Chapter 12

Coupled dynamical equations

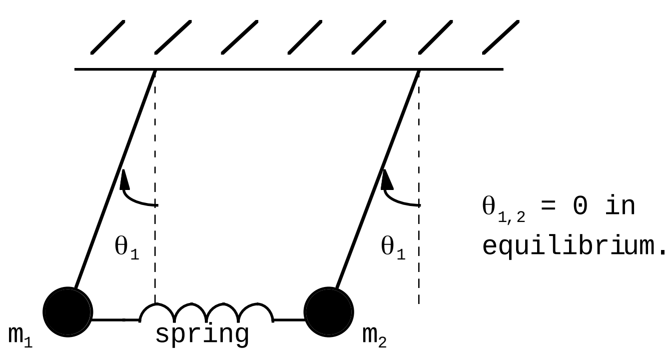
In a number of cases we have derived coupled linear differential equations. We have treated such systems as special cases, but it is now time to develop some general techniques for handling them. Mathematically, we will encounter here both the use of complex numbers (which were used in the Focault pendulum discussion and various differential equation solutions) as well as the eigenvalue-eigenvector matrix algebra of the last chapter. Examples of systems these techniques will cover: conservative, linear mechanical or electrical oscillations, molecular vibrations, approximate planetary motions, and many more. We will also use stability analysis and the concept of generalized coordinates.

I will first introduce the theory and then will work out a number of examples to illustrate the general techniques.

Let us consider a system with generalized coordinates q_j ($j = 1,\ldots, n$; n = no. of "degrees of freedom"), related formally to the $x_{\alpha i}$ by

$$x_{\alpha i} = x_{\alpha i}(q_j) \Rightarrow \frac{\partial x_{\alpha i}}{\partial t} = 0. \tag{12.1}$$

Usually, the q_j we will consider will be measured from equilibrium positions or lengths. For example,



Let us also only consider $U = U(q_j)$. This means we are limiting our discussion to conservative systems for which H = T + U = const. in time. Another limitation of the following discussion will be the assumption that the system is linear. That is, we will only consider potentials that are quadratic in the generalized coordinates, or are approximately quadratic. For small oscillations, we have

$$U(q_1, q_2, \ldots, q_n) = U_0 + \sum_k \frac{\partial U}{\partial q_k}\bigg|_0 q_k + \frac{1}{2} \sum_{j,k} \underbrace{\frac{\partial^2 U}{\partial q_j \partial q_k}}_{\substack{= A_{jk} = A_{kj} \\ \text{(constants)}}} \bigg|_0 q_j q_k + \ldots, \tag{12.2}$$

where higher order terms will be neglected. In addition, we will assume or require that (q_{10}, q_{20},...etc. are equilibrium q_i's):

1. $U_0 = U(q_1 = q_{10},\ q_2 = q_{20}, \ldots) = 0$.

2. $\dfrac{\partial U}{\partial q_k}\bigg|_0 = 0$ for each k.

3. $\dfrac{\partial^2 U}{\partial Q^2}\bigg|_0 > 0$, where Q is any linear combination of the generalized coordinates, q_i.

Condition 1 is no restriction in the applicability of the analysis since the absolute value of the potential U is always arbitrary up to an overall constant. Conditions 2 & 3 above are just conditions for stable equilibrium in a system with n degrees of freedom. Notice that condition 3 above (which is a sufficient, rather than necessary, condition for stability) is equivalent to the requirement that $U(q_i) > 0$ for quadratic potentials, given $U_0 = 0$. We can show this as follows. Assume ($i = 1, \ldots, n$)

$$Q = \sum_i x_i q_i \,, \tag{12.3}$$

where the x_i are arbitrary constants. Then

$$\frac{\partial^2 U}{\partial Q^2} = \frac{\partial}{\partial Q} \sum_i \frac{\partial U(q_i)}{\partial q_i} \frac{\partial q_i}{\partial Q} \,,$$

but $\frac{\partial q_i}{\partial Q} = \frac{1}{x_i}$, so

$$\frac{\partial^2 U}{\partial Q^2} = \frac{\partial}{\partial Q} \sum_i \frac{\partial U(q_i)}{\partial q_i} \frac{1}{x_i} = \sum_{i,j} \frac{\partial^2 U}{\partial q_i \partial q_j} \frac{1}{x_i x_j} \,. \tag{12.4}$$

Condition 3 now gives,

$$\sum_{i,j} \frac{\partial^2 U}{\partial q_i \partial q_j}\bigg|_0 \frac{1}{x_i x_j} > 0 \,. \tag{12.5}$$

(The sum in (12.5) is only over the nonzero x_i values.) Since the x_i are arbitrary (but nonzero) constants, this is the same as

$$U(\tilde{q}_i) > 0, \tag{12.6}$$

where $\tilde{q}_i = \frac{1}{x_i}$ (see Eq.(12.2)).

Because of our condition $\frac{\partial x_{\alpha i}}{\partial t} = 0$, we know from Chapter 7 for a system of point particles, for example, that we may write (the m_{jk} here were called a_{jk} there)

$$T = \frac{1}{2} \sum_{j,k} m_{jk} \dot{q}_j \dot{q}_k \,, \tag{12.7}$$

where

$$m_{jk} = \sum_{\alpha,i} m_\alpha \frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k} = m_{kj}. \tag{12.8}$$

