Take

$$\vec{F}_{\text{eff}} \simeq \vec{F} - m\left(\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) + 2\vec{\omega} \times \dot{\vec{r}}\right),$$

To $0^{\underline{\text{th}}}$ order, we have

$$\vec{R} = \underset{\substack{\uparrow \\ \text{radius of the Earth}}}{R_E}\hat{e}_3 ,$$

$$\Rightarrow \begin{cases} r_1(t) = 0, \\ r_2(t) = -\left(v_0 \cos\alpha\right)t \text{ (to the West if } v_0 > 0\text{)}, \\ r_3(t) = -\dfrac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t \text{ (does not include effect of rotation).} \end{cases}$$

10.22

Work out $-\vec{\omega} \times (\vec{\omega} \times \vec{R})$ term:

$$\vec{\omega} \times \vec{R} = \left(-\omega \cos \lambda \hat{e}_1 + \omega \sin \lambda \hat{e}_3\right) \times \left(R_E \hat{e}_3\right),$$

$$= \omega R_E \cos \lambda \hat{e}_2 ,$$

$$\Rightarrow \quad -\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) = -\left(-\omega \cos \lambda \hat{e}_1 + \omega \sin \lambda \hat{e}_3\right) \times \left(\omega R_E \cos \lambda \hat{e}_2\right) ,$$

$$= \omega^2 R_E \cos^2 \lambda \hat{e}_3 + \underset{\substack{\uparrow \\ \text{term we are interested in}}}{\omega^2 R_E \sin \lambda \cos \lambda \hat{e}_1} .$$

Work out Coriolis term:

$$2\vec{\omega} \times \dot{\vec{r}} \simeq -2\left(-\omega \cos \lambda \hat{e}_1 + \omega \sin \lambda \hat{e}_3\right)$$
$$\times \left(-v_0 \cos \alpha \hat{e}_2 \left(-gt + v_0 \sin \alpha\right)\hat{e}_3\right)$$

$$-2\vec{\omega} \times \dot{\vec{r}} \simeq -2\omega \cos \lambda\, v_0 \cos \alpha \hat{e}_3 - 2\omega \cos \lambda \left(-gt + v_0 \sin \alpha\right)\hat{e}_2$$
$$\underset{\substack{\uparrow \\ \text{This is the term} \\ \text{we are interested in}}}{-2\omega \sin \lambda\, v_0 \cos \alpha \hat{e}_1}$$

Plugging these results back into our $\vec{F}_{eff}$ equation, we find

$$\left(\vec{F}_{eff}\right)_1 = m\ddot{r}_1 \simeq \omega^2 R_E \sin \lambda \cos \lambda - 2\omega \sin \lambda\, v_0 \cos \alpha .$$

If our initial condition is that $r_1(t) = 0$ at $t = 0$, then

$$r_1(t) \simeq \frac{1}{2}\left(\omega^2 R_E \sin \lambda \cos \lambda - 2\omega \sin \lambda\, v_0 \cos \alpha\right)t^2 .$$

Now eliminate the time, t, by using the $0^{\underline{th}}$ order equation,

$$r_3(t) = 0 = -\frac{1}{2} g t^2 + (v_0 \sin\alpha) t\,,$$

$$\Rightarrow t \simeq \frac{2 v_0 \sin\alpha}{g}\,.$$

This is the approximate time it takes for the projectile to hit the ground. Therefore

$$r_1(t) \simeq \frac{1}{2}\left(\omega^2 R_E \sin\lambda\cos\lambda - 2\omega\sin\lambda\, v_0\cos\alpha\right)\frac{4 v_0^2 \sin^2\alpha}{g^2}$$

