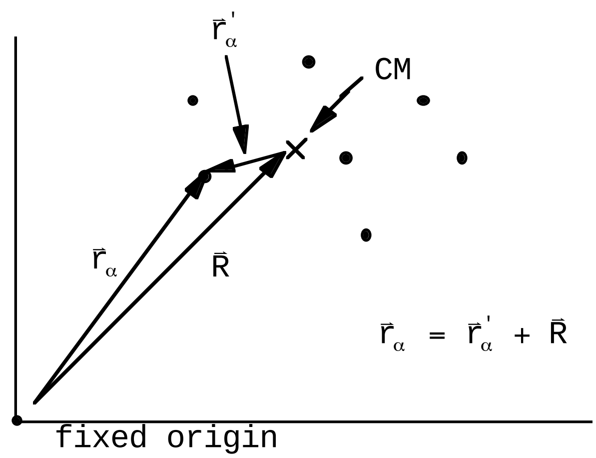
Chapter 9

Multi-particle conservation laws

Multiparticle conservation theorems now. A generalization of results for momentum and energy reached in Ch.2 for a single particle.

Latin indices (a,b,c, etc.) : vector indices
Greek indices (α,β,γ, ...) : particle number

Consider an arbitrary system of particles:



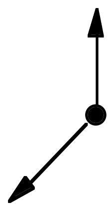
Center of mass located by $\left(M = \sum_\alpha m_\alpha\right)$

$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha . \tag{9.1}$$

As we have seen before

$$\sum_\alpha m_\alpha \vec{r}_\alpha^{\,'} = 0 . \tag{9.2}$$

Particle α:



$\vec{f}_{\alpha\beta}$: internal force on α due to β

$\vec{F}^{(e)}_{\alpha}$: external force

Total force on α:

$$\vec{F}_{\alpha} = \vec{F}^{(e)}_{\alpha} + \sum_{\beta} \vec{f}_{\alpha\beta}\,. \tag{9.3}$$

Call $\vec{f}_{\alpha} = \sum_{\beta} \vec{f}_{\alpha\beta}$. From Newton's third law

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}\,. \quad \text{(weak form)} \tag{9.4}$$

Also assume

$$\vec{f}_{\alpha\alpha} = 0. \tag{9.5}$$

(Newton's mechanics not equipped to handle self-interactions which, however, really do exist!) Newton's second law:

$$\dot{\vec{p}}_{\alpha} = \vec{F}^{(e)}_{\alpha} + \vec{f}_{\alpha}\,, \tag{9.6}$$

or

$$\frac{d^2}{dt^2}\left(m_{\alpha}\vec{r}_{\alpha}\right) = \vec{F}^{(e)}_{\alpha} + \sum_{\beta} \vec{f}_{\alpha\beta}\,. \tag{9.7}$$

Sum on α:

$$\frac{d^2}{dt^2}\left(\sum_{\alpha} m_{\alpha}\vec{r}_{\alpha}\right) = \sum_{\alpha} \vec{F}^{(e)}_{\alpha} + \sum_{\alpha,\beta} \vec{f}_{\alpha\beta}\,,$$

$$\sum_{\alpha,\beta} \vec{f}_{\alpha\beta} = \underbrace{\vec{f}_{12} + \vec{f}_{21}}_{0} + \ldots = 0,$$

$$\Rightarrow \frac{d^2}{dt^2}\left(M\vec{R}\right) = \sum_{\alpha} \vec{F}_{\alpha}^{(e)} \equiv \vec{F}^{(e)}. \tag{9.8}$$

Moral: Center of mass moves as if the total external force were acting on the entire mass of system concentrated at the center of mass. (Internal forces have no effect on CM motion.) Of course, if $\vec{F}^{(e)} = 0$, then

$$\frac{d^2}{dt^2}\left(M\vec{R}\right) \equiv \dot{\vec{P}} = 0, \tag{9.9}$$

and total linear momentum is conserved.

Likewise, the angular momentum of the $\alpha^{\underline{th}}$ particle about the origin is

$$\vec{L}_{\alpha} = \vec{r}_{\alpha} \times \vec{p}_{\alpha}. \tag{9.10}$$

Summing this (we already did this in (8.7):

$$\vec{L} = \sum_{\alpha} \vec{L}_{\alpha} = \sum_{\alpha} \vec{r}_{\alpha} \times \vec{p}_{\alpha} = \sum_{\alpha} \left(\vec{r}_{\alpha} \times m_{\alpha}\dot{\vec{r}}_{\alpha}\right)$$

$$= \sum_{\alpha} \left(\vec{r}_{\alpha}^{\,\prime} + \vec{R}\right) \times m_{\alpha}\left(\dot{\vec{r}}_{\alpha}^{\,\prime} + \dot{\vec{R}}\right). \tag{9.11}$$

$$\vec{L} = \vec{R} \times \vec{P} + \sum_{\alpha} \vec{r}_{\alpha}^{\,\prime} \times \vec{p}_{\alpha}^{\,\prime}. \qquad \left(\vec{p}_{\alpha}^{\,\prime} \equiv m_{\alpha}\dot{\vec{r}}_{\alpha}^{\,\prime}\right) \tag{9.12}$$

Moral: Total angular momentum about a coordinate axis is the angular momentum of the system as if it were concentrated at the center of mass, plus the angular momentum of motion about the center of mass.

Now the rate of change of $\vec{L}_\alpha$ is

$$\dot{\vec{L}}_\alpha = \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \vec{r}_\alpha \times \left(\vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta} \right)$$

$$\Rightarrow \dot{\vec{L}} = \sum_\alpha \dot{\vec{L}}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} + \sum_{\alpha,\beta} \vec{r}_\alpha \times \vec{f}_{\alpha\beta} \,. \tag{9.13}$$

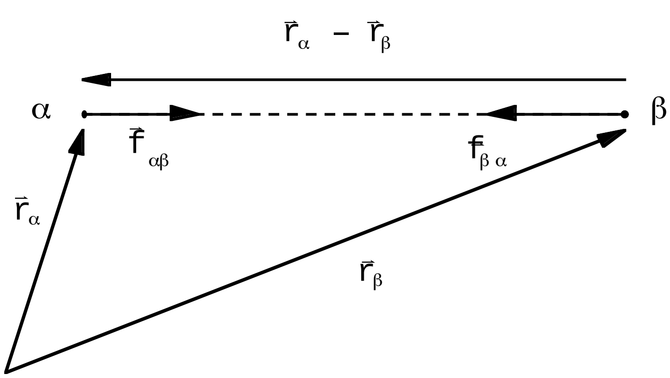
① ② (assume $\vec{f}_{\alpha\alpha} = 0$)

$$① = \sum_\alpha \underbrace{\vec{r}_\alpha \times \vec{F}_\alpha^{(e)}}_{\vec{N}_\alpha^{(e)}} \equiv \vec{N}^{(e)}, \tag{9.14}$$

$$② = -\sum_{\alpha,\beta} \vec{r}_\alpha \times \vec{f}_{\beta\alpha} = -\sum_{\alpha,\beta} \vec{r}_\beta \times \vec{f}_{\alpha\beta} \,. \tag{9.15}$$

Therefore, we may write

$$② = \frac{1}{2} \sum_{\alpha,\beta} \left(\vec{r}_\alpha - \vec{r}_\beta \right) \times \vec{f}_{\alpha\beta} = 0 \,. \quad \text{(see figure below)} \tag{9.16}$$



I am showing an attractive force case. I am also assuming the "strong form" of Newton’s second law, which says that $\vec{f}_{\alpha\beta}$ and $\vec{f}_{\beta\alpha}$ lie along the line connecting the two particles. (Violated, for example, in electromagnetism.) Under these circumstances then

$$\dot{\vec{L}} = \vec{N}^{(e)}. \tag{9.17}$$

Moral: So $\vec{L}$ = const. in time if $\vec{N}^{(e)} = 0$.

As before (in (8.4))

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2} \sum_\alpha m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha'\right)^2$$

$$= \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'^{\,2}. \tag{9.18}$$

Moral: Kinetic energy, like angular momentum, consists of two parts: the kinetic energy of the center of mass, plus the kinetic energy of motion about the center of mass.

Now, what about work being done on the system? Define

$$W_{12} \equiv \sum_\alpha \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha = \sum_\alpha \int_1^2 m_\alpha \ddot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha dt$$

$$= \frac{1}{2} \sum_\alpha m_\alpha \int_1^2 \frac{d}{dt} \dot{\vec{r}}_\alpha^{\,2} = \sum_\alpha \frac{1}{2} m_\alpha\left(\vec{v}_{2\alpha}^2 - \vec{v}_{1\alpha}^2\right),$$

$$\Rightarrow \quad W_{12} = T_2 - T_1 \quad \text{(useful in a bit)} \tag{9.19}$$

Assume conservative external and internal forces:

$$\vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha U_\alpha(\vec{x}_\alpha), \tag{9.20}$$

$$\vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}\left(\left|\vec{r}_\alpha - \vec{r}_\beta\right|\right) \tag{9.21}$$

Then, alternatively

$$W_{12} = \sum_\alpha \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha = \overset{①}{\sum_\alpha \int_1^2 \vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha} + \overset{②}{\sum_{\alpha,\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha}. \tag{9.22}$$

$$① \;=\; -\sum_{\alpha}\int_1^2 \left(\vec{\nabla}_\alpha U_\alpha\right)\cdot d\vec{r}_\alpha \;=\; -\sum_{\alpha,i}\int_1^2 \frac{\partial U_\alpha}{\partial x_{\alpha,i}}\, dr_{\alpha,i}$$

$$= -\sum_{\alpha}\int_1^2 dU_\alpha \;=\; \sum_{\alpha}\left(U_{1\alpha} - U_{2\alpha}\right). \tag{9.23}$$

② is more complicated. On one hand

$$② \;=\; \sum_{\alpha,\beta}\int_1^2 \vec{f}_{\alpha\beta}\cdot d\vec{r}_\alpha \;=\; -\sum_{\alpha,\beta}\int_1^2 \vec{f}_{\beta\alpha}\cdot d\vec{r}_\alpha \;=\; -\sum_{\alpha,\beta}\int_1^2 \vec{f}_{\alpha\beta}\cdot d\vec{r}_\beta\,,$$

$$\Rightarrow\quad ② \;=\; \frac{1}{2}\sum_{\alpha,\beta}\int_1^2 \vec{f}_{\alpha\beta}\cdot\left(d\vec{r}_\alpha - d\vec{r}_\beta\right). \tag{9.24}$$

On the other hand the chain rule gives

$$d\bar{U}_{\alpha\beta} \;=\; \sum_i \Bigg[\underbrace{\frac{\partial\bar{U}_{\alpha\beta}}{\partial r_{\alpha,i}}\, dr_{\alpha,i}}_{\left(\vec{\nabla}_\alpha\bar{U}_{\alpha\beta}\right)\cdot d\vec{r}_\alpha} \;+\; \underbrace{\frac{\partial\bar{U}_{\alpha\beta}}{\partial r_{\beta,i}}\, dr_{\beta,i}}_{\left(\vec{\nabla}_\beta\bar{U}_{\alpha\beta}\right)\cdot d\vec{r}_\beta}\Bigg],$$

$$\Rightarrow\quad d\bar{U}_{\alpha\beta} \;=\; \underbrace{\left(\vec{\nabla}_\alpha\,\bar{U}_{\alpha\beta}\right)}_{-\vec{f}_{\alpha\beta}}\cdot d\vec{r}_\alpha \;+\; \underbrace{\left(\vec{\nabla}_\beta\,\bar{U}_{\alpha\beta}\right)}_{-\vec{f}_{\beta\alpha}=\vec{f}_{\alpha\beta}}\cdot d\vec{r}_\beta,$$

$$\Rightarrow\quad d\bar{U}_{\alpha\beta} \;=\; -\vec{f}_{\alpha\beta}\cdot\left(d\vec{r}_\alpha - d\vec{r}_\beta\right). \tag{9.25}$$

$$\Rightarrow\quad ② \;=\; \frac{1}{2}\sum_{\alpha,\beta}\int_1^2 \vec{f}_{\alpha\beta}\cdot\left(d\vec{r}_\alpha - d\vec{r}_\beta\right) \;=\; -\frac{1}{2}\sum_{\alpha,\beta}\int_1^2 d\bar{U}_{\alpha\beta}\,,$$

$$\Rightarrow\quad ② \;=\; -\frac{1}{2}\sum_{\alpha,\beta}\left(\bar{U}_{2\alpha\beta} - \bar{U}_{1\alpha\beta}\right). \tag{9.26}$$