Note that in general $m_{jk} = m_{jk}(q_i)$ (functions of the q_i), unlike the A_{jk} above, which are just constants. However, consistent with the restriction to quadratic terms in U, we will restrict ourselves to quadratic terms in the $\dot{q}_j$. This means that we will expnad

$$m_{jk}(q_i) = m_{jk}\Big|_0 + \underbrace{\sum_\ell \frac{\partial m_{jk}}{\partial q_\ell}\Big|_0 q_\ell + \ldots,}_{\text{neglect}} \tag{12.9}$$

and neglect everything except the first term, which is assumed non-zero. The reason is that terms like $\sim q_\ell \dot{q}_j \dot{q}_k$ in the Euler-Lagrange equations produce nonlinear terms in the differential equations. (Neglecting such nonzero terms give additional restrictions on what is meant by "small oscillations".)
Notice also that, in general

$$\frac{\partial T}{\partial q_\ell} = \frac{1}{2}\sum_{j,k} \frac{\partial m_{jk}}{\partial q_\ell}\dot{q}_j\dot{q}_k,$$

so that with (12.9) we have

$$\frac{\partial T}{\partial q_\ell} = 0.$$

After these restrictions, the general form of our Lagrangian in generalized coordinates, q_i, is

$$L = T - U = \frac{1}{2}\sum_{j,k} \underset{\substack{\uparrow \\ \text{real constants}}}{m_{jk}}\, \dot{q}_j\dot{q}_k - \frac{1}{2}\sum_{j,k} \underset{\substack{\uparrow \\ \text{real constants}}}{A_{jk}}\, q_j q_k. \tag{12.10}$$

Equations of motion are given by

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0. \qquad (12.11)$$

We have

$$\frac{\partial U}{\partial q_k} = \sum_j A_{jk} q_j\,, \qquad (A_{jk} = A_{kj}) \qquad (12.12)$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_j m_{jk} \dot{q}_j\,. \qquad (m_{jk} = m_{kj}) \qquad (12.13)$$

$$\Rightarrow \sum_j \left(A_{jk} q_j + m_{jk} \ddot{q}_j\right) = 0. \qquad (12.14)$$

Although we have derived these equations in the context of particle oscillations, the small oscillations of many realistic rigid body systems can be so characterized as long as the Lagrangian may be put into the form of (12.10) above.

Eigenvalue/eigenvector solution

We will use complex number analysis to simplify the solution of this sytem of equations. Under conditions 1, 2, 3 above, we know that the solutions are oscillations. Therefore assume

$$q_j(t) = \underbrace{a_j}_{\text{complex (involves two undetermined constants)}} e^{i\underbrace{\omega}_{\text{real frequency}} t} \qquad (12.15)$$

where the real part of the right hand side is understood. Plugging this back above, we find

$$\sum_j \left(A_{jk} - \omega^2 m_{jk}\right) a_j = 0. \qquad (12.16)$$

[This is almost the same form as the equations which determine principal axes:

$$\sum_j \left(I_{ij} - I'\delta_{ij}\right)\omega_j = 0.]$$

Again, the condition that there is a nontrivial solution for the a_j is that the determinant of the matrix (A_{ij} - ω^2 m_{ij}) be zero:

$$\det\begin{pmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} \ldots \\ A_{21} - \omega^2 m_{21} & A_{22} - \omega^2 m_{22} \ldots \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} = 0. \qquad (12.17)$$

this gives an $n^{\underline{th}}$ order equation in ω^2, the roots of which will be labeled at ω_r^2, r = 1,2,...n. These are called the <u>eigenfrequencies</u> or <u>characteristic frequencies</u>. In general, the above is more difficult than solving the analogous equation for the principal moments of inertial since the ω^2 terms (the analog of the priciple moments, I') enter off-diagonal elements of the matrix also. However, one is solving the same <u>mathematical</u> problem in either case.

Analogy:

	Inertia tensor		Coupled Equations
<u>eigenvalues</u>:	I_i	$\leftrightarrow$	ω_r^2
	(i = 1,2,3)		(r = 1,2,...,n)
	(principle axes)		(oscillatory modes)
<u>eigenvectors</u>:	$\vec{\omega}^i$ (ω_j^i)	$\leftrightarrow$	$\vec{a}^r$ (a_j^r)
	j = {1,2,3}		j = {1,2,...,n}

(vector components) (vector components)

Just as there are directions in physical space which render the inertia tensor diagonal, there are directions in mode space, associated with the eigen frequencies ω_r^2, for which the motions uncouple. We get the $\bar{a}^r$ by the same procedure as in the last chapter: substitute a known ω_r^2 in the algebraic equations, and then solve for the ratios

$$a_1^r : a_2^r : a_3^r : \ldots : a_n^r$$

for a given eigenvalue, ω_r^2. As before, the overall normalization of the $\bar{a}^r$ (chosen real) are arbitrary. This makes physical sense here also since this corresponds to the amplitude of the motion, which is not determined by the equations of motion but by the initial condition. Since there are n $\bar{a}_r$'s, we can require n condtions to arbitrarily normalize them. In the inertia tensor case, we required that

$$\vec{\omega}^i \cdot \vec{\omega}^i = 1 \ , \ i = 1,2,3.$$

Here, we require,

$$\sum_{j,k} m_{jk} a_j^r a_k^r = 1 \ , \ r = 1,2,\ldots n.. \qquad (12.18)$$

(That this combination is always positive can be shown from the positivity of T, the kinetic energy.) Also, we can prove that (just a generalization to an n-dimensional space of the similar proof in Ch.11 for the inertia tensor)

$$\sum_{j,k} m_{jk} a_j^r a_k^s = 0 \ , \ (r \neq s) \qquad (12.19)$$

for $\omega_r^2 \neq \omega_s^2$, and that for degenerate roots ($\omega_r^2 = \omega_s^2$) we may always choose this equation to hold. The above two conditions can be written together as

$$\sum_{j,k} m_{jk}\, a_j^r a_k^s = \delta_{rs}\,. \tag{12.20}$$

Also, we can show that the ω_r^2 are all real. In fact one can show, under conditions 1,2,3 above, that $\omega_r^2 > 0$.[1] Without loss of generality, one may further choose $\omega_r > 0$.