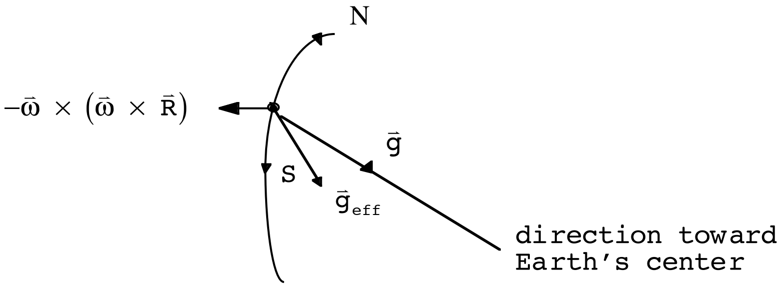
Question: which effect is larger? Depends on initial velocity, v_o. "Break even" velocity is

$$(v_0)_{BE} = \frac{\omega R_E \cos\lambda}{2\cos\alpha}\,. \qquad \left(\frac{\omega R_E}{2} \simeq 232\,\frac{\text{meters}}{\text{sec}}\right)$$

Depends on latitude, $\cos\alpha$. (In this problem I have not been very careful about taking care of the Earth's curvature. The above considerations only hold for short range projectiles.)

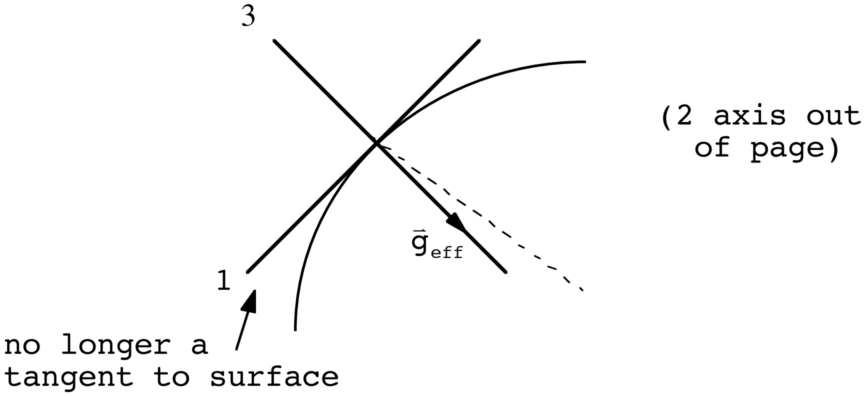
Deflections for dropped objects

Let me now introduce another coordinate system which is useful for calculating the deflection of projectiles relative to the <u>gravitational</u> <u>vertical</u>. Remember:



$$\vec{g}_{eff} = \vec{g} - \vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right).$$

Take new, skewed axes along $\vec{g}_{eff}$:



Then since

$$\vec{F}_{eff} = \underbrace{\vec{F} - m\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right)}_{m\vec{g}_{eff}} - m\left(\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \dot{\vec{r}}\right),$$

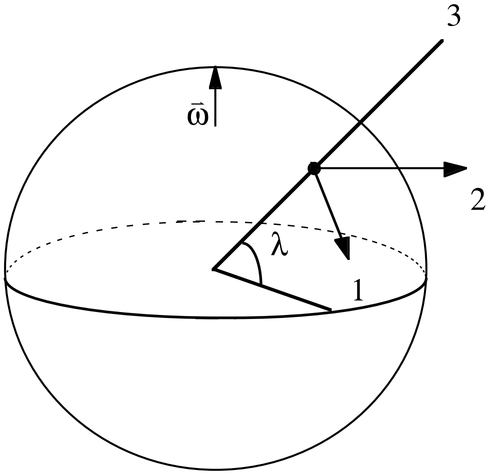
$$\vec{F}_{eff} = m\vec{a}_{eff},$$

$$\Rightarrow \vec{a}_{eff} = \vec{g}_{eff} - \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\vec{\omega} \times \dot{\vec{r}}. \qquad (10.46)$$

where now $\vec{g}_{eff} = g_{eff}\,\hat{e}_3$ only. This is useful in discussing the deflection of particles relative to the local gravitational vertical, which can be established with a plumbob, say. For example, if we had done the projectile problem above in the "skewed" frame, the term proportional to R_E in the form for $r_1(t)$ would have been absent. Then, to first order in ω:

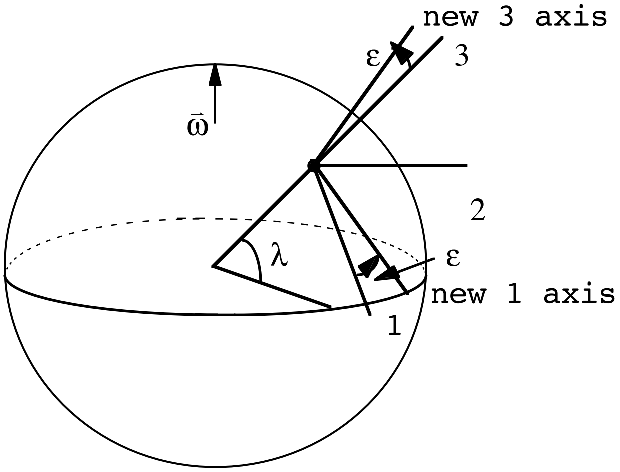
$$\vec{a}_{\text{eff}} \simeq -g_{\text{eff}}\,\hat{e}_3 - 2\vec{\omega} \times \dot{\vec{r}}\,. \qquad (10.47)$$

In this new frame, we need to find $\vec{\omega}$. We had



$$\begin{cases} \omega_1 = -\omega\cos\lambda \\ \omega_2 = 0 \\ \omega_3 = \omega\sin\lambda \end{cases}$$

After we "skew" it by an angle, it looks like (rotation is around 2 axis):



10.26

Clearly, we have

$$\begin{cases} \omega_1 = -\omega \cos(\lambda + \varepsilon), \\ \omega_2 = 0, \\ \omega_3 = \omega \sin(\lambda + \varepsilon). \end{cases}$$

However, you will find in Marion prob.9-9 that this is a small quantity (smaller than .002 radians) and will be neglected below.

Now imagine dropping an object from a height, h.

<u>In 0th order:</u>

$$\ddot{r}_3 = -g_{eff}, \qquad \ddot{r}_{1,2} = 0,$$

$$\Rightarrow \dot{\bar{r}} = -g_{eff} t \hat{e}_3 \;, \bar{r} = -\frac{1}{2} g_{eff} t^2 \hat{e}_3 \;\; (\text{if } \bar{r} = 0 \text{ at } t = 0)$$

<u>1st order (in ω) correction</u>:

$$\left.\begin{array}{l} \omega_1 \simeq -\omega \cos\lambda, \\ \omega_2 = 0, \\ \omega_3 \simeq \omega \sin\lambda. \end{array}\right\} \text{as explained above}$$

Then,

$$-2\bar{\omega} \times \dot{\bar{r}} \simeq 2\left(-\omega \cos\lambda \hat{e}_1 + \omega \sin\lambda \hat{e}_3\right) \times \left(-g_{eff}\, t\hat{e}_3\right),$$

$$= 2\omega \cos\lambda g_{eff} + \hat{e}_2 .$$

We now get

$$\ddot{r}_3 = -g_{eff},$$

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 \simeq 2\,\omega g_{\text{eff}}\, t \cos\lambda\,.$$

Integrating twice on $\ddot{r}_2$, we get

$$r_2(t) \simeq \frac{1}{3}\,\omega g_{\text{eff}}\, t^3 \cos\lambda\,.$$

Of course, we have

$$t^2 \simeq \frac{2h}{g_{\text{eff}}}\,,$$

$$\Rightarrow\ r_2(t) \simeq \frac{1}{3}\,\omega\left(\frac{8h^3}{g_{\text{eff}}}\right)^{1/2} \cos\lambda\,.$$

Since $r_2 > 0$, the deflection is to the East. You will study this problem further (to second order in ω^2) in a further HW problem.

Focault pendulum

Last problem: the Foucault pendulum. Again, use our "skewed" coordinate system. Only changes:

$$g \rightarrow \vec{F}_{\text{eff}} = \underset{\substack{\uparrow \\ \text{new external force}}}{\vec{F}} + m\vec{g}_{\text{eff}} - m\big[\vec{\omega} \times \underset{\substack{\uparrow \\ \text{small (ignore)}}}{(\vec{\omega} \times \vec{r})} + 2\vec{\omega} \times \dot{\vec{r}}\big]\, g_{\text{eff}}$$