Therefore

$$W_{12} = \left.\left(-\sum_{\alpha} U_{\alpha} - \frac{1}{2}\sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \overline{U}_{\alpha\beta}\right)\right|_1^2 . \tag{9.27}$$

Define the total potential energy,

$$U \equiv \sum_{\alpha} U_{\alpha} + \sum_{\alpha<\beta} \overline{U}_{\alpha\beta} = \sum_{\alpha} U_{\alpha} + \frac{1}{2}\sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \overline{U}_{\alpha\beta}, \tag{9.28}$$

then

$$W_{12} = -U\Big|_1^2 = U_1 - U_2 . \tag{9.29}$$

Combining this with $W_{12} = T_2 - T_1$,

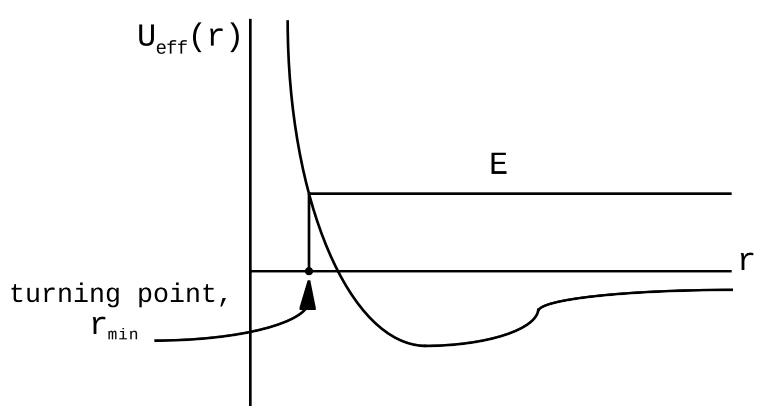
$$\Rightarrow \quad T_1 + U_1 = T_2 + U_2 . \tag{9.30}$$

Moral: Total energy is conserved.

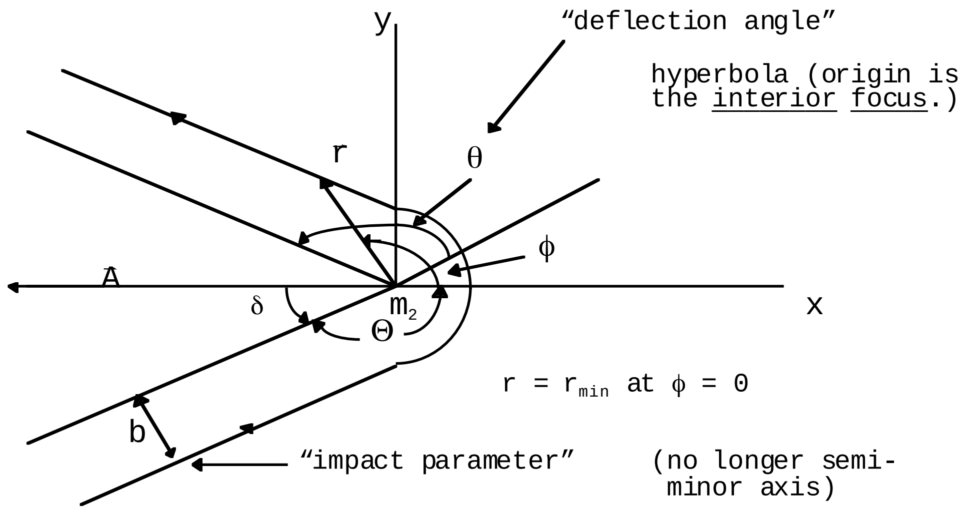
Coulomb scattering

I will discuss scattering next since it builds on the results of the last chapter.

Consider $E > 0$ ($\varepsilon > 1$) motion in an attractive Coulomb field. Picture:



Follow the course of $\bar{r}$ during scattering from m_2's point of view.



Equation of orbit (an hyperbola, $\varepsilon > 1$):

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \varepsilon \cos \phi\right), \tag{9.31}$$

$$\frac{1}{r_{min}} = \frac{1}{\alpha}(1 + \varepsilon), \tag{9.32}$$

$$\varepsilon = \left(1 + \frac{2E\ell^2}{\mu k^2}\right)^{1/2}, \tag{9.33}$$

$$\alpha = \frac{\ell^2}{\mu k}. \tag{9.34}$$

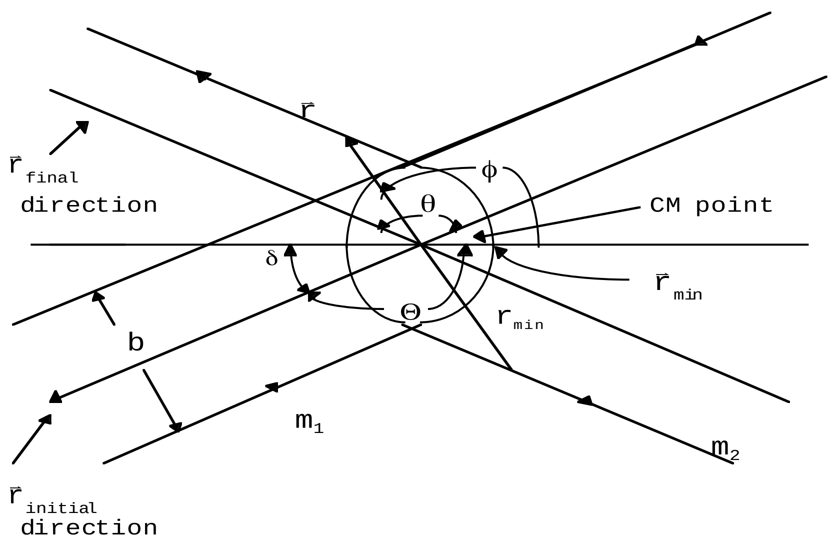
Notice: $2\delta + \theta = \pi$. Also

$$\delta + \Theta = \pi,$$

$$\Rightarrow \Theta = \frac{\pi + \theta}{2}. \qquad (9.35)$$

One gets $\Theta = \dfrac{\pi - \theta}{2}$ for the repulsive force case. (where the scattering center is now the exterior focus of the hyperbola.)

Scattering event looks completely different from CM frame:



Paths are also hyperbolas here ($\vec{r}_1{}', \vec{r}_2{}'$ are just rescaling of $\vec{r}$.) No matter which picture you prefer, it is clear that θ, Θ are quantities relating to the direction of $\vec{r}$ and therefore can be thought of as being measured in the CM frame. However, we will actually define scattering angles with respect to velocity vectors, and so take on different values in alternate inertial frames.

$\theta \equiv$ positive angle between initial and final velocity vectors for either m_1 or m_2 in CM frame

Things are simpler in the CM; however, this is not the usual experimental situation. Eventually, we will learn how to translate our results in the CM to other frames of reference. Notice that only a certain range of angles for ϕ are now permitted for $\varepsilon > 1$:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \varepsilon \cos\phi\right),$$

$$\Rightarrow \frac{1}{\infty} = \frac{1}{\alpha}\left(1 + \varepsilon \cos\Theta\right),$$

$$\Rightarrow \cos\Theta = -\frac{1}{\varepsilon} \quad \text{(allows 2 symmetric values of } \Theta\text{; take the + value)} \tag{9.36}$$

$\Rightarrow$ only angles $\cos\phi > -\dfrac{1}{\varepsilon}$ are allowed. (Values greater than this would say that r is negative.) Go back to the Runge-Lenz vector to see it from another viewpoint:

$$\vec{A} = \vec{\mathfrak{L}} \times \vec{p} + \mu k \hat{e}_r,$$

$$\Rightarrow \vec{A} \cdot \vec{p} = \mu k \hat{e}_r \cdot \vec{p}\,.$$

Remember, $\vec{A}$ is directed along the symmetry axis, in the direction opposite to $\vec{r}_{min}$. In particular, if we consider the initial situation with $r \to \infty$,

$$\vec{A} \cdot \vec{p}_\infty = \mu k \hat{e}_{r\infty} \cdot \vec{p}_\infty\,.$$

$$\vec{A} \cdot \vec{p}_\infty = A p_\infty \cos\Theta\,,\ \hat{e}_{r\infty} \cdot \vec{p}_\infty = -p_\infty\,,$$

$$\Rightarrow A p_\infty \cos\Theta = -\mu k p_\infty\,,$$

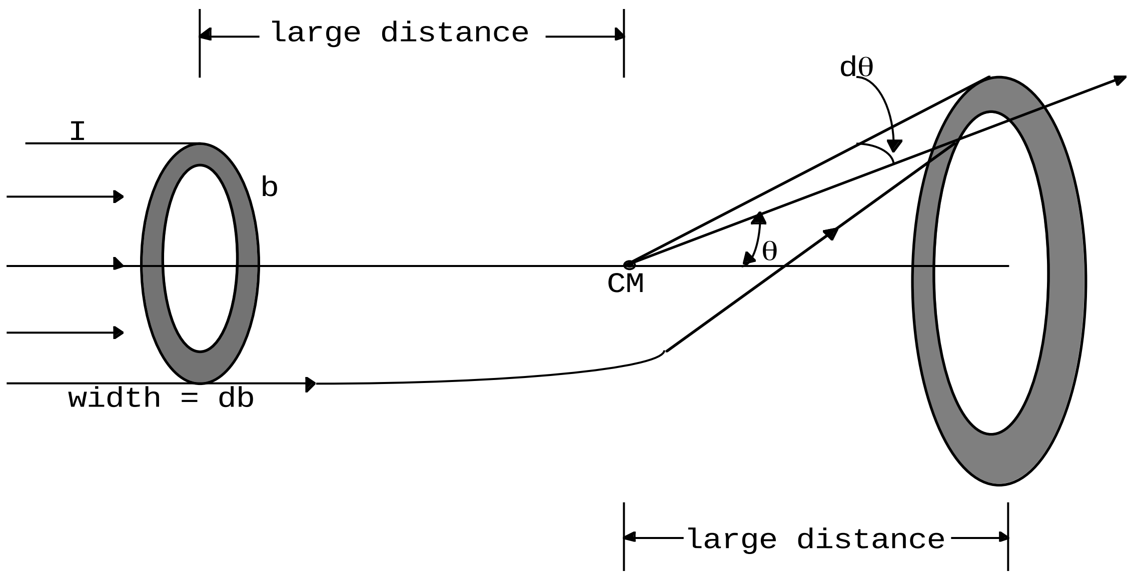
$$\Rightarrow \cos\Theta = -\frac{\mu k}{A} = -\frac{1}{\varepsilon}\text{, as before.}$$

Differential cross sections

Need some more concepts for scattering. Let

I = flux in the incident beam (# particles per unit area per unit time)

dN = number of particles through a ring of radius b and width db in the incident beam per unit time.



