

PROPERTIES OF THE SOLUTIONS OF THE FOURTH-ORDER BESSEL-TYPE DIFFERENTIAL EQUATION

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ABSTRACT. The structured Bessel-type functions of arbitrary even-order were introduced by Everitt and Markett in 1994; these functions satisfy linear ordinary differential equations of the same even-order. The differential equations have analytic coefficients and are defined on the whole complex plane with a regular singularity at the origin and an irregular singularity at the point of infinity. They are all natural extensions of the classical second-order Bessel differential equation. Further these differential equations have real-valued coefficients on the positive real half-line of the plane, and can be written in Lagrange symmetric (formally self-adjoint) form.

In the fourth-order case, the Lagrange symmetric differential expression generates self-adjoint unbounded operators in certain Hilbert function spaces. These results are recorded in many of the papers here given as references.

It is shown in the original paper of 1994 that in this fourth-order case one solution exists which can be represented in terms of the classical Bessel functions of order 0 and 1. The existence of this solution, further aided by computer programs in Maple, led to the existence of a linearly independent basis of solutions of the differential equation.

In this paper a new proof of the existence of this solution base is given, on using the advanced theory of special functions in the complex plane. The methods lead to the development of analytical properties of these solutions, in particular the series expansions of all solutions at the regular singularity at the origin of the complex plane.

1. INTRODUCTION AND MAIN IDEAS

The fourth-order Bessel-type differential equation has the Lagrange symmetric (formally self-adjoint) form

$$(1.1) \quad [xy''(x)]'' - [(9x^{-1} + 8M^{-1}x)y'(x)]' = \Lambda xy(x) \text{ for all } x \in (0, \infty).$$

Here, $M \in (0, \infty)$ is a given parameter and $\Lambda \in \mathbb{C}$ is the spectral parameter. This equation belongs to a sequence of structured ordinary differential equations of even order with analytic coefficients in the complex plane \mathbb{C} , which were introduced in 1994 by Everitt and Markett [7].

Equation (1.1) has many features in common with the higher-order differential equations associated with the Laguerre- and Jacobi-type orthogonal polynomials, see the diagram in [2, Section 1] or [3, Section 3]. On the other hand, each higher-order Bessel-type equation

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may be seen as an extension of the classical second-order Bessel equation with parameter $\nu = 0$,

$$(1.2) \quad -[xy'(x)]' = \lambda^2 xy(x) \text{ for all } x \in (0, \infty).$$

Concerning the theory of Bessel equations and functions we refer to Watson [17] or other standard books on higher transcendental functions [1, Chapter 9], [12, Chapter VII]. In particular, there is the following limit relation between the two equations (1.2) and (1.1). If we connect the two spectral parameters Λ and λ by

$$(1.3) \quad \Lambda = \lambda^2(\lambda^2 + 8M^{-1})$$

and multiply equation (1.1) by $M > 0$, we arrive at equation (1.2) by formally letting M tend to zero. More and deeper relationships are presented and discussed later on.

In a series of papers [2], [5], [6], [8] and [9] we investigated the spectral theoretical aspects of the fourth-order equation (1.1) in certain weighted Lebesgue, and Lebesgue-Stieltjes spaces. This study exhibited remarkable new phenomena and analytic structures, for example see the most recent account in [9]. Among others we introduced a generalized Hankel transform [6] and two versions of the Fourier-Bessel-type series [8], [9]. Some of these features have been deduced by applying general results on singular linear differential equations [13] and [14], but at various stages of our approach it is also useful and sometimes even indispensable, to have particular knowledge of the underlying special function solutions of equation (1.1).

There are recent applications of the fourth-order Bessel-type differential equations to the study of properties of the biharmonic partial differential equation in the plane, when considered in polar co-ordinates; see [4].