The general solution is now given by (real part still understood)

$$q_j(t) = \sum_{r=1}^{n} \beta_r\, a_j^r\, e^{i\omega_r t}\,, \tag{12.21}$$

where the a_j^r are now all determined (up to an overall sign) and β_r are so called “scale factors,” which in general, are complex. (We still have not built in the initial conditions, so we still need two arbitrary constants for a second order differential equation.) The β_r are the amplitudes associated with the $r^{\underline{th}}$ eigenmode as determined by the initial conditions. If only a single β_r is nonzero,

$$\beta_r = 0,\ r \neq k,\ \beta_k \neq 0,$$

then only a single eigenmode of the system has been excited and the solution of the motion is particularly simple.

$$q_j(t) = \beta_k\, a_j^k\, e^{i\omega_k t} \ \text{(no k sum)}$$

Or, introducing the “normal coordinates”,

$$n_k \equiv \beta_k\, e^{i\omega_k t}\,, \tag{12.22}$$

we have

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[1] Notice that we could replace condition 3 with the alternate condition that the system of equations, (12.17), for small oscillations yield only real nonzero eigenfrequencies, ω^r. This was the same condition for stability seen in Ch.2.

$$q_j(t) = n_k a_j^k . \qquad (12.23)$$

In general,

$$q_j(t) = \sum_k n_k a_j^k . \qquad (12.24)$$

Expressed in normal coordinates, one has (using the orthogonality condition, (12.20), as well as (12.16))

$$T = \frac{1}{2} \sum_r \dot{n}_r^2 \ , \ U = \frac{1}{2} \sum_r \omega_r^2 n_r^2 \, ,$$

$$\frac{\partial L}{\partial n_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{n}_r}\right) = 0,$$

$$\Rightarrow \ddot{n}_r + \omega_r^2 n_r = 0. \qquad (12.25)$$

This in turn shows that the energy associated with each normal mode is constant since for the $r^{\underline{th}}$ mode,

$$\int dt\, \dot{n}_r \times \left(\ddot{n}_r + \omega_r^2 n_r = 0\right),$$

$$\Rightarrow E = \text{constant} = \frac{1}{2}\dot{n}_r^2 + \frac{1}{2}\omega_r^2 n_r^2$$

$$= T_r + U_r \ . \qquad (12.26)$$

Final cookbook recipe for solving a system for small oscillations:

1. Write the Lagrangian, L, of the system in terms of generalized coordinates, q_i, and find the A_{jk} and m_{jk} either by using the explicit formulas ($A_{jk} = -\left.\frac{\partial^2 L}{\partial q_j \partial q_k}\right|_0$, $m_{jk} = \left.\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}\right|_0$) or by comparing to the form of the Lagrangian, Eq.(12.10).

2. Form the matrix $\left(A_{jk} - \omega^2 m_{jk}\right)$ and find the n eigenvalues, ω_r^2.

3. Determine the ratios $a_1^r : a_2^r : a_3^r : ... : a_n^r$ and normalize the a_i^r according to Eq.(12.20).

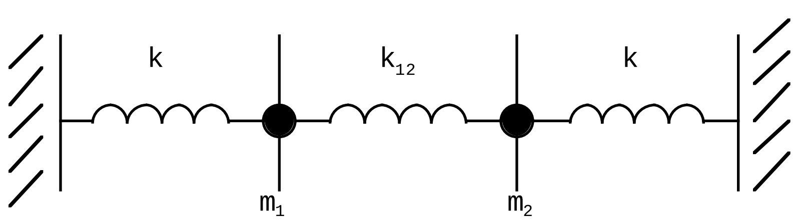
4. Write the general solution for the q_i as in Eq.(12.21). Physical interpretation follows from an examination of the motion of the normal modes (setting all β_r=1).

5. Apply the initial conditions and find the β_r.

If T and U for a system are not given, but the equations of motion are, then obviously we may skip step 1 and instead form the characteristic equation using the ansatz Eq.(12.15) for the q_i. As a faster procedure, note that we may also skip the normalization condition, (12.20), and determine $q_j(t) = \sum_r a_j^r e^{i\omega_r t}$ (β_r absorbed in the new a_j^r) directly from the ratios in step 3 and the initial conditions.

Example

Enough of the theory, let's do some examples to make this more understandable. Coupled masses example (this is mathematically the same for small oscillations as the coupled pendulum problem at the start of this Chapter with $M=m_1=m_2$, $x \to \ell\theta$, and $k = \frac{mg}{\ell}$, ℓ being the pendulum length):



$m_1 = m_2 = M$

x_1, x_2 (playing the role of the q's) measured from <u>equilibrium</u> positions. Step ①:

$$T = \frac{1}{2} M \left(\dot{x}_1^2 + \dot{x}_2^2\right).$$

$$\Rightarrow \quad m_{11} = m_{22} = M \quad , \quad m_{12} = m_{21} = 0,$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k_{12}\left(x_2 - x_1\right)^2,$$

$$\Rightarrow A_{11} = \frac{\partial^2 U}{\partial x_1^2}\bigg|_0 = k + k_{12}, \quad A_{22} = \frac{\partial^2 U}{\partial x_2^2}\bigg|_0 = k + k_{12},$$

$$A_{12} = \frac{\partial^2 U}{\partial x_1 \partial x_2}\bigg|_0 = -k_{12}, \quad A_{21} = -k_{12}.$$

Step ②:

$$\det \begin{pmatrix} k + k_{12} - \omega^2 M & -k_{12} \\ -k_{12} & k + k_{12} - \omega^2 M \end{pmatrix} = 0,$$

$$\Rightarrow \left(k + k_{12} - \omega^2 M\right)^2 - k_{12}^2 = 0,$$

$$\Rightarrow \omega_1 = \left(\frac{k + 2k_{12}}{M}\right)^{1/2} \quad , \quad \omega_2 = \left(\frac{k}{M}\right)^{1/2}.$$

Step ③:

Equations for eigenvectors, r = 1 case:

$$\begin{cases} \left(A_{11} - \omega_1^2 m_{11}\right) a_1^1 + \left(A_{21} - \omega_1^2 m_{21}\right) a_2^1 = 0, \\ \left(A_{12} - \omega_1^2 m_{12}\right) a_1^1 + \left(A_{22} - \omega_1^2 m_{22}\right) a_2^1 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} -k_{12} a_1^1 - k_{12} a_2^1 = 0 \\ -k_{12} a_1^1 - k_{12} a_2^1 = 0 \end{cases} \Rightarrow \underline{\underline{a_1^1 = -a_2^1}}.$$

12.12

r = 2 case:

$$k_{12}a_1^2 - k_{12}a_2^2 \Rightarrow \underline{\underline{a_1^2 = a_2^2}}\,.$$

Normalization:

$$\sum_{j,k} m_{kj}\, a_j^r a_k^s = \delta_{rs}\,.$$

overall sign
undetermined

r = s = 1: $$M\left(a_1^1\right)^2 + M\left(a_2^1\right)^2 = 1 \Rightarrow a_1^1 = +\frac{1}{\sqrt{2M}}$$

r = s = 2: $$M\left(a_1^2\right)^2 + M\left(a_2^2\right)^2 = 1 \Rightarrow a_1^2 = +\frac{1}{\sqrt{2M}}$$

same comment

Step ④:

$$\begin{cases} x_1(t) = \dfrac{1}{\sqrt{2M}}\,\mathrm{Re}\left(\beta_1\, e^{i\omega_1 t} + \beta_2\, e^{i\omega_2 t}\right), \\ x_2(t) = \dfrac{1}{\sqrt{2M}}\,\mathrm{Re}\left(-\beta_1\, e^{i\omega_1 t} + \beta_2\, e^{i\omega_2 t}\right). \end{cases}$$

[Note we have ($n_r(t) = \beta_r\, e^{i\omega_r t}$)

$$x_1 = \sum_r a_1^r n_r = a_1^1 n_1 + a_1^2 n_2\,,$$

$$\Rightarrow x_1 = \frac{1}{\sqrt{2M}}\left(n_1 + n_2\right).$$

$$x_2 = \sum_r a_2^r n_r = a_2^1 n_1 + a_2^2 n_2\,,$$

$$\Rightarrow x_2 = \frac{1}{\sqrt{2M}}\left(-n_1 + n_2\right).$$

Thus $n_1 = \sqrt{\dfrac{M}{2}}\left(x_1 - x_2\right)$, $n_2 = \sqrt{\dfrac{M}{2}}\left(x_1 + x_2\right)$.

Physical interpretation of the modes follows from these results. Set $n_1 = 0 \Rightarrow x_1 = x_2$ for mode 2 (“symmetrical mode”). Set $n_2 = 0$ $\Rightarrow x_1 = -x_2$ (“antisymmetrical mode:).]

Step ⑤: As an example, consider the initial condition,

$$x_1(0) = -x_2(0) = A, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0.$$

Then

real part of β_1, β_2

$$A = \frac{1}{\sqrt{2M}}\left(\beta_{1r} + \beta_{2r}\right)$$

$$-A = \frac{1}{\sqrt{2M}}\left(-\beta_{1r} + \beta_{2r}\right)$$

$$\Rightarrow \beta_{2r} = 0 \quad , \quad \beta_{1r} = \sqrt{2M}\,A$$

Also

imaginary part

$$\dot{x}_1(0) = -\frac{1}{\sqrt{2M}}\left(\omega_1\beta_{1i} + \omega_2\beta_{2i}\right) = 0,$$

$$\dot{x}_2(0) = \frac{1}{\sqrt{2M}}\left(\omega_1\beta_{1i} - \omega_2\beta_{2i}\right) = 0,$$

$$\Rightarrow \beta_{1i} = \beta_{2i} = 0.$$

(If we had chosen $\dot{x}_1(0) = -\dot{x}_2(0) \neq 0$, then we would have had $\beta_{2i} = 0$, $\beta_{1i} = -\dfrac{\sqrt{2M}}{\omega_1}\dot{x}(0)$, and mode 1 would still be the only mode excited.) Complete solution:

$$x_1 = A\cos(\omega_1 t) \quad , \quad x_2 = -A\cos(\omega_1 t).$$

If we had chosen $x_1(0) = x_2(0)$, $\dot{x}_1(0) = \dot{x}_2(0)$ would have excited the symmetrical mode (2) alone. General motion is a linear combination of the two modes. Complicated to understand, but simplifies in the cases $k_{12} \ll k$ ("weak coupling") and $k_{12} \gg k$ ("strong coupling"). Let's examine these cases.

Weak/strong coupling

Let's rewrite the above general solution to prepare to discuss weak coupling. Introduce

$$\omega_0 \equiv \frac{\omega_1 + \omega_2}{2} \quad , \quad \underset{\text{“beats”}}{\underset{\uparrow}{\omega_b}} \equiv \frac{\omega_1 - \omega_2}{2}.$$

Can show (mucho algebra) that the above general solutions for $x_1(t)$, $x_2(t)$ may be written as (solve for ω_1, ω_2 in terms of ω_0, ω_b, substitute above and use trigonometric identities),

$$x_1(t) = \frac{A_1}{\sqrt{2M}}\cos(\omega_0 t)\cos(\omega_b t) - \frac{A_2}{\sqrt{2M}}\cos(\omega_0 t)\sin(\omega_b t)$$
$$+ \frac{A_3}{\sqrt{2M}}\sin(\omega_0 t)\cos(\omega_b t) - \frac{A_4}{\sqrt{2M}}\sin(\omega_0 t)\sin(\omega_b t),$$

$$x_2(t) = -\frac{A_4}{\sqrt{2M}}\cos(\omega_0 t)\cos(\omega_b t) - \frac{A_3}{\sqrt{2M}}\cos(\omega_0 t)\sin(\omega_b t)$$
$$+ \frac{A_2}{\sqrt{2M}}\sin(\omega_0 t)\cos(\omega_b t) + \frac{A_1}{\sqrt{2M}}\sin(\omega_0 t)\sin(\omega_b t).$$