$$dN = \left(2\pi b|db|\right)I. \qquad (9.37)$$

The particles passing through the ring are scattered through the angles between θ and $\theta + d\theta$. dN becomes a scattering concept when we assume that $b = b(\theta)$:

$$dN(\theta) = \left(2\pi b(\theta)\left|db(\theta)\right|\right)I. \qquad (9.38)$$

$\dfrac{dN(\theta)}{I}$ = number of particles scattered into $(\theta,\ \theta + d\theta)$ per unit time per incident flux

This is now a quantity which is independent of I. We now define the differential cross section as (take $d\theta$ positive)

$$\frac{d\sigma(\theta)}{d\theta} \equiv \frac{dN}{d\theta}\frac{1}{I} = 2\pi\, b(\theta)\left|\frac{db(\theta)}{d\theta}\right|. \qquad (9.39)$$

Intrinsically positive. We usually use <u>solid</u> <u>angle</u>,

$$d\Omega \equiv \sin\theta\, d\theta d\phi\,. \qquad (9.40)$$

We will assume azimuthal symmetry here, so

$$d\Omega = 2\pi \sin\theta d\theta\,,$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\theta) = \frac{b}{\sin\theta}\left|\frac{db(\theta)}{d\theta}\right|. \qquad (9.41)$$

Rutherford scattering in the center of mass frame

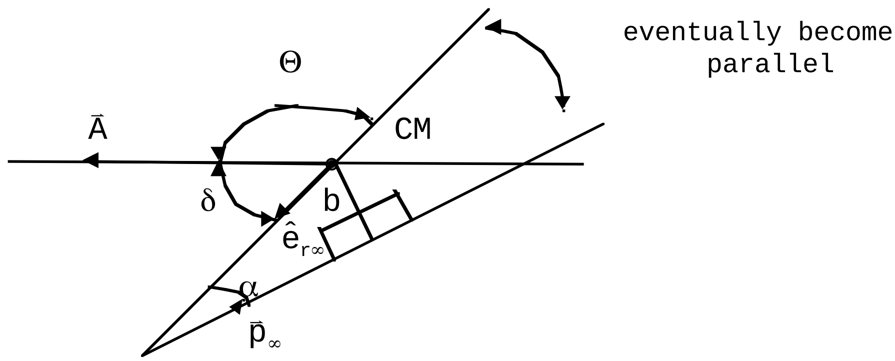
For the Coulomb problem, remember

$$A^2 = \mu^2 k^2 + 2\mu\ell^2 E, \quad (A = \mu k\varepsilon)$$

$$\Rightarrow 1 = \left(\frac{\mu k}{A}\right)^2 + \frac{2\mu\ell^2 E}{A^2}\,.$$

But $\cos\Theta = -\dfrac{1}{\varepsilon} = -\dfrac{\mu k}{A}$,

$$\Rightarrow 1 = \cos^2\Theta + \frac{2\mu\ell^2 E}{A^2}\,.$$



At great distances,

$$E = \text{kinetic} + \overset{0}{\text{potential}},$$

$$\Rightarrow E = \frac{p_\infty^2}{2\mu} \text{ (two body problem form)},$$

$$\ell = |\bar{r} \times \bar{p}| = b\, p_\infty .$$

$$\Rightarrow \sin\Theta = \frac{\ell\sqrt{2\mu E}}{A} = \frac{\ell\, p_\infty}{A} = \frac{b\, p_\infty^2}{A},$$

$$\Rightarrow \tan\Theta = -\frac{\frac{b\, p_\infty^2}{A}}{\frac{\mu k}{A}} = -\frac{b\, p_\infty^2}{\mu k}.$$

But

$$\Theta = \frac{\theta + \pi}{2} \Rightarrow \tan\Theta = -\cot\frac{\theta}{2},$$

$$\Rightarrow \cot\frac{\theta}{2} = \frac{b p_\infty^2}{\mu k} \tag{9.42}$$

Provides the necessary connection between b and θ for the Coulomb problem. Now get

$$\frac{db}{d\theta} = \frac{\mu k}{p_\infty^2}\,\frac{d}{d\theta}\cot\frac{\theta}{2},$$

$$= -\frac{\mu k}{2p_\infty^2}\;\frac{1}{\sin^2\frac{\theta}{2}}.$$

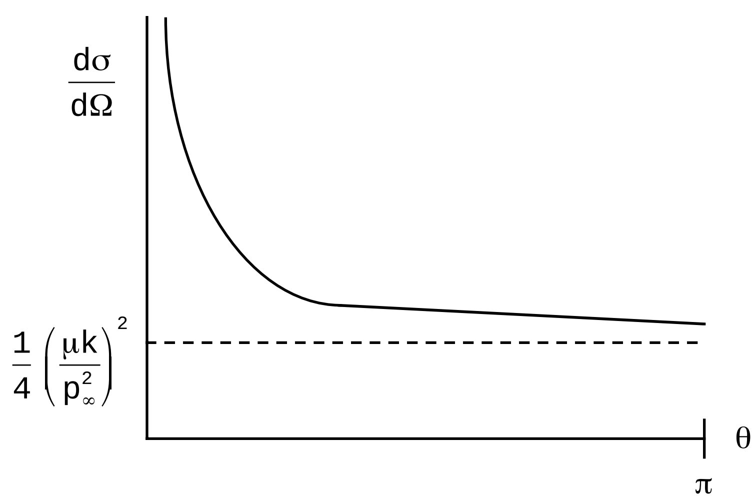
So

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{2}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin\theta}\cdot\frac{1}{\sin^2\frac{\theta}{2}}\;\cot\frac{\theta}{2},$$

$$\sin\theta = 2\sin\frac{\theta}{2}\;\cos\frac{\theta}{2},$$

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \quad \text{. “Rutherford formula”} \qquad (9.43)$$

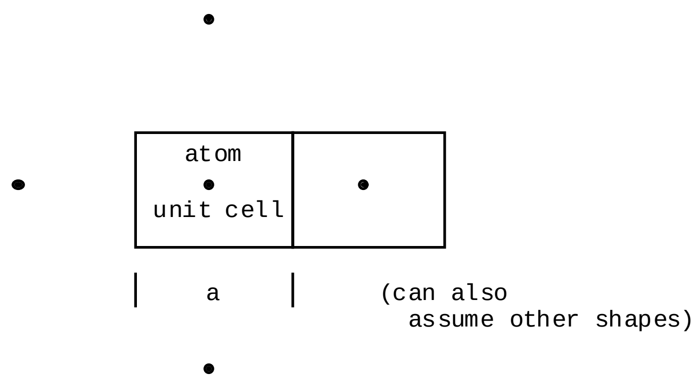
(CM frame)



True also for <u>repulsive</u> Coulomb force. Indeed, unchanged by quantum mechanics (non-relativistic),except that $k \to \pm(Zze^2)$ for electrically charged particles. (Interpretation of it in quantum mechanics is completely different, however.)

First measured experimentally by H. Geiger and E. Marsden. By careful observation in a darkened room, they found (by counting!) that approximately one in every eight thousand α-particles (Helium nuclei) was backscattered from a thin gold target. Let's see if we can understand the $\frac{1}{8000}$ factor backscattering fraction from the above formula.

<u>Model</u>: (each atomic layer)



Assume layers are randomly oriented with respect to each other (i.e., one atom in front does not "shadow" an atom behind.)

Each α-particle can be imagined to pass through the unit cell shown as may times as there are layers of atoms. We will take "backscattering" to mean $\frac{\pi}{2} < \theta < \pi$.

$$\frac{d\sigma}{d\Omega} = \frac{dN}{d\Omega}\frac{1}{I} = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}},$$

$$dN = I\,d\sigma = \frac{I}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{d\Omega}{\sin^4\frac{\theta}{2}},$$

$$N_{back} = I\sigma_{back} = \frac{1}{4} I\left(\frac{\mu k}{p_\infty^2}\right)^2 \int_{\pi/2}^{\pi} \frac{d\Omega}{\sin^4\frac{\theta}{2}}.$$

For this rough estimate, we will set $\sin^4\frac{\theta}{2} \to 1$ over this range of θ, and since the solid angle corresponding to $\frac{\pi}{2} < \theta < \pi$ is 2π, we then have approximately,

$$N_{back} \cong \frac{\pi}{2} I \left(\frac{\mu k}{p_\infty^2}\right)^2 .$$

For scattering α's off gold foil:

$$k \to Zze^2, \mu \cong m_\alpha,$$

$$Z = 79, z = 2,$$

$$\Rightarrow N_{back} \cong \frac{\pi}{2} I \left(\frac{Zze^2 m_\alpha}{m_\alpha^2 v_\alpha^2}\right)^2 .$$

Let us assume that (nonrelativistic formula)

$$\frac{1}{2} m_\alpha v_\alpha^2 \cong 5\text{MeV} . \quad \text{(I looked it up)}$$

($1\,\text{MeV} \cong 1.6 \times 10^{-6}\text{erg}$). We also need

$$|e| = 4.803 \times 10^{-10}\text{esu}$$
$$m_\alpha = 6.68 \times 10^{-24}\text{gm}$$

$$\Rightarrow p_\infty = m_\alpha v_\alpha = 1.034 \times 10^{-14}\text{gm}\,\frac{\text{cm}}{\text{sec}} .$$

Still need I. In our case (imagine a single particle passing through the sample; also imagine multiplying both sides of the above equation for N_{back} by the total time of the experiment so that N_{back} is a pure number):

I = number scattering processes per alpha particle / unit cell

$$\Rightarrow I = \frac{1}{a^2} n,$$

where n is the number of atomic layers in the sample thickness. Sample thickness was ~ $1\mu = 10^{-4}$cm (also looked this up). So

$$n = \frac{10^{-4}\mathrm{cm}}{a},$$

$$\Rightarrow I = \frac{10^{-4}\mathrm{cm}}{a^3}$$

Some other calculations giving a:

$$\rho_{gold} = 19.28 \frac{\mathrm{gm}}{\mathrm{cm}^3}$$

$$\underset{\substack{\downarrow}}{\underline{\text{Avagadro}}}$$

$$N_A = 6.023 \times 10^{23} \frac{\mathrm{amu}}{\mathrm{gm}}, \quad m_{gold} \simeq 197\,\mathrm{amu}$$

$$\left(\frac{\mathrm{atoms}}{\mathrm{cm}^3}\right)_{gold} = \rho_{gold} \times \left(\frac{\#\ \mathrm{atoms}}{\mathrm{gm}}\right) = 19.28\left(\frac{6.023 \times 10^{23}}{197}\right)$$

$$= 5.9 \times 10^{22} \frac{\mathrm{atoms}}{\mathrm{cm}^3}.$$

This means

$$a \simeq \left(5.9 \times 10^{22}\right)^{-1/3} = 2.57 \times 10^{-8}\mathrm{cm}$$

$$\Rightarrow I \simeq 5.90 \times 10^{18}\mathrm{cm}^{-2}.$$

Putting all the pieces together now gives us

$$\Rightarrow N_{back} \simeq 4.76 \times 10^{-5} \simeq \frac{1}{21{,}000}.$$

too small ~ 2. One reason: did not integrate the cross section. This means we have underestimated the number of interactions. (But see the homework problem.)