Say

$$\vec{F} = F_1\,\hat{e}_1 + F_2\,\hat{e}_2 + F_3\,\hat{e}_3\,,$$

$$\vec{g}_{\text{eff}} = -g_{\text{eff}}\,\hat{e}_3\,,$$

10.28

$$\vec{\omega} = -\omega \cos\lambda \, \hat{e}_1 + \omega \sin\lambda \, \hat{e}_3 \, .$$

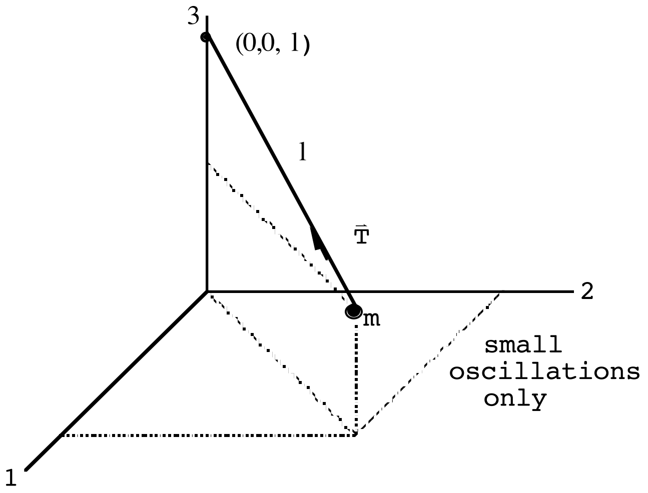
Put it all together:

$$m\ddot{r}_1 = F_1 + 2m\omega\dot{r}_2 \sin\lambda \, ,$$

$$m\ddot{r}_2 = F_2 - 2m\omega\left(\dot{r}_2 \sin\lambda + \dot{r}_3 \cos\lambda\right),$$

$$m\ddot{r}_3 = F_3 - mg_{\text{eff}} + 2m\omega\dot{r}_2 \cos\lambda \, .$$

Consider:



$$F_1 = -\frac{r_1}{\ell}T, \quad F_2 = -\frac{r_2}{\ell}T, \quad F_3 = \frac{\left(\ell - r_3\right)}{\ell}T.$$

Neglect $\ddot{r}_3$ and $\frac{r_3}{\ell} \simeq 0$. $3^{\underline{rd}}$ $eq^{\underline{n}}$ above becomes

$$\Rightarrow T \simeq mg_{\text{eff}} - \underbrace{2m\omega\dot{r}_2 \cos\lambda}_{\text{small but} \neq 0} \, .$$

(not a constant)

Substitute this value of T above:

$$m\ddot{r}_1 = \frac{-r_1}{\ell}\underset{\substack{\uparrow \\ \text{small}}}{\left[2m\omega\dot{r}_2 \cos\lambda\right]} + 2m\omega\dot{r}_2 \sin\lambda ,$$

$$m\ddot{r}_2 = \frac{-r_2}{\ell}\underset{\substack{\uparrow \\ \text{small}}}{\left[2m\omega\dot{r}_2 \cos\lambda\right]} - 2m\omega\left[\dot{r}_1 \sin\lambda + \underset{\substack{\uparrow \\ \text{small}}}{\dot{r}_3 \cos\lambda}\right]$$

Set $\alpha^2 \equiv \frac{g_{\text{eff}}}{\ell}$. Then

$$\ddot{r}_1 + \alpha^2 r_1 = 2\omega\dot{r}_2 \sin\lambda ,$$

$$\ddot{r}_2 + \alpha^2 r_2 = -2\omega\dot{r}_1 \sin\lambda .$$

There is a dynamic coupling between the motions. Can solve both at once using complex numbers:

$$\left(\ddot{r}_1 + i\ddot{r}_2\right) + \alpha^2\left(r_1 + ir_2\right) = 2\omega\left(\dot{r}_2 - i\dot{r}_1\right)\sin\lambda ,$$

$$= -2\omega i\left(\dot{r}_1 + i\dot{r}_2\right)\sin\lambda ,$$

or

$$\dot{q} + \alpha^2 q = -2\omega i\dot{q} \sin\lambda .$$

where $q \equiv r_1 + ir_2$. This is solved by assuming

$$q = A\,e^{\lambda t} ,$$

where A, λ are to be determined. Substitute above:

$$\Rightarrow \lambda^2 + 2i\omega\lambda \sin\lambda + \alpha^2 = 0 ,$$

$$\Rightarrow \lambda = \frac{1}{2}\left[-2i\omega \sin\lambda \pm \sqrt{-4\omega^2 \sin^2\lambda - 4\alpha^2}\right],$$

$$\lambda = -i\omega \sin\lambda \pm i\sqrt{\omega^2 \sin^2\lambda + \alpha^2}\,.$$