where

$$A_1 = \beta_{1r} + \beta_{2r} \quad , \quad A_2 = \beta_{1i} - \beta_{2i},$$

$$A_3 = -\beta_{1i} - \beta_{2i} \quad , \; A_4 = \beta_{1r} - \beta_{2r}$$

(Please confirm this.) Notice that there are still only 4 undetermined (real) constants, A_1, A_2, A_3, A_4. Let us now plug in the initial conditions,

$$x_1(0) = D, \; x_2(0) = 0, \; \dot{x}_1(0) = \dot{x}_2(0) = 0.$$

These initial conditions excite both modes 1 and 2, as opposed to the previous set. Plugging in above, we find

$$A_1 = \sqrt{2M}\; D, \; A_2 = A_3 = A_4 = 0.$$

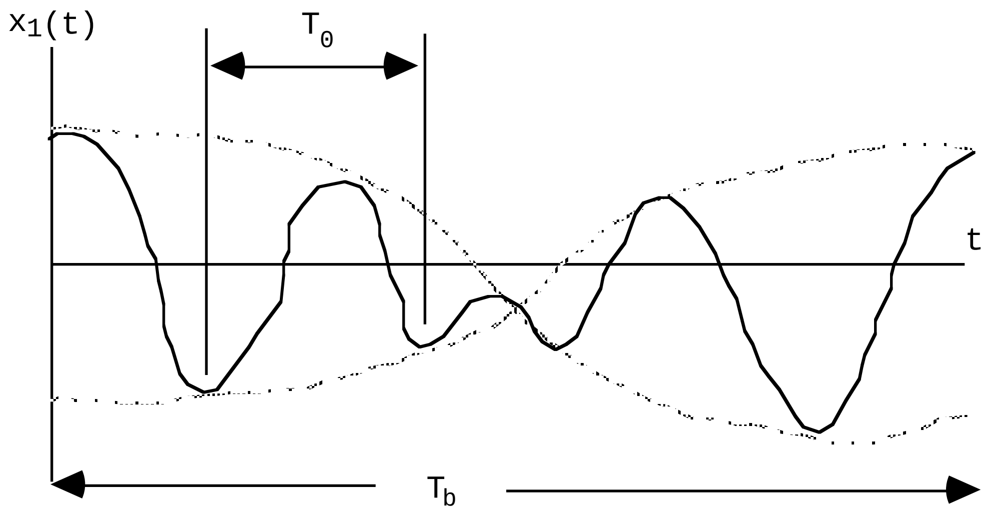
So the general solution is

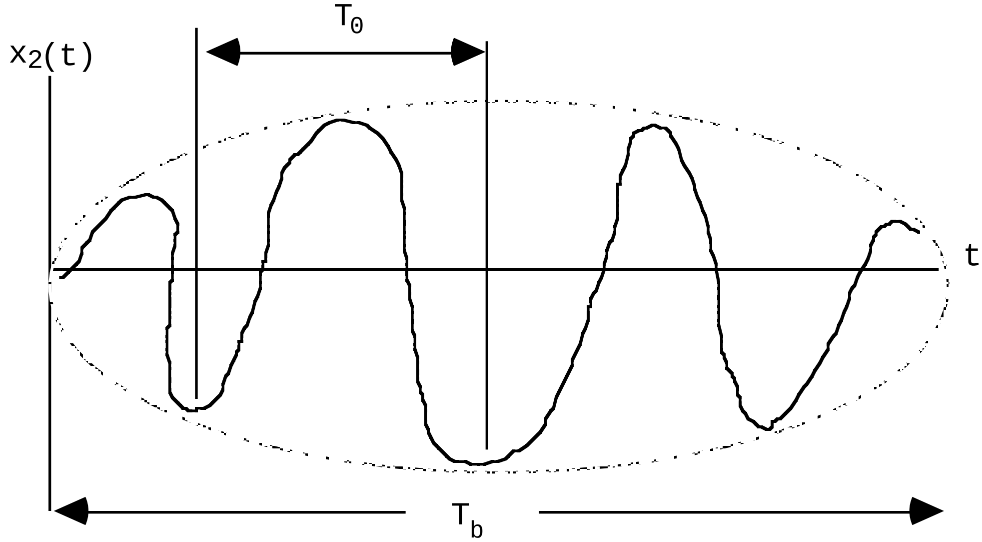
$$x_1(t) = D\cos(\omega_0 t)\cos(\omega_b t),$$
$$x_2(t) = D\sin(\omega_0 t)\sin(\omega_b t).$$

This form is convenient for discussing the case of weak coupling, $k_{12} << k$. Defining $\varepsilon \equiv \frac{k_{12}}{2k}$, we have

$$\omega_0 \simeq \sqrt{\frac{k}{M}}(1+\varepsilon) \quad , \quad \underset{\substack{\uparrow \\ \text{small compared to } \omega_0}}{\omega_b} \simeq \varepsilon\sqrt{\frac{k}{M}}\,.$$

Of course $\sqrt{\frac{k}{M}}$ is the decoupled frequency. Motion looks like (sorry for the bad graphs):





$$T_0 = \frac{2\pi}{\omega_0}, \; T_b = \frac{2\pi}{\omega_b}; \; T_b \gg T_0 \text{ in weak coupling.}$$

Phenomenon of <u>beats</u> occurs anytime a linear combination of very close frequencies is present. (Listen for them the next time you take a ride on a two propellar airplane.) This corrresponds to an approximate multiplicative modulation of the uncoupled motion.