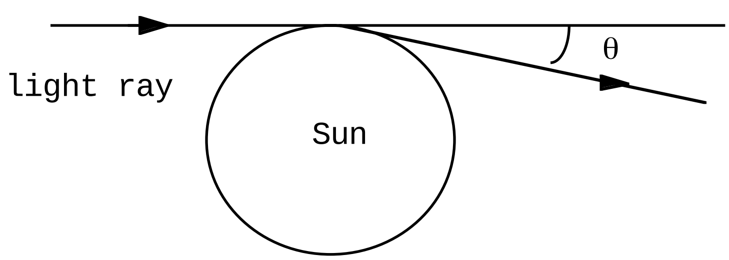
Simple treatment of light deflection

Since we talked a little about one general relativity effect last chapter (perihelion procession), will indicate another result here. For very small angular defections, the above relationship between b and θ becomes

$$b \simeq \frac{\mu k}{p_\infty^2} \frac{1}{\theta/2}$$

$$\Rightarrow \theta \simeq \frac{2\mu k}{p_\infty^2} \frac{1}{b} \qquad \left(\mu = \frac{m_1 m_2}{m_1 + m_2}, k = G m_1 m_2\right)$$

Use this to model a light ray grazing the radius of the Sun:



Therefore, take $b = R_\odot$, $m_2 = M_\odot$ (Sun's radius, mass.) What to do about $\mu \simeq m_1$, p_∞? For light,

$$p = \frac{h\nu}{c}.$$

Use the relativistic connection between mass and energy to replace (E = pc)

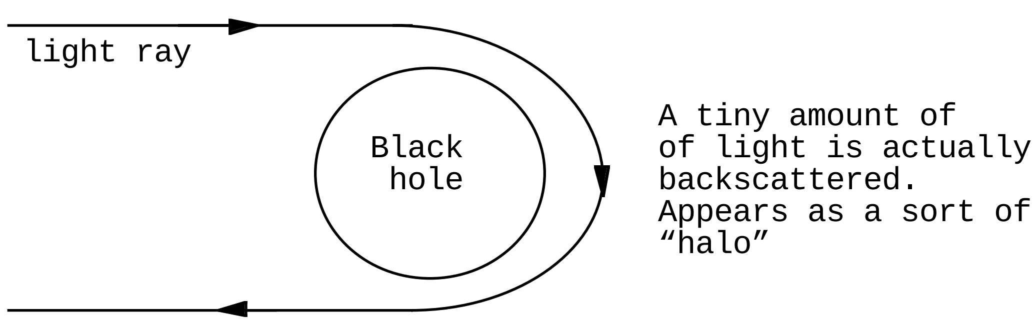
$$m_1 \rightarrow \frac{E}{c^2} = \frac{h\nu}{c^2},$$

$$\Rightarrow \theta \approx \frac{2\left(\frac{h\nu}{c^2}\right) GM_\odot \left(\frac{h\nu}{c^2}\right)}{\left(\frac{h\nu}{c^2}\right)^2 R_\odot} = \frac{2GM_\odot}{c^2 R_\odot}.$$

Almost correct, but too small by a factor of 2! (This was, in fact, Einstein’s original result, which he modified later.) Einstein’s general relativity gives the correct result:

$$\theta \approx \frac{4GM_\odot}{c^2 R_\odot} = 1.75'' \text{ sec of arc}.$$

Another new thing about light scattering in general relativity: glories.



Cross section cookbook

We have been discussing a special case of scattering, the inverse square force law. We need some cookbook formulas for doing other force laws. Will give differential cross section in the CM frame. Need general connection between b and θ. Start:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r).$$

Can "reduce the problem to quadratures." Solve for $\dot{r}$:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\big(E - U(r)\big) - \frac{\ell^2}{2\mu^2 r^2}}\,, \qquad (9.44)$$

$\uparrow$ have to pick correct root ($U_{eff}(r)$ picture good for this)

$$d\phi = \frac{d\phi}{dt}\frac{dt}{dr}dr = \frac{\dot{\phi}}{\dot{r}}dr\ , \text{ but } \dot{\phi} = \frac{\ell}{\mu r^2}\ , \text{ so}$$

$$d\phi = \pm\frac{\ell/r^2\ dr}{\sqrt{2\mu\big(E - U(r)\big) - \frac{\ell^2}{r^2}}}. \qquad (9.45)$$

Assuming $E > 0$, $\left(E = \frac{p_\infty^2}{2\mu}\right)$ and <u>integrating</u> on dr from r_{min} to $r = \infty$, we get ($\ell = b\,p_\infty$):

Θ defined positive
$\downarrow$

$$\Theta = +b\int_{r_{min}}^{\infty}\frac{dr/r}{\sqrt{\left(r^2 - b^2\right) - \frac{2\mu}{p_\infty^2}r^2U(r)}}. \qquad (9.46)$$

Warning: r_{min} itself is a function of b, in general.

Cookbook steps:

1. Evaluate $r_{min}(b)$.

2. Do integral.

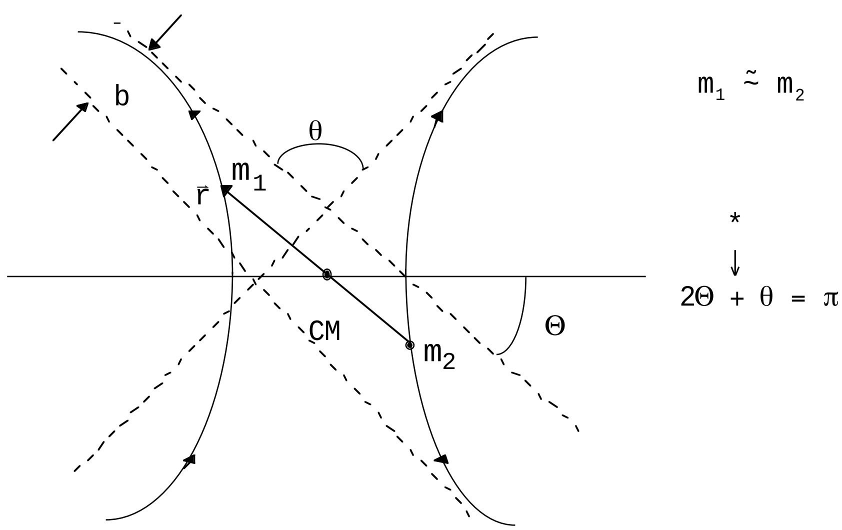
3. Use $\Theta = \dfrac{\pi + \theta}{2}$ or $\Theta = \dfrac{\pi - \theta}{2}$ in the attractive or repulsive cases, respectively, to find $b(\theta)$.

4. Plug in $\dfrac{d\sigma}{d\Omega} = \dfrac{b}{\sin\theta}\left|\dfrac{db}{d\theta}\right|$.

Another warning: does not work for all potentials, $U(r)$. Most require $\lim\limits_{r\to\infty} U(r) = 0$.

Just to get a feeling for using the cookbook method, do the repulsive Coulomb case (different from attractive case).

Picture:



Jump to result: (you will verify this in a problem)

$$\cos\Theta = \frac{\dfrac{k\mu}{p_\infty^2 b}}{\sqrt{1 + \left(\dfrac{k\mu}{p_\infty^2 b}\right)^2}}\,. \qquad (9.47)$$

Can write as

$$\cos^2\Theta = \frac{\left(\dfrac{k\mu}{p_\infty^2 b}\right)^2}{1 + \left(\dfrac{k\mu}{p_\infty^2 b}\right)^2},$$

$$\Rightarrow \tan^2\Theta = \left(\frac{p_\infty^2 b}{k\mu}\right)^2 \text{ or } b^2 = \left(\frac{k\mu}{p_\infty^2}\right)^2 \tan^2\Theta.$$

Choose $b = +\left(\dfrac{k\mu}{p_\infty^2}\right)\tan\Theta\,,\ \Theta = \dfrac{\pi-\theta}{2},$

$\uparrow$

different from attractive case

$$\Rightarrow b = \frac{k\mu}{p_\infty^2}\cot\frac{\theta}{2},$$

same as before.

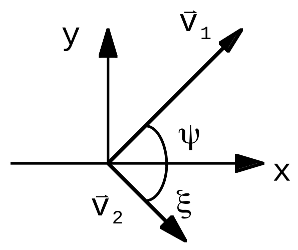
Connection between Lab and CM frames

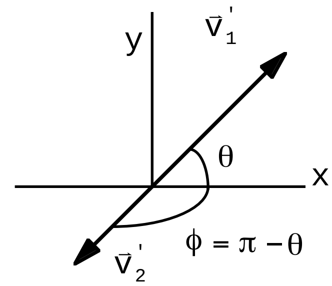
Only problem: cross sections usually not measured in the CM frame.

Lab Frame (usual expt. situation) CM Frame ($\vec{P} = 0$)

initial:

final:





Only showing initial and final velocities; not discussing dynamics but kniematics. We will assume <u>elastic</u> collisions - no heat generated or mass/energy lost. We will conserve both momentum and energy. We will also assume that all motion takes place in one plane (azimuthal symmetry, as we assumed before in the cross section discussion). We will take the angles ψ, ξ, etc. as positive; if not, we can always re-orient our axis so that they are.

Only difference between the two viewpoints: observed by two people who have a relative velocity, $\bar{V}$. All primed and umprimed quantities are related by $\bar{V}$:

<u>initial</u> <u>final</u>

$$\bar{u}_1 = \bar{u}_1' + \bar{V} \qquad (9.48a) \qquad \bar{v}_1 = \bar{v}_1' + \bar{V} \qquad (9.49a)$$

$$\bar{u}_2 = \bar{u}_2' + \bar{V} = 0 \qquad (9.48b) \qquad \bar{v}_2 = \bar{v}_2' + \bar{V} \qquad (9.49b)$$

$$\Rightarrow \bar{u}_2' = -\bar{V}$$

<u>notation</u>

$\bar{u}$ - initial velocity

$\bar{v}$ - final velocity

1,2 - which particle

prime, unprimed - CM, Lab frame, respectively

By definition,

$$\vec{R} = \frac{1}{M} \sum_i m_i \underset{\text{lab coordinates}}{\vec{r}_i},$$

$$\vec{R} = \frac{1}{m_1 + m_2} \left(m_1 \vec{r}_1 + m_2 \vec{r}_2 \right),$$

$$\Rightarrow \vec{V} = \frac{1}{m_1 + m_2} \left(m_1 \vec{u}_1 + m_2 \vec{u}_2 \right), \quad \left(\vec{u}_2 = 0 \right)$$

$$\vec{V} = \frac{m_1 \vec{u}_1}{m_1 + m_2}. \qquad \left(= -\vec{u}_2' \right) \tag{9.50}$$

In the CM coordinate system $\left(\vec{P}\right)^{\text{after}}_{\text{before}} = 0$. Therefore

$$m_1 \vec{u}_1' + m_2 \vec{u}_2' = 0 \Rightarrow m_1 u_1' = m_2 u_2', \tag{9.51}$$

$$m_1 \vec{v}_1' + m_2 \vec{v}_2' = 0 \Rightarrow m_1 v_1' = m_2 v_2'. \tag{9.52}$$

(u_1', v_1' etc. are magnitudes only.) Assume any potential that exists between the particles $\to 0$ as distances $\to \infty$; then, far enough apart, the energy is purely kinetic. Let

T_0' = total energy in CM frame

$$\left(T_0'\right)_{\text{before}} = \left(T_0'\right)_{\text{after}},$$

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2,$$

$$\Rightarrow m_1 u_1'^2 + \left(\frac{m_1}{m_2} \right)^2 m_2 u_1'^2 = m_1 v_1'^2 + m_2 \left(\frac{m_1}{m_2} \right)^2 v_1'^2,$$

$$\Rightarrow u_1' = v_1'. \tag{9.53}$$

Also

$$m_1\left(\frac{m_1}{m_2}\right)^2 u_2'^2 + m_2 u_2'^2 = \left(\frac{m_1}{m_2}\right)^2 m_1 v_2'^2 + m_2 v_2'^2,$$

$$\Rightarrow u_2' = v_2'. \qquad (9.54)$$

Thus, to summarize:

$$u_1' = v_1' = \frac{m_2}{m_1} u_2' = \frac{m_2}{m_1} v_2'.$$

$\Rightarrow$ only 1 unknown velocity magnitude in the CM frame. Other unknown: $\theta\,(\phi = \pi - \theta)$. Let's say we measure these 2 things in a given collision. How are they related to quantities in the Lab frame? From before,

$$\bar{V} = \frac{m_1 \bar{u}_1}{m_1 + m_2} = -\bar{u}_2',$$

but

$$u_2' = \frac{m_1}{m_2} u_1',$$

$$\Rightarrow u_1 = \frac{m_1 + m_2}{m_1} \cdot \frac{m_1}{m_2} u_1' = \left(1 + \frac{m_1}{m_2}\right) \underbrace{\underline{\underline{u_1'}}}_{\text{known or measured, say}}. \qquad (9.55)$$