But $\omega^2 \sin^2 \lambda \ll \alpha^2 \left(= \frac{g}{\ell}\right)$ for the Earth, so

$$\lambda \simeq -i\omega \sin\lambda \pm i\alpha\,.$$

General solution: (A,B real)

$$g(t) = \left\{Ae^{i\alpha t} + Be^{-i\alpha t}\right\} e^{-i\omega t \sin\lambda}\,.$$

A,B are fixed by initial conditions. Write it out:

$$A(\cos\alpha t + i\sin\alpha t) + B(\cos\alpha t - i\sin\alpha t)\,,$$

$$= C_1 \cos\alpha t + iC_2 \sin\alpha t,$$

$$C_1 = A + B\ , C_2 = A - B\,,$$

$$\Rightarrow q(t) = \left(C_1 \cos\alpha t + iC_2 \sin\alpha t\right) e^{-i\omega t \sin\lambda}\,.$$

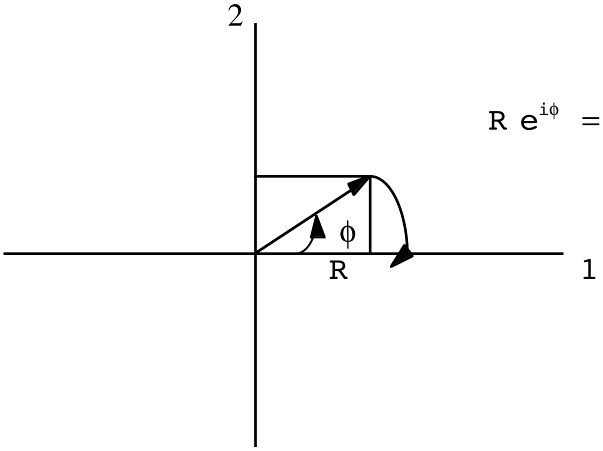
There are 2 parts to the motion with different frequencies since $\omega \ll \alpha$. Let's say $e^{-i\omega t \sin\lambda} \simeq 1$. Then, since

$$r_1 = \mathrm{Re}\, q(t)\,, \qquad r_2 = \mathrm{Im}\, q(t)$$

$$\simeq C_1 \cos\alpha t \qquad \simeq C_2 \cos\alpha t$$

$$\Rightarrow \frac{r_1^2}{C_1^2} + \frac{r_2^2}{C_2^2} = 1\,.$$

Eq$^{\underline{n}}$ of an ellipse. However, we made r_1 and r_2 coordinates in the complex plane. The factor $e^{-i\omega t \sin\lambda}$ is just a <u>rotation</u> in the complex plane:



$$R\, e^{i\phi} = R(\cos\phi + i\sin\phi)$$

a <u>counter-clockwise</u>
rotation

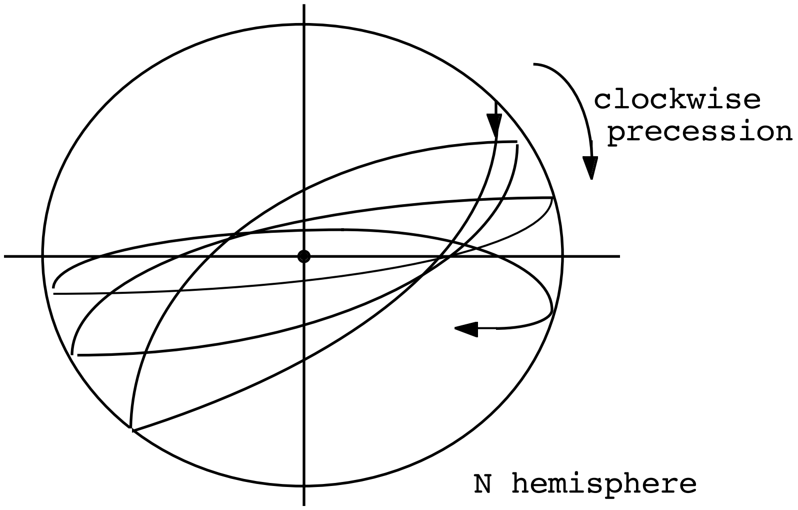
The real and imaginary parts of the complex number g(t) are just the r_1, r_2 components of the real motion. Therefore, a rotation in the complex plane is also describing a rotation of the vector $\bar{r} = r_1\,\hat{e}_1 + r_2\,\hat{e}_2$ in real space. Because of the minus sign in $e^{-i\omega t \sin\lambda}$, this is a <u>clockwise</u> rotation in coordinate space. (It would be counter-clockwise in the Southern hemisphere.) Time it takes to complete a rotation:

$$2\pi = \omega T \sin\lambda \Rightarrow T = \frac{2\pi}{\omega\sin\lambda},$$

$$\omega = \frac{2\pi}{1\ \text{day}} \Rightarrow T = \frac{1}{\sin\lambda}\ \text{days}.$$

Goes around once a day at $\lambda = 90^0$ (N or S poles) and does not precess at all at the equator ($\lambda = 0^0$). Actual motion looks like:

10.32



Looks like the precession of the orbit of a planet under general relativity, but the forces here certainly are not central. The “force” that makes it precess, in fact, is purely fictional.

Chapter 10 Problems

1.(a) I showed in the notes that instantaneously

$$\vec{r}\,' = \vec{r} + \vec{R},$$

$$\dot{\vec{r}}\,' = \dot{\vec{r}} + \vec{R} + \vec{\omega} \times \vec{r},$$

$$\ddot{\vec{r}}\,' = \ddot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}).$$

Find the relation between $\dddot{\vec{r}}\,' = \dot{\vec{a}}\,'$ and $\dddot{\vec{r}} = \dot{\vec{a}}$. ($\dddot{\vec{R}} = \dot{\vec{A}}$; assume all derivatives of $\vec{\omega}$ to be zero.) Ans:

$$\dot{\vec{a}}\,' = \dot{\vec{a}} + \dot{\vec{A}} + 2\vec{\omega} \times \vec{a} + 3\vec{\omega} \times (\vec{\omega} \times \dot{\vec{r}}) + \vec{\omega} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r})).$$

(b) Continue this process and find the relation between $\ddot{\vec{a}}\,'$ and $\ddot{\vec{a}}$.

2. Consider the statement in the footnote of Marion and Thorton, p.348. Is it correct? Find out by computing $\ddot{\vec{R}}_f$ for the situation shown in his Fig. 9-7. (Take the inertial axes at the Earth's center, instantaneously parallel to the non inertial axes). If Marion is right, compute the small deflection caused by keeping $\ddot{\vec{R}}_f$; if he is wrong, tell me the real reason this term may be neglected in this example.

3. Show that the angular deviation ε of a plumb line from the true vertical at a point on the Earth's surface at a latitude λ is

$$\varepsilon \simeq \frac{r_0\omega^2 \sin\lambda\cos\lambda}{g - r_0\omega^2\cos^2\lambda},$$

where r_0 is the Earth's radius and g is acceleration due to gravity.

4.(a) By balancing centrifical "force" and gravitational force, find the orbital velocity of an object in a circular orbit just above the Moon's surface.

$$(R_{moon} = 1.74 \times 10^{8}\ \text{cm},\ M_{moon} = 7.35 \times 10^{25}\ \text{gm},$$

$$G = 6.67 \times 10^{-8}\ \text{cm}\ \frac{\text{dyne}\cdot\text{cm}^2}{\text{gm}^2}\ \cdot)$$

Newton's gravitational constant

(b) At Waco's latitude, $\lambda = 31.5^\circ$, how many hours does it take for the plane of a Focault pendulum to complete a revolution?