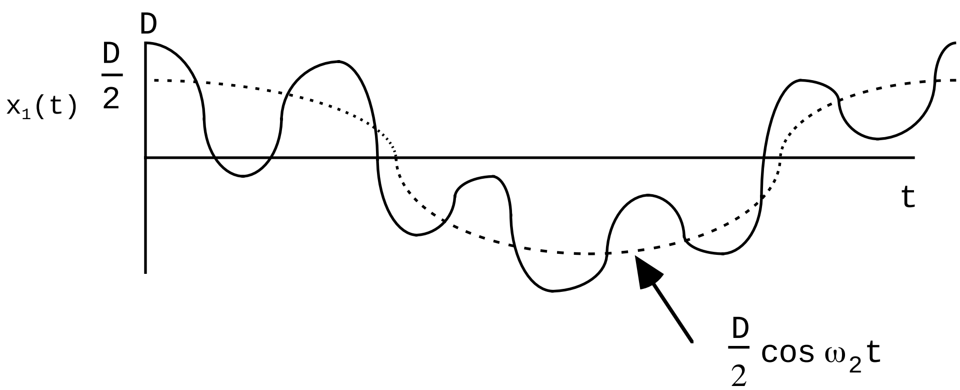
To discuss strong coupling, let us recast the above solution (still exact for the given BC’s) as

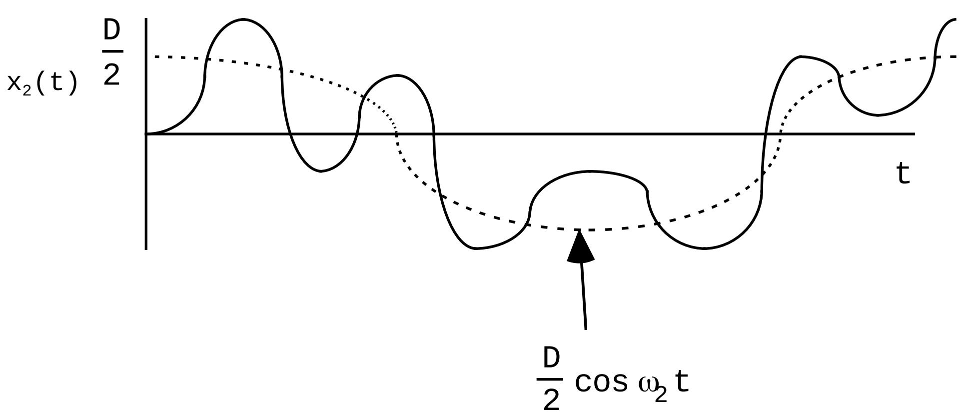
$$\text{also confirm}\quad \begin{cases} x_1(t) = \dfrac{D}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right], \\ x_2(t) = \dfrac{D}{2}\left[-\cos\omega_1 t + \cos\omega_2 t\right]. \end{cases}$$

Then, for strong coupling, $k_{12} >> k$, we have $\left(\tilde{\varepsilon} \equiv \left(\dfrac{k}{2k_{12}}\right)^{1/2}\right)$

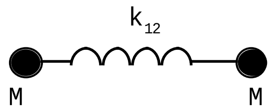
$$\omega_1 \simeq \sqrt{\frac{2k_{12}}{M}}\left(1 + \frac{\tilde{\varepsilon}^2}{2}\right), \quad \omega_2 \simeq \tilde{\varepsilon}\,\omega_1 .$$

We find the motion looks like:

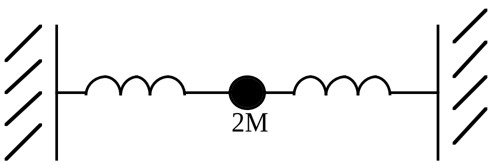




Now get an <u>additive</u> modulation. The two frequencies here are understandable from:



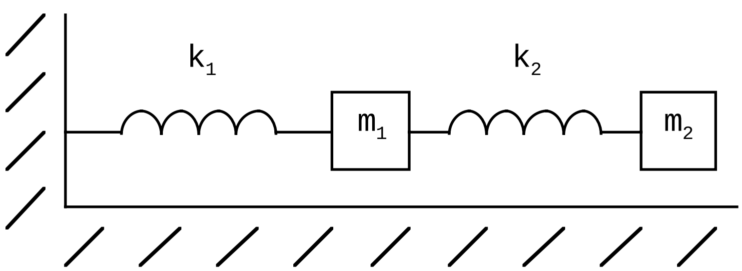
$$\Rightarrow \qquad \omega = \sqrt{\frac{2k_{12}}{M}} \simeq \omega_1 \, ,$$



$$\omega = \sqrt{\frac{2k}{2M}} = \sqrt{\frac{k}{M}} = \omega_2 \, .$$

Example using mechanical/electrical analogy

Let’s do one more example. It is a coupled system we encountered before, but we did not solve it because the equations were coupled. We had in Ch.3:



(no wall on this side)

x_1 : length of spring 1
x_2 : length of spring 2

(These lengths are compared to the equilibrium lengths.)

$$m_1\ddot{x}_1 = -k_1x_1 + k_2x_2 \qquad ①$$

$$m_2\left(\ddot{x}_1 + \ddot{x}_2\right) = -k_2x_2 \qquad ②$$

Reminder of the mechanical/electrical analogy:

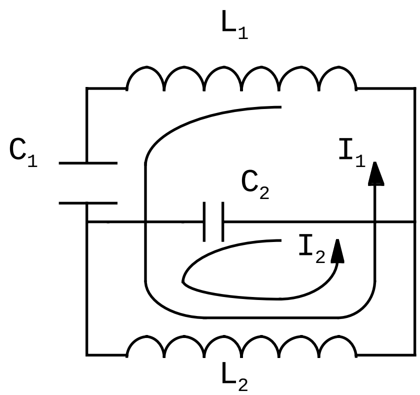
$$F \leftrightarrow V \ , \quad \dot{x} \leftrightarrow I \ , \quad k \leftrightarrow \frac{1}{C}$$

$$x \leftrightarrow q \ , \quad m \leftrightarrow L \ ,$$

$$① \quad \Rightarrow \quad L_1\ddot{q}_1 + \frac{1}{C_1}q_1 - \frac{1}{C_2}q_2 = 0,$$

$$② \quad \Rightarrow \quad L_2\left(\ddot{q}_1 + \ddot{q}_2\right) + \frac{1}{C_2}q_2 = 0.$$