Not specified yet: $\bar{v}_1, \bar{v}_2$. We have,

$$\bar{v}_1 = \bar{v}_1' + \bar{V},$$

$$\bar{v}_2 = \bar{v}_2' + \bar{V}.$$

Have to start invoking angles now:

$$v_1^2 = v_1'^2 + V^2 + 2\bar{V} \cdot \bar{v}_1', \qquad \left(\begin{array}{c} \bar{V} = -\bar{u}_2' \\ \Rightarrow V = u_2' = \dfrac{m_1}{m_2} u_1' \end{array} \right)$$

$$v_2^2 = v_2'^2 + V^2 + 2\bar{V} \cdot \bar{v}_2'.$$

9.26

refer to the figures $\rightarrow$ $\begin{cases} 2\bar{V}\cdot\bar{v}_1' = 2Vv_1'\cos\theta = 2\dfrac{m_1}{m_2}u_1'u_1'\cos\theta \\ 2\bar{V}\cdot\bar{v}_2' = 2Vv_2'\cos(\pi-\theta) = -2Vv_2'\cos\theta \end{cases}$

$$= 2\frac{m_1}{m_2}u_1'\frac{m_1}{m_2}u_1'\cos\theta$$

$$v_1^2 = u_1'^2 + \left(\frac{m_1}{m_2}\right)^2 u_1'^2 + 2\frac{m_1}{m_2}u_1'^2\cos\theta,$$

$$\Rightarrow \quad v_1^2 = u_1'^2\left[1+\left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right)\cos\theta\right]. \tag{9.56}$$

$$v_2^2 = \left(\frac{m_1}{m_2}u_1'\right)^2 + \left(\frac{m_1}{m_2}u_1'\right)^2 - 2\left(\frac{m_1}{m_2}u_1'\right)^2\cos\theta,$$

$$v_2^2 = 2\left(\frac{m_1}{m_2}u_1'\right)^2(1-\cos\theta),$$

$$v_2^2 = 4u_1'^2\left(\frac{m_1}{m_2}\right)^2\sin^2\frac{\theta}{2}. \tag{9.57}$$

Therefore v_1, v_2 are known if u_1', θ are known. Only things left: ψ, ξ

① $\bar{v}_1 = \bar{v}_1' + \bar{V}$,

② $\bar{v}_2 = \bar{v}_2' + \bar{V}$.

① x : $v_1\cos\psi = v_1'\cos\theta + V$, (9.58)

y : $v_1\sin\psi = v_1'\sin\theta$. (9.59)

② x : $v_2\cos\xi = v_2'\underbrace{\cos(\pi-\theta)}_{-\cos\theta} + V$, (9.60)

y : $-v_2\sin\xi = -v_2'\underbrace{\sin(\pi-\theta)}_{\sin\theta}$. (9.61)

Divide $\frac{y}{x}$ from ①:

$$\tan\psi = \frac{v_1' \sin\theta}{v_1' \cos\theta + V} = \frac{\sin\theta}{\cos\theta + V/v_1'} . \tag{9.62}$$

But $$\frac{V}{v_1'} = \frac{\frac{m_1}{m} u_1'}{u_1'} = \frac{m_1}{m_2} \quad \Rightarrow \quad \tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2} . \tag{9.63}$$

Says that the connection between θ and ψ is <u>not</u> <u>unique</u> under some circumstances. Look at $\frac{m_1}{m_2} >> 1$:

$$\tan\psi \simeq \frac{m_2}{m_1} \sin\theta . \tag{9.64}$$

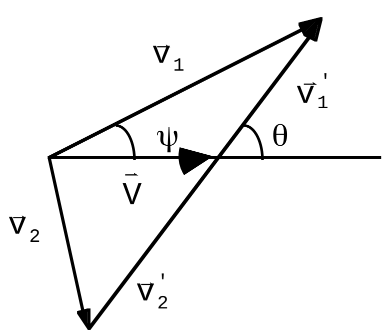
Given ψ, 2 values for θ. Other extreme, $\frac{m_1}{m_2} << 1$:

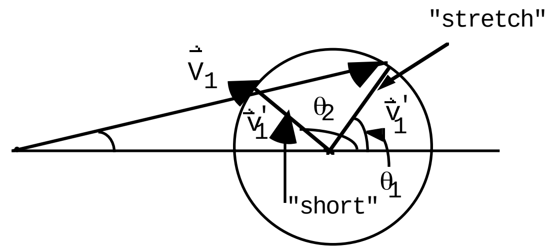
$$\tan\psi \simeq \tan\theta ,$$

$$\Rightarrow \quad \psi \simeq \theta . \tag{9.65}$$

One value of $\psi \Rightarrow$ one value of θ. Geometry helps:

$\frac{V}{v_1'} = \frac{m_1}{m_2} < 1$ $\qquad\qquad$ $\frac{V}{v_1'} = \frac{m_1}{m_2} > 1$



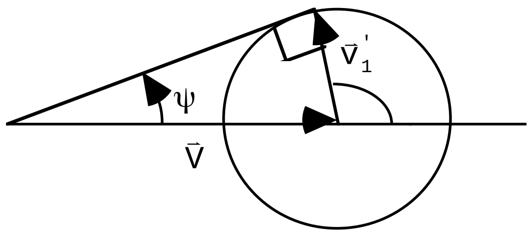


one value of $\psi \Rightarrow$
one value of θ

one value of $\psi \Rightarrow$
two values of θ

However, ψ <u>and</u> v_1 determine θ uniquely. Also, one value of $\theta \Rightarrow$ always one value of ψ. Can now see what the two extreme cases above correspond to geometrically.

More geometry : $\frac{m_1}{m_2} > 1$ case:



There is a max value of ψ in this case:

$$\sin \psi_{max} = \frac{v_1'}{V} = \frac{m_2}{m_1}. \tag{9.66}$$

In this case one value of $\psi \Rightarrow$ one value of θ. Rutherford case: $\frac{m_1}{m_2} << 1$, essentially scattering individual α-particles off the entire sample since the atomic centers are fixed **=>** unique determination.

Special case: $m_1 = m_2$

$$\tan \psi = \frac{\sin \theta}{\cos \theta + 1} = \frac{2 \sin \theta/2 \cos \theta/2}{1 + \cos^2 \theta/2 - \sin^2 \theta/2},$$

$$" \quad = \frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2} = \tan \theta/2,$$

$$\Rightarrow \psi = \theta/2. \tag{9.67}$$

Now consider equation ② above. Divide $\frac{y}{x}$:

$$\frac{v_2 \sin\xi}{v_2 \cos\xi} = \frac{v_2' \sin\theta}{-v_2' \cos\theta + V},$$

$$\Rightarrow \tan\xi = \frac{\sin\theta}{-\cos\theta + \frac{V}{v_2'}}.$$

However, $V = v_2'$,

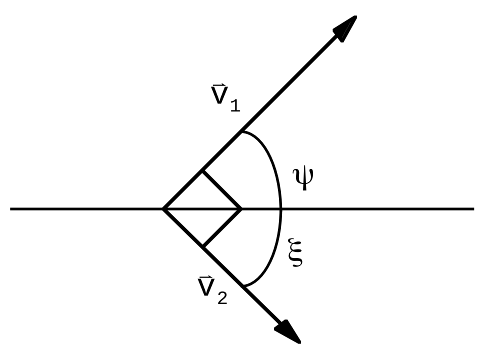
$$\tan\xi = \frac{\sin\theta}{-\cos\theta + 1} = \frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2},$$

$$\Rightarrow \tan\xi = \cot\theta/2 = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right),$$

$$\Rightarrow \xi = \frac{\pi - \theta}{2} \overset{\phi = \pi - \theta}{\Rightarrow} \xi = \frac{\phi}{2}. \qquad (9.68)$$

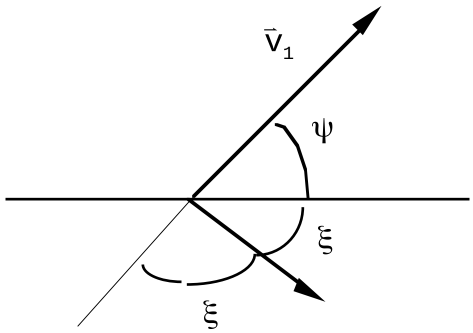
For the $m_1 = m_2$ case we had $\psi = \theta/2$, so

$$\xi + \psi = \frac{\pi}{2}\ ,\ \left(m_1 = m_2\right) \qquad (9.69)$$



In the $\frac{m_1}{m_2} << 1$

case:



$$2\xi = \pi - \theta,$$

$$\theta \approx \psi,$$

$$\Rightarrow 2\xi + \psi \approx \pi. \tag{9.70}$$

Let's find relationships between the various kinetic energies. Definitions first:

$$\begin{pmatrix} T_0 \\ T_0' \end{pmatrix} = \text{total K.E. in} \begin{pmatrix} \text{lab} \\ \text{CM} \end{pmatrix} \text{frame.}$$

$$\begin{pmatrix} T_1 \\ T_1' \end{pmatrix} = \text{final K.E. of } m_1 \text{ in} \begin{pmatrix} \text{lab} \\ \text{CM} \end{pmatrix} \text{frame.}$$

Similarly for T_2, T_2'. Of course

$$T_0' = T_1' + T_2' \quad , \quad T_0 = T_1 + T_2 .$$

Now

$$T_0 = \frac{1}{2} m_1 u_1^2, \tag{9.71}$$

$$T_0' = \frac{1}{2}\left(m_1 u_1'^2 + m_1 u_2'^2\right) \tag{9.72}$$

$$T_1' = \frac{1}{2} m_1 v_1'^2, \qquad \begin{pmatrix} T_2' = \frac{1}{2} m_2 v_2'^2 \\ \\ T_2 = \frac{1}{2} m_2 v_2^2 \end{pmatrix}$$

$$T_1 = \frac{1}{2} m_1 v_1^2,$$

Let's try to express them all in terms of T_0. Remember

$$u_1 = u_1'\left(1 + \frac{m_1}{m_2}\right) \quad , \quad u_1' = \frac{m_2}{m_1} u_2',$$

$$\Rightarrow \quad T_0' = \frac{1}{2}\left(\frac{m_1}{\left(1 + \frac{m_1}{m_2}\right)^2} + \frac{m_2\left(\frac{m_1}{m_2}\right)^2}{\left(1 + \frac{m_1}{m_2}\right)^2}\right) u_1^2,$$

$$\Rightarrow \quad T_0' = \frac{1}{2}\left[\frac{m_1 m_2^2 + m_2 m_1^2}{\left(m_1 + m_2\right)^2}\right] u_1^2 = \frac{1}{2}\underbrace{\mu}_{\text{reduced mass}} u_1^2 .$$

$$\Rightarrow T_0' = \frac{\mu}{m_1} T_0 . \tag{9.73}$$

so $T_0' < T_0$ always. Also

$$T_1' = \frac{1}{2} m_1 v_1'^2 = \frac{1}{2} m_1 u_1'^2 = \frac{\frac{1}{2} m_1 u_1^2}{\left(1 + \frac{m_1}{m_2}\right)^2},$$

$$\Rightarrow T_1' = \left(\frac{m_2}{m_1 + m_2}\right)^2 T_0 . \tag{9.74}$$

$$T_2' = \frac{1}{2} m_2 v_2'^2 = \frac{1}{2} m_2\left(\frac{m_1}{m_2} u_1'\right)^2 = \frac{\frac{1}{2} m_2\left(\frac{m_1}{m_2}\right)^2 u_1^2}{\left(1 + \frac{m_1}{m_2}\right)^2},$$

$$\Rightarrow T_2' = \frac{m_1 m_2}{\left(m_1 + m_2\right)^2} T_0 . \tag{9.75}$$

9.32

We have $\frac{T_1}{T_0} = \frac{v_1^2}{u_1^2}$. Go back to earlier v_1 expression:

$$v_1 = u_1' \sqrt{1 + \left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right)\cos\theta}\,,$$

$$\Rightarrow \quad v_1 = \frac{u_1}{\left(1 + \left(\frac{m_1}{m_2}\right)\right)} \sqrt{1 + \left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right)\cos\theta}\,,$$

$$\Rightarrow \quad \left(\frac{v_1}{u_1}\right)^2 = \left(\frac{m_2^2 + m_1^2 + 2m_1 m_2 \cos\theta}{(m_1 + m_2)^2}\right) = \frac{T_1}{T_0}\,. \tag{9.76}$$