Circuit looks like:



Assume (do not need to know T or U here since we already have the equations of motion)

$$q_{1,2}(t) = a_{1,2}\, e^{i\omega t},$$

$$\begin{cases} \left(\dfrac{1}{C_1} - L_1\omega^2\right)a_1 - \dfrac{1}{C_2}a_2 = 0, \\ -L_2\omega^2 a_1 + \left(\dfrac{1}{C_2} - L_2\omega^2\right)a_2 = 0. \end{cases}$$

$$\Rightarrow \det \begin{pmatrix} \dfrac{1}{C_1} - L_1\omega^2 & -\dfrac{1}{C_2} \\ -L_2\omega^2 & \dfrac{1}{C_2} - L_2\omega^2 \end{pmatrix} = 0,$$

$$\Rightarrow \left(\frac{1}{C_1} - L_1\omega^2\right)\left(\frac{1}{C_2} - L_2\omega^2\right) - \frac{1}{C_2} L_2\omega^2 = 0,$$

or $$L_1L_2\omega^4 - \frac{1}{C_2} L_1\omega^2 - \frac{1}{C_2} L_2\omega^2 - \frac{1}{C_1} L_2\omega^2 + \frac{1}{C_1C_2} = 0.$$

A quadratic equation in ω^2. Roots are:

$$\omega^2 = \frac{1}{2}\left[\frac{1}{C_2L_2} + \frac{1}{C_2L_1} + \frac{1}{C_1L_1}\right] \pm \frac{1}{2}\sqrt{\left(\frac{1}{C_2L_2} + \frac{1}{C_2L_1} + \frac{1}{L_1C_1}\right)^2 - \frac{4}{C_1C_2L_1L_2}}.$$

Because the above is so complicated, let us just look at a special case: $C_1 = C_2$, $L_1 = L_2$. Then

$$\omega^2 = \frac{1}{LC}\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right).$$

Call

$$\omega_1^2 = \frac{3 + \sqrt{5}}{2LC} \quad , \quad \omega_2^2 = \frac{3 - \sqrt{5}}{2LC}.$$

These are the normal mode frequencies of this circuit. We will skip the step of normalizing the eigenvectors. The general solution is (real part understood)

$$\left.\begin{aligned} q_1(t) &= a_1^1 e^{i\omega_1 t} + a_1^2 e^{i\omega_2 t} \\ q_2(t) &= a_2^1 e^{i\omega_1 t} + a_2^2 e^{i\omega_2 t} \end{aligned}\right\} \text{8 real constants}$$

where the $\bar{a}^i$ are in general complex. To find the eigenvector relations, substitute ω_1^2, ω_2^2 back into the eigenvector equations:

$$\left(\frac{1}{C} - L\omega_1^2\right) a_1^1 - \frac{1}{C} a_2^1 = 0,$$

$$\Rightarrow a_2^1 = -\left(\frac{1+\sqrt{5}}{2}\right) a_1^1 .$$

Likewise

$$\Rightarrow a_2^2 = -\left(\frac{1-\sqrt{5}}{2}\right) a_1^2 .$$

We now have,

$$\left.\begin{aligned} q_1(t) &= a_1^1 e^{i\omega_1 t} + a_1^2 e^{i\omega_2 t} \\ q_2(t) &= -\left(\frac{1+\sqrt{5}}{2}\right) a_1^1 e^{i\omega_1 t} - \left(\frac{1-\sqrt{5}}{2}\right) a_1^2 e^{i\omega_2 t} \end{aligned}\right\} \text{ 4 real constants.}$$

Using $n_{1,2} = e^{i\omega_{1,2}t}$ as the normal mode variables, we find

$$n_1 = \left(\frac{\sqrt{5}-1}{2} q_1 - q_2\right) \frac{1}{\sqrt{5}\, a_1^1},$$

$$n_2 = \left(\frac{1+\sqrt{5}}{2} q_1 + q_2\right) \frac{1}{\sqrt{5}\, a_1^2}.$$

Mode 2 occurs when $n_1 = 0$, so

$$\Rightarrow \quad q_2 = \frac{\sqrt{5}-1}{2} q_1 .$$

Mode 1 occurs when

$$q_2 = -\left(\frac{1+\sqrt{5}}{2}\right) q_1 .$$

These are like the modes for the coupled masses we saw in our first example, except the oscillation amplitudes are unsymmetrical. We can now build in the initial conditions. Let's say

$$\dot{q}_1(0) = \dot{q}_2(0) = 0 \quad , \quad q_1(0) = q_{10},\ q_2(0) = q_{20}\,.$$

Plugging in above, we get (remember the a_1^1, a_1^2 are complex)

$$0 = \mathrm{Re}\ [i\omega_1 a_1^1 + i\omega_2 a_1^2],$$

$$0 = \mathrm{Re}\ \left[-\left(\frac{1+\sqrt{5}}{2}\right) i\omega_1 a_1^1 - \left(\frac{1-\sqrt{5}}{2}\right) i\omega_2 a_1^2\right],$$

$$\Rightarrow \quad (a_1^1)_I = (a_1^2)_I = 0.$$

imaginary part

Likewise

real part

$$q_{10} = (a_1^1)_R + (a_1^2)_R,$$

$$q_{20} = -\left(\frac{1+\sqrt{5}}{2}\right)\left(a_1^1\right)_R - \left(\frac{1-\sqrt{5}}{2}\right)\left(a_1^2\right)_R,$$

$$\begin{cases} \left(a_1^1\right)_R = \dfrac{1}{\sqrt{5}}\left[-q_{20} + \dfrac{\sqrt{5}-1}{2}\, q_{10}\right], \\ \left(a_1^2\right)_R = \dfrac{1}{\sqrt{5}}\left[q_{20} + \dfrac{1+\sqrt{5}}{2}\, q_{10}\right]. \end{cases}$$

Full solution:

$$q_1(t) = \frac{1}{\sqrt{5}}\left[-q_{20} + \frac{\sqrt{5}-1}{2}\, q_{10}\right]\cos\omega_1 t + \frac{1}{\sqrt{5}}\left[q_{20} + \frac{1+\sqrt{5}}{2}\, q_{10}\right]\cos\omega_2 t\,,$$

$$q_2(t) = \frac{1}{\sqrt{5}}\left[-q_{10} + \frac{1+\sqrt{5}}{2}\, q_{20}\right]\cos\omega_1 t + \frac{1}{\sqrt{5}}\left[q_{10} + \frac{\sqrt{5}-1}{2}\, q_{20}\right]\cos\omega_2 t\,.$$

Whew!

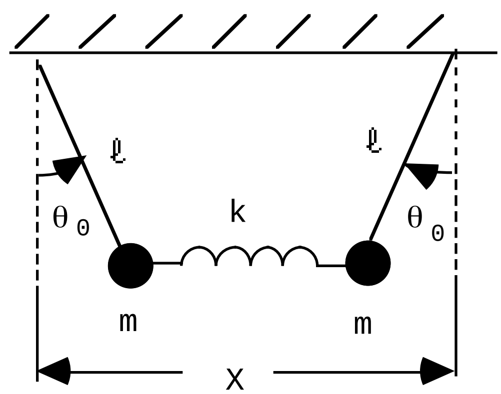
Chapter 12 Problems

1. Prove that the squared characteristic angular frequencies, ω_r^2, defined from (see Eq.(12.16); no sum on r)

$$\omega_r^2 = \frac{\sum_{i,j} a_i^r A_{ij} a_j^r}{\sum_{i,j} a_i^r m_{ij} a_j^r},$$

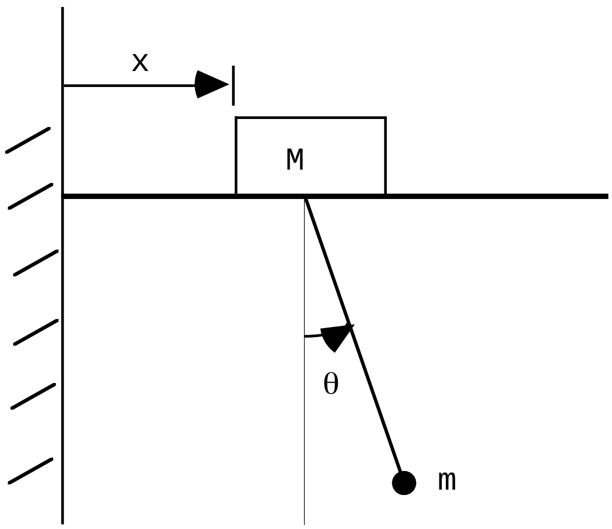
are positive, given the equilibrium conditions in the text.

2. A double pendulum system is arranged such that at equilibrium the pendulums from which the masses, m, are hung are displaced at an angle, θ_0, from vertical as shown. The pendulum lengths are ℓ and the spring constant is k. The unstretched length of the spring is L and X is the distance between the attachment points, as shown.



Find the eigenfrequencies of the system for small oscillations about θ_0. [Hints: First, show that the equilibrium angle, θ_0, is determined by $Y \equiv X-L = 2\ell \sin\theta_0 + \frac{mg}{k} \tan\theta_0$. Then, argue that the potential, U, is given for angles θ_1, θ_2 by $U(\theta_1,\theta_2) = mg\ell(1-\cos(\theta_0+\theta_1)) + mg\ell(1 - \cos(\theta_0-\theta_2)) + \frac{1}{2} k(Y - \ell \sin(\theta_0+\theta_1) - \ell \sin(\theta_0-\theta_2))^2$. Expand for small angles and solve for the eigenfrequencies.]

3. A mass M moves horizontally along a smooth rail. A pendulum is hung from M with a weightless rod with a mass m at it's end. The mass M is located a distance x from the wall.

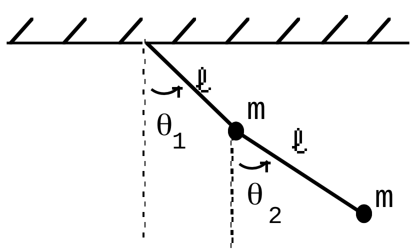


The Lagrangian may be written (you do not have to derive this)

$$L \simeq \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}(b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}) - \frac{mgb}{2}\,\theta^2 \,.$$

in terms of x and θ ($\theta \ll 1$). Find the characteristic frequencies of this system and determine the normal modes. Qualitatively describe the motion of these modes.

4. In the first semester we considered a double pendulum, consisting of two equal masses connected to each other and a horizontal support



by weightless rods of length ℓ. For small oscillations, the equations of motion we found were,

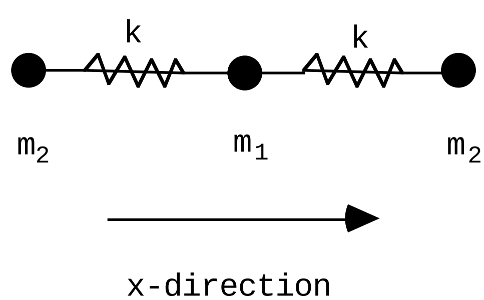
$$\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 + \frac{g}{\ell}\theta_1 = 0,$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{\ell}\theta_2 = 0.$$

(a) Find the characteristic frequencies of the system.

(b) Solve for the normal coordinates, n, in terms of θ_1, and θ_2. Describe the conditions which excite these modes.

5. Three masses, arrayed as shown, are coupled together in a straight line with two springs, both with spring constant, k. This is a one-dimensional problem, so motion can only occur along the x-direction.



(a) Find the squared eigenfrequencies (ω_r^2) of the system.

(b) Find the corresponding eigenvectors (they need not be normalized). Descibe the motion associated with each of the normal modes.