What about in terms of ψ? Must express $\cos\theta$ as a function of ψ.

$$\tan\psi = \frac{\sin\theta}{\cos\theta + x}\,. \qquad \left(x \equiv \frac{m_1}{m_2}\right) \tag{9.77}$$

[“Aside 1”:

$$\Rightarrow x = \frac{\sin\theta}{\tan\psi} - \cos\theta\,,$$

$$x = \frac{\sin\theta\cos\psi - \cos\theta\sin\psi}{\sin\psi}\,,$$

$$x = \frac{\sin(\theta - \psi)}{\sin\psi}\,, \quad \text{useful later.}\ ]$$

$$\Rightarrow \quad \tan^2\psi = \frac{1 - \cos^2\theta}{\cos^2\theta + x^2 + 2x\cos\theta}\,,$$

$$\tan^2\psi\left(\cos^2\theta + x^2 + 2x\cos\theta\right) = 1 - \cos^2\theta\,,$$

$$\cos^2\theta\underbrace{\left(\tan^2\psi + 1\right)}_{\uparrow\ \frac{1}{\cos^2\psi}} + \cos\theta\left(2x\tan^2\psi\right) + \left(x^2\tan^2\psi - 1\right) = 0\,.$$

A quadratic equation in $\cos\theta$. Solution to $Ax^2 + Bx + C = 0$ is

$$x = \frac{-B \pm \sqrt{B^2 - 4ac}}{2A},$$

so

$$\cos\theta = \frac{-2x\tan^2\psi \pm \sqrt{4x^2\tan^4\psi + \dfrac{4\left(1 - x^2\tan^2\psi\right)}{\cos^2\psi}}}{\dfrac{2}{\cos^2\psi}},$$

$$\Rightarrow \quad \cos\theta = -x\sin^2\psi \pm \cos^2\psi\sqrt{x^2\tan^4\psi + \frac{\left(1 - x^2\tan^2\psi\right)}{\cos^2\psi}}\,. \qquad (9.78)$$

Inside the square root:

$$x^2\tan^2\psi\underbrace{\left(\tan^2\psi - \frac{1}{\cos^2\psi}\right)}_{-1} = -x^2\tan^2\psi\,,$$

$$\Rightarrow \cos\theta = -x\sin^2\psi \pm \cos^2\psi\sqrt{\frac{1}{\cos^2\psi} - x^2\tan^2\psi}\,,$$

$$\Rightarrow \quad \cos\theta = -x\sin^2\psi \pm \underset{\uparrow}{x\cos\psi}\sqrt{\frac{1}{x^2} - \sin^2\psi}\,. \qquad (9.79)$$

would have written $|\cos\psi|$, but because of the ± signs, this does not matter.

From above

9.34

$$\frac{T_1}{T_0} = \frac{\left[1 + \frac{1}{x^2} + \frac{2}{x}\cos\theta\right]}{\left(1 + \frac{1}{x}\right)^2}, \tag{9.80}$$

so

$$\frac{T_1}{T_0} = \frac{\left[1 + \frac{1}{x^2} - 2\sin^2\psi \pm 2\cos\psi\sqrt{\frac{1}{x^2} - \sin^2\psi}\right]}{\left(1 + \frac{1}{x}\right)^2}. \tag{9.81}$$

Notice the numerator is a perfect square:

$$\left(\cos\psi \pm \sqrt{\frac{1}{x^2} - \sin^2\psi}\right)^2 = \cos^2\psi + \frac{1}{x^2} - \sin^2\psi$$

$$\pm 2\cos\psi\sqrt{\frac{1}{x^2} - \sin^2\psi},$$

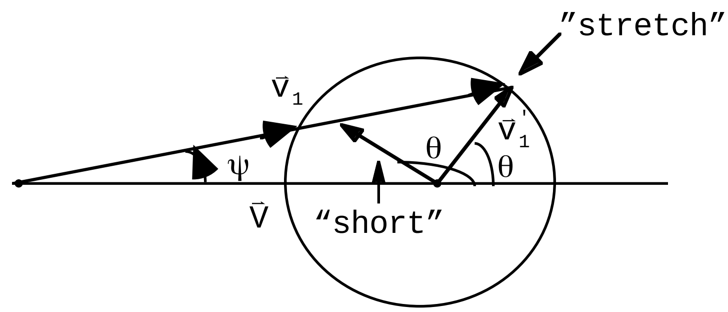
$$= 1 + \frac{1}{x^2} - 2\sin^2\psi \pm 2\cos\psi\sqrt{\frac{1}{x^2} - \sin^2\psi}.$$

$$\Rightarrow \frac{T_1}{T_0} = \frac{1}{\left(1 + \frac{1}{x}\right)^2}\left[\cos\psi \pm \sqrt{\frac{1}{x^2} - \sin^2\psi}\right]^2 \tag{9.82}$$

Since $\frac{T_1}{T_0} = \left(\frac{v_1}{u_1}\right)^2$,

$$\Rightarrow v_1 = \frac{u_1}{(1 + x)}\left[x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}\right]. \tag{9.83}$$

The ± corresponds to the 2 possibilites in Fig. 8-11(b). Remember, for $\frac{m_1}{m_2} = x > 1$, $\psi_{max} < \frac{\pi}{2} \Rightarrow \cos\psi > 0$. So we have



$$v_1 = \frac{u_1}{(1+x)}\Big[x\cos\psi \overset{\text{"stretch"}}{\underset{\text{"short"}}{\pm}} \sqrt{1 - x^2\sin^2\psi}\Big] \quad (x > 1)$$

Confirmation: "short" case v_1 should $\to 0$ as $x \to 1^+$ (through larger values of x):

$$v_{1\,\text{short}} \to \frac{u_1}{2}\underbrace{\Big[\cos\psi - \sqrt{1 - \sin^2\psi}\Big]}_{\cos\psi - |\cos\psi|} = 0.$$

$\Rightarrow$ Only the "stretch" case survives for $x < 1$. (Can see that for $x < 1$ the negative root would make v_1 negative.) The same interpretation applies to the expression for $\frac{T_1}{T_0}$.

["Aside 2": From ①, y equation:

$$v_1 \sin\psi = v_1' \sin\theta$$

$$\Rightarrow \frac{\sin\theta}{\sin\psi} = \frac{v_1}{v_1'} = \frac{\dfrac{u_1}{\left(1+\frac{1}{x}\right)}\left[\cos\psi \pm \sqrt{\dfrac{1}{x^2\sin^2\psi}}\right]}{\dfrac{u_1}{(1+x)}},$$

$$\frac{\sin\theta}{\sin\psi} = x\cos\psi \; \underset{\substack{\uparrow \\ \text{"short"}}}{\overset{\substack{\text{"stretch"} \\ \downarrow}}{\pm}} \; \sqrt{1 - x^2\sin^2\psi}\,. \;\Big]$$

$\Big[$ "Aside 3": $\quad \cos\theta + x\sin^2\psi = \pm\cos\psi\sqrt{1 - x^2\sin^2\psi}\,,$

$$\Rightarrow \frac{\cos\theta}{\cos\psi} + x\,\frac{\sin^2\psi}{\cos\psi} = \pm\sqrt{1 - x^2\sin^2\psi}\,.$$

Now $x = \dfrac{\sin(\theta - \psi)}{\sin\psi}$ from "Aside 1". Therefore the ℓhs of the above becomes

$$\ell.\text{h.s} = \frac{\cos\theta}{\cos\psi} + \frac{\sin(\theta - \psi)\sin\psi}{\cos\psi}\,,$$

$$= \frac{\cos\theta}{\cos\psi} + \frac{\sin\theta\cos\psi\sin\psi - \sin\psi\cos\theta\sin\psi}{\cos\psi}\,,$$

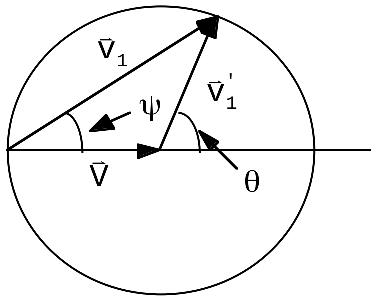
$$= \frac{\cos\theta}{\cos\psi}\left(1 - \sin^2\psi\right) + \sin\theta\sin\psi\,,$$

$$= \cos\theta\cos\psi + \sin\theta\sin\psi = \cos(\theta - \psi)\,.$$

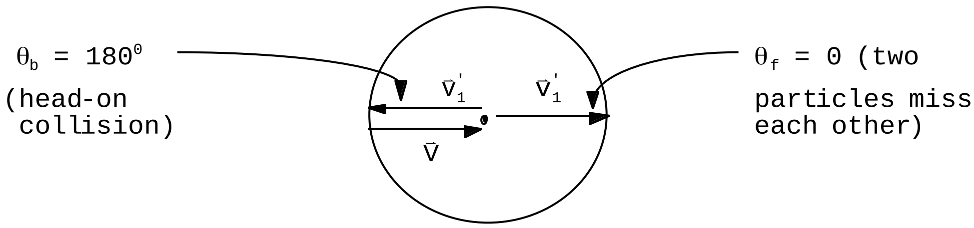
Therefore

$$\cos(\theta - \psi) = \underset{\substack{\uparrow \\ \text{"short"}}}{\overset{\substack{\text{"stretch"} \\ \downarrow}}{\pm}}\sqrt{1 - x^2\sin^2\psi}\,.\Big]$$

We get very simple results when $m_1 = m_2$. Then, for $\psi \neq 0$, we have only one θ value:



However, when ψ = 0, we can have two solutions for $\vec{v}_1'$:



The head on collision is also characterized by $\psi \to 90^0$.(Remember, $\psi = \frac{\theta}{2}$ for x = 1.) We have for x = 1,

$$\frac{T_1}{T_0} = \frac{1}{(1+1)^2}\left[2\cos\psi\right]^2 = \cos^2\psi\,. \qquad (9.84)$$

This goes to zero as $\psi \to 90^0$, which means in this limit m_1 comes to a complete halt after the collision. This fact is useful in nuclear reactor modulators, which slow down or stop neutrons. It says that the best way of stopping free neutrons is a material which contains light nuclei (like deuterium, so-called heavy water.) Obviously,

$$\frac{T_2}{T_0} = \sin^2\psi\,, \qquad (9.85)$$

in this limit.

A kinematical example in the Lab frame

Let’s do an example. Let's say a particle of mass m_1 scatters elastically from one of mass m_2 at rest in the Lab frame. The ratio $v_1/u_1 = f$ is given. Find the angle ψ through which m_1 is scattered. We have

$$v_1 = \frac{u_1}{(1 + x)}\left[x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}\right].$$

Thus,

$$\Rightarrow f(1 + x) = x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}$$

$$\Rightarrow f^2(1 + x)^2 + x^2\cos^2\psi - 2f\,x(1 + x)\cos\psi$$

$$= 1 - x^2\sin^2\psi$$

$$\Rightarrow f^2(1 + x)^2 + x^2 - 2f\,(1 + x)\,x\cos\psi = 1.$$

Solve for $\cos\psi$ in terms of x and f:

$$\cos\psi = \frac{x^2 - 1 + f^2(1 + x)^2}{2f\,x(1 + x)}.$$

To find the meaningful range for x, write

$$-1 \le \cos\psi \le 1.$$

Plug $\cos\psi = 1$ into the above (note 0<f<1 and that x is positive):

$$\Rightarrow x(\cos\psi = 1) = \frac{f + 1}{1 - f}.$$

The other limit is for $\cos\psi = -1$, which we get by simply letting $f \to -f$. Thus $\Rightarrow\ x(\cos\psi = 1) = \dfrac{1}{x(\cos\psi = -1)}$.

Rutherford scattering in the Lab frame

Let's find the differential cross sections in the Lab frame. Remember the meaning of $d\sigma$:

$$d\sigma = \frac{dN}{I} = 2\pi\, b|db|\,. \tag{9.86}$$

This is unchanged going from one frame to another. (At this stage there is no reference to angles.) Then

$$b = b(\theta)\,,\ \text{CM frame} \quad \text{or} \quad b = b(\psi)\,,\ \text{Lab frame.}$$

The connection between Lab and CM cross sections is,

$$\Rightarrow \underset{\text{Lab}}{\underbrace{\frac{d\sigma}{d\Omega}(\psi)}} = \underset{\text{CM}}{\underbrace{\left(\frac{d\sigma}{d\Omega}(\theta)\right)}} \frac{d\Omega(\theta)}{d\Omega(\psi)} \tag{9.87}$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta)\, \frac{\sin\theta d\theta}{\sin\psi d\psi} \tag{9.88}$$

Result of “Aside 1”:

$$\frac{\sin(\theta - \psi)}{\sin\psi} = x\,. \qquad \text{Do } \frac{d}{d\psi} \text{ on both sides:}$$

$$-\frac{\sin(\theta - \psi)}{\sin^2\psi}\cos\psi + \frac{\cos(\theta - \psi)}{\sin\psi}\left(\frac{d\theta}{d\psi} - 1\right) = 0,$$

$$\Rightarrow \frac{d\theta}{d\psi} - 1 = \frac{\sin(\theta - \psi)\cos\psi}{\cos(\theta - \psi)\sin\psi}\,. \tag{9.89}$$

Use

$$\sin(\theta - \psi) = x \sin \psi , \qquad \text{“Aside 1” again}$$

$$\cos(\theta - \psi) = +\sqrt{1 - x^2 \sin^2 \psi} , \qquad \text{“Aside 3”}.$$

$\uparrow$

specializing to $\underline{x < 1}$.

We find

$$\frac{d\theta}{d\psi} = 1 + \frac{x \sin \psi \cos \psi}{\sqrt{1 - x^2 \sin^2 \psi}\ \sin \psi}, \tag{9.90}$$

$$\Rightarrow \frac{d\theta}{d\psi} = \frac{x \cos \psi + \sqrt{1 - x^2 \sin^2 \psi}}{\sqrt{1 - x^2 \sin^2 \psi}}. \tag{9.91}$$

so (x < 1 case; using "Aside 2" now in (9.88):

$$\frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta) \ \frac{\sin\theta}{\sin\psi} \frac{d\theta}{d\psi} ,$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta(\psi)) \frac{\left[x\cos\psi + \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}}. \tag{9.92}$$

Also from $\dfrac{\sin(\theta - \psi)}{\sin\psi} = x$,

$$\Rightarrow \theta = \psi + \sin^{-1}(x \sin \psi) \tag{9.93}$$

If $x < 1$, can do an expansion in powers of x. To $0^{\underline{th}}$ order,

$$\theta \simeq \psi,$$

$$\frac{d\sigma}{d\Omega}(\psi) \simeq \frac{d\sigma}{d\Omega}(\theta)\Big|_{\theta=\psi} ,$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) \simeq \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4 \frac{\psi}{2}} . \tag{9.94}$$

Remember, p_∞ is measured in the CM frame. Connection with other quantities:

$$p_\infty = \mu \left|\dot{\bar{r}}_{\text{initial}}\right| = \mu\left(u_1' + u_2'\right),$$

$$m_1 u_1' = m_2 u_2' \Rightarrow u_2' = \frac{m_1}{m_2} u_1',$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

$$\Rightarrow p_\infty = \frac{m_1 m_2}{m_1 + m_2}\left(1 + \frac{m_1}{m_2}\right) \underset{\uparrow}{u_1'} = m_1 u_1' = \mu u_1 .$$

$$u_1' = \frac{u_1}{\left(1 + \frac{m_1}{m_2}\right)}$$

Notice for "b" large enough in the $x = \frac{m_1}{m_2} = \frac{V}{v_1'} > 1$ case, we must use the “stretch” case. When b is smaller than the value that causes scattering into angle $\psi_{MAX} = \sin^{-1}\left(\frac{m_2}{m_1}\right)$, then we must use the “short” result. Thus the analog for x > 1 is

$$\frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta(\psi)) \times \begin{cases} \dfrac{\left[x \cos\psi + \sqrt{1 - x^2 \sin^2\psi}\right]^2}{\sqrt{1 - x^2 \sin^2\psi}}, & b \geq b_0 \\[2ex] \dfrac{\left[x \cos\psi - \sqrt{1 - x^2 \sin^2\psi}\right]^2}{\sqrt{1 - x^2 \sin^2\psi}}, & b \leq b_0 \end{cases} \tag{9.95}$$

where $b_0 = b\left(\psi = \sin^{-1}\left(\frac{m_2}{m_1}\right)\right)$. Now

$$\frac{p_\infty^2}{2\mu} = \frac{\mu^2 u_1^2}{2\mu} = \frac{\mu u_1^2}{2} = T_0',$$

But

$$T_0' = \frac{m_2}{m_1 + m_2} T_0 = \frac{1}{1 + x} T_0$$

Therefore for x << 1 we have

$$\frac{d\sigma}{d\Omega}(\psi) \approx \frac{1}{16}\left(\frac{k}{T_0}\right)^2 \frac{1}{\sin^4 \frac{\psi}{2}}. \qquad (9.96)$$

written entirely in terms of Lab quantities. Now let's see what the first order correction to this expression (in x) is.

$$\theta = \theta_0 + x\theta_1 + \ldots,$$

$$\Rightarrow \theta_0 + x\theta_1 = \sin^{-1}(x \sin \psi) + \psi,$$

$$\sin^{-1} y = y + y\frac{3}{6} + \frac{3}{40} y^5 + \ldots, |y| < 1,$$

$$\Rightarrow \theta_0 + x\theta_1 \approx x \sin \psi + \psi,$$

$$\Rightarrow \begin{cases} \theta_0 = \psi \\ \theta_1 = \sin \psi. \end{cases}$$

$$\frac{\left[x \cos \psi + \sqrt{1 - x^2 \sin^2 \psi}\right]^2}{\sqrt{1 - x^2 \sin^2 \psi}} \approx \frac{\left[1 + x \cos \psi\right]^2}{1} \approx 1 + 2x \cos \psi,$$

$$\sin\frac{\theta}{2} \simeq \sin\left(\frac{\psi}{2} + \frac{x}{2}\sin\psi\right),$$

$$\simeq \sin\frac{4}{2} + \frac{x}{2}\sin\psi\cos\frac{\psi}{2},$$

$$= \sin\frac{4}{2} + x\sin\frac{\psi}{2}\cos^2\frac{\psi}{2},$$

$$= \sin\frac{4}{2}\left(1 + x\cos^2\frac{\psi}{2}\right),$$

$$\Rightarrow \sin^4\frac{\theta}{2} \simeq \sin^4\frac{\psi}{2}\left(1 + 4x\cos^2\frac{\psi}{2}\right).$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) \simeq \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\psi}{2}} \underbrace{\frac{(1 + 2x\cos\psi)}{\left(1 + 4x\cos^2\frac{\psi}{2}\right)}},$$

$$\simeq 1 + x\left(2\cos\psi - 4\cos^2\frac{\psi}{2}\right),$$

$$= 1 + x\left(-2\cos^2\frac{\psi}{2} - 2\sin^2\frac{\psi}{2}\right) = 1 - 2x.$$

Also

$$\frac{\mu k}{p_\infty^2} = \frac{k}{2\left(\frac{p_\infty^2}{2\mu}\right)} = \frac{k}{2T_0'} = \frac{k(1 + x)}{2T_0},$$

$$\Rightarrow \left(\frac{\mu k}{p_\infty^2}\right)^2 \simeq \left(\frac{k}{2T_0}\right)^2(1 + 2x).$$

Finally, then to first order in x:

$$\frac{d\sigma}{d\Omega}(\psi) \simeq \frac{1}{16}\left(\frac{k}{T_0}\right)^2 \frac{1}{\sin^4\frac{\psi}{2}} \underbrace{(1 + 2x)(1 - 2x)}_{\simeq 1}.$$

First order correction vanishes! Will not torture you with the next order, $\sim x^2$, correction.

Total cross section

Total cross section:

$$\sigma \equiv \int_{\substack{\text{all} \\ \text{angles}}} d\sigma = \int \frac{d\sigma}{d\Omega}(\theta)\, d\Omega \,. \qquad (9.97)$$

$$\left(d\Omega = \sin\theta d\theta d\phi\right)$$

For Rutherford:

$$\sigma = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \int_0^{2\pi} d\phi \int_{0^+}^{\pi} \frac{\overset{2\sin\theta/2 \cos\theta/2}{\overset{\downarrow}{\sin\theta}}\, d\theta}{\sin^4 \frac{\theta}{2}} \,.$$

(strictly speaking, the point $\theta = 0$ is excluded in the integration)

$$\sigma = 2\pi \left(\frac{\mu k}{p_\infty^2}\right)^2 \int_{0^+}^{\pi} \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} \frac{d\theta}{2},$$

$$\sigma = 3\pi \left(\frac{\mu k}{p_\infty^2}\right)^2 \left[-\frac{1}{2\sin^2\frac{\theta}{2}}\right]\Bigg|_{0^+}^{\pi} \rightarrow +\infty.$$

Understandable: particles are <u>always</u> deflected regardless of b value. Can see it in:

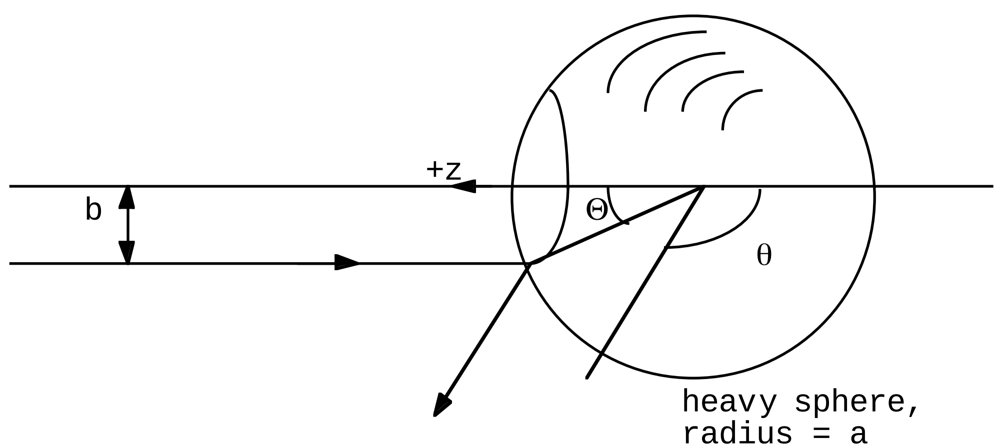
$$d\sigma = 2\pi\, b db,$$

$$\Rightarrow \sigma = 2\pi \int_0^{b_{max}} b\,db = \pi b_{max}^2. \tag{9.98}$$

where b_{max} is the maximum impact parameter which suffers an angular deflection $\neq 0$. Thus, in classical mechanics, the only type of force laws for which σ is finite are those of the form

$$F(r) = 0 \quad , \quad r > a$$

where a is some finite value of separation. Example of this possibility ($\theta = \psi$ here since the sphere is infinitely heavy):



Clearly, this problem has azimuthal symmetry if we take the +z axis as shown. We have

$$b = a \sin\Theta,$$

$$2\Theta + \theta = \pi.$$

9.46

<u>notice</u>: -sign
(repulsive scattering)

$$\Theta = \frac{\pi - \theta}{2},$$

$$b = a \sin\left(\frac{\pi - \theta}{2}\right) = a \cos \frac{\theta}{2}.$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left|\frac{db}{d\theta}\right| \quad ; \quad \frac{db}{d\theta} = -\frac{a}{2} \sin \frac{\theta}{2};$$

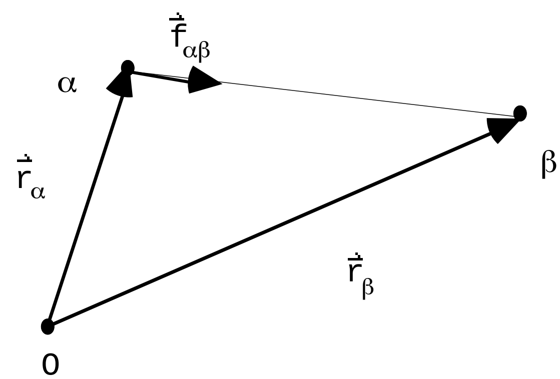
$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{a \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \cdot \frac{a}{2} \sin \frac{\theta}{2} = \frac{1}{4} a^2.$$

$$\Rightarrow \sigma = \int \frac{1}{4} a^2 d\Omega = \pi a^2 .$$

Just what we expect.

Chapter 9 Problems

1. A system of point particles interact via forces which follow the "strong form" of Newton's second law. That is, the force of β on α, $\vec{f}_{\alpha\beta}$, points along the instantaneous line connecting them, as shown.



Given the usual connection between fixed ($\vec{r}_\alpha$) and center of mass ($\vec{r}\,'_\alpha$) coordinates,

$$\vec{r}_\alpha = \vec{r}\,'_\alpha + \vec{R},$$

($\vec{R} = \frac{1}{M}\sum_\alpha \vec{r}_\alpha$, M is total mass) and the total force on α,

$$\dot{\vec{p}}_\alpha = \underbrace{\vec{F}^{(e)}_\alpha}_{\left(\substack{\text{external}\\ \text{force on } \alpha}\right)} + \underbrace{\sum_\beta \vec{f}_{\alpha\beta}}_{\left(\substack{\text{internal}\\ \text{force}}\right)},$$

show that the total torque, $\dot{\vec{L}} = \sum_\alpha \dot{\vec{L}}_\alpha$ ($\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$), for an external force of the form,

$$\vec{F}^{(e)}_\alpha = m_\alpha \vec{g},$$

is simply given by

$$\dot{\vec{L}} = \vec{R} \times \vec{F}^{(e)},$$

where $\vec{F}^{(e)} = \sum_{\alpha} \vec{F}^{(e)}_{\alpha}$ is the total external force.

2. The Rutherford differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{\mu k}{p_{\infty}^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} .$$

Therefore, the backscattering cross section is

$$\sigma_{back} = \frac{1}{4}\left(\frac{\mu k}{p_{\infty}^2}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \frac{1}{\sin^4\frac{\theta}{2}} .$$

In class I estimated the integral by setting $\left.\frac{1}{\sin^4\frac{\theta}{2}}\right|_{\theta=\pi} = 1$,

which resulted in an underestimate of σ_{back}. Now do this integral exactly. By what factor was the original estimate off? Will this improve the agreement with Rutherford's experiment?

3. Given that (T_0' is total CM energy)

$$b = \frac{C}{(T_0')^2} \cdot \frac{1}{\theta^2},$$

exactly for some unknown central force law, find

(a) the CM differential cross section, $\frac{d\sigma}{d\Omega}(\theta)$.

(b) the number of particles backscattered (that is, with angles θ such that $\frac{\pi}{2} \leq \theta \leq \pi$) in the CM frame. (Call this N_{back}). Assume a known incoming particle flux, I.

4. Do the integration in (9.46) leading to (9.47) of the notes.

5. Consider scattering off of a weak potential U(r) such that $U(b) \ll E$, where b is the impact parameter and E is the total energy. Show that

$$r_{min} \approx b\left(1 + \frac{U(b)}{2E}\right)$$

which shows that $r_{min} < b$ for an attractive potential $(U(b) < 0)$ and vice versa, as one would expect.

6. Look up and plug values in ($M_\odot$, $R_\odot$ are the Sun's mass, radius)

$$\Delta\theta = \frac{4GM_\odot}{c^2}\frac{1}{R_\odot},$$

to get the angular deflection for starlight in <u>seconds of arc.</u>

7. Consider a head-on collision as seen from an unknown reference frame. Assume the ratio $x = \frac{m_1}{m_2}$ is known. It is observed that m_1 comes to a complete stop after the interaction. Assuming energy and momentum conservation, find the after/before ratio of m_2's kinetic energies in terms of x.

8. Derive

$$\tan\psi = \frac{\sin 2\xi}{\frac{m_1}{m_2} - \cos 2\xi}.$$

9. Derive the relation

9.50

$$\sin\psi = \frac{\sin\phi}{\sqrt{1+x^2-2x\cos\phi}}$$

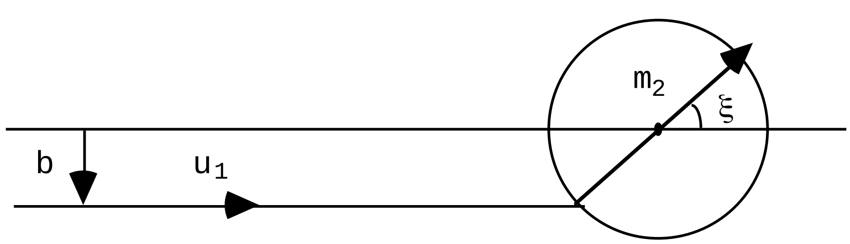
where $x = \frac{m_1}{m_2}$ and the angles ψ and ϕ are defined in the notes.

10. We know from Eq.(9.95) of the notes that the Laboratory cross section when $x > 1$ ($x \equiv \frac{m_1}{m_2}$), where m_1 is the incident and m_2 the target particle, has the form

$$\frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta(\psi)) \times \begin{cases} \dfrac{\left[x\cos\psi + \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}}, & b \geq b_0 \\ \dfrac{\left[x\cos\psi - \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}}, & b \leq b_0 \end{cases}$$

where $b_0 = b\left(\psi = \sin^{-1}\left(\frac{m_2}{m_1}\right)\right)$. Evaluate the critical impact parameter variable, b_0, for the Coulomb scattering potential, $U(r) = \pm \frac{k}{r}$, in terms of k, μ, p_∞ and x.

11. Consider scattering of a point mass m_1 off of a hard sphere of radius a and mass m_2. (The ratio $x = \frac{m_1}{m_2}$ is arbitrary.)



radius = a

The angle of the sphere's (m_2's) deflection in the laboratory frame (m_2 initially stationary) is given by $\xi = \sin^{-1}\left(\frac{b}{a}\right)$, where b is the impact parameter.

(a) Show that the cross section evaluated in the CM frame of reference is a constant.

(b) Find the deflection angle θ for m_1 in the CM frame if the impact parameter is $b = \frac{a}{\sqrt{2}}$.

12. Show that (9.92) of the notes can be written more simply as

$$\frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta)\ \left(\frac{v_1}{v_1'}\right)^2\ \frac{v_1 v_1'}{\vec{v}_1' \cdot \vec{v}_1'}.$$

13. Given a *center of mass* differential cross section,

$$\frac{d\sigma}{d\Omega} = A\cos^2\theta\ ,$$

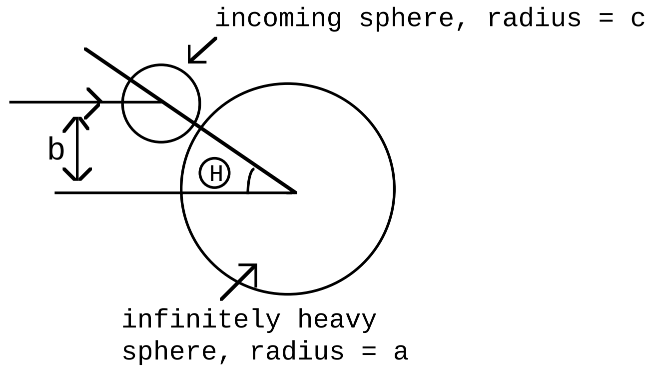
(θ is the deflection angle of m_1 in the CM frame) and a particle flux, I, find the number of backscattered particles ($\frac{\pi}{2} < \psi < \pi$) per unit time in the *lab* frame. Assume $m_2 >> m_1$ where m_1 is the mass of the incident particle.

14. <u>Given</u> a CM differential cross section

$$\frac{d\sigma}{d\Omega}(\theta) = a\left(\frac{1 + \cos\left(\frac{\theta}{2}\right)}{2}\right)$$

a particle flux I, and $m_1=m_2$, find the number of incident particles <u>backscattered</u> ($\pi \le \psi \le \frac{\pi}{2}$) per unit time in the <u>lab frame</u> (call it N_{back}). [Hint: first get the laboratory differential cross-section.]

15. Hard sphere scattering:



Assuming a symmetrical scattering event, find $\frac{d\sigma}{d\Omega}$ and the total cross section.

Other Problems

16. Escape velocity is the minimum speed a particle needs to escape a planet or star, starting from it's surface. Previously, we estimated the angular deflection of a light beam traveling near the Sun, treating the light as if it were an ordinary massive particle. In the same spirit, find a formula for the maximum radius of a star of mass M from which light, traveling at the speed of light, c, may no longer escape. (This is called the "Schwartzchild radius" of the star.)

17. We estimated in the notes the fraction of 5 MeV alpha particles backscattered ($\frac{\pi}{2} < \theta < \pi$) from a target made of gold foil and found that about one in every 21,000 particles should backscatter. (This was changed to about one in every

10,500 after we corrected the integral.) Suppose Rutherford wanted to make a lead shield to protect the other experiments in his lab. Note that for lead, $Z = 82$ and $m_{Pb} = 207$ amu, and that the density is $\rho_{Pb} = 11.4\ \frac{\text{gm}}{\text{cm}^3}$. Estimate the minimum thickness of the lead that would shield against these alpha particles (in centimeters).