The present paper deals with a full basis of four linearly independent solutions of equation (1.1) denoted by

$$\left\{ J_\lambda^{0,M}, Y_\lambda^{0,M}, I_\lambda^{0,M}, K_\lambda^{0,M} \right\}.$$

These functions may be defined in terms of the Bessel functions of the first and second kind, J_ν, Y_ν , as well as of the corresponding modified Bessel functions, I_ν, K_ν , where $\nu = 0$ or 1 ; see [3, Section 3] or [8, (6.1)]. For $\lambda \in \mathbb{C}$ with $\arg(\lambda) \in (-\pi/2, \pi/2)$ and $M > 0$ we set

$$(1.4) \quad a_\lambda = a_\lambda(M) := 1 + \frac{1}{4}M\lambda^2 \text{ and } c_\lambda^2 = c_\lambda^2(M) := \lambda^2 + 8M^{-1},$$

whence

$$\lambda^2 = 4M^{-1}(a_\lambda - 1) \text{ and } c_\lambda^2 = 4M^{-1}(a_\lambda + 1).$$

Then, for all $x \in (0, \infty)$,

$$(1.5) \quad J_\lambda^{0,M}(x) = a_\lambda J_0(\lambda x) + 2(1 - a_\lambda)(\lambda x)^{-1} J_1(\lambda x)$$

$$(1.6) \quad Y_\lambda^{0,M}(x) = a_\lambda Y_0(\lambda x) + 2(1 - a_\lambda)(\lambda x)^{-1} Y_1(\lambda x)$$

$$(1.7) \quad I_\lambda^{0,M}(x) = -a_\lambda I_0(c_\lambda x) + 2(1 + a_\lambda)(c_\lambda x)^{-1} I_1(c_\lambda x)$$

$$(1.8) \quad K_\lambda^{0,M}(x) = -a_\lambda K_0(c_\lambda x) - 2(1 + a_\lambda)(c_\lambda x)^{-1} K_1(c_\lambda x).$$

The Bessel functions involved are defined by any of the following representations:

$$(1.9) \quad J_0(z) := {}_0F_1\left(-; 1; -\frac{1}{4}z^2\right) = \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k}}{k!k!}$$

$$(1.10) \quad \begin{aligned} Y_0(z) &:= \frac{\pi}{2} \lim_{\nu \rightarrow 0} \frac{J_\nu(z) \cos(\nu z) - J_{-\nu}(z)}{\sin(\nu z)} = [\log(\tfrac{1}{2}z) + \gamma] J_0(z) - \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k}}{k!k!} h_k \\ &:= \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k}}{k!k!} [\log(\tfrac{1}{2}z) - \psi(k+1)] \end{aligned}$$

$$(1.11) \quad I_0(z) := {}_0F_1\left(-; 1; \frac{1}{4}z^2\right) = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k}}{k!k!}$$

$$(1.12) \quad \begin{aligned} K_0(z) &:= \frac{\pi}{2} \lim_{\nu \rightarrow 0} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu z)} = -[\log(\tfrac{1}{2}z) + \gamma] I_0(z) + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k}}{k!k!} h_k \\ &= -\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k}}{k!k!} [\log(\tfrac{1}{2}z) - \psi(k+1)]. \end{aligned}$$

The symbol ψ represents the logarithmic derivative of the gamma function Γ , and so

$$(1.13) \quad \psi(1) = -\gamma, \psi(k+1) = h_k - \gamma \text{ and } h_k := \sum_{j=1}^k (1/j) \text{ for all } k \in \mathbb{N}$$

where γ is Euler's constant. These settings are to be found in [12, 7.2.1, 7.2.4 and 7.2.5] or [17, Chapter III], up to an additional factor $\frac{1}{2}\pi$ in the definition of the Bessel function $Y_0(z)$, which is in accordance with the definition introduced by Schäfli [17, 3.54]. Here and in the following

$${}_pF_q \text{ for } p, q \in \mathbb{N}_0,$$

denotes the generalized hypergeometric series [11, Chapter IV]. Concerning the Bessel functions of order $\nu = 1$ we note that (cf. [17, 3.56 and 3.71])

$$(1.14) \quad \begin{aligned} J_1(z) &= -J'_0(z) = \frac{z}{2} {}_0F_1\left(-; 2; -\frac{1}{4}z^2\right) = \frac{z}{2} \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k}}{k!(k+1)!} \\ Y_1(z) &= -Y'_0(z) = -z^{-1} J_0(z) + [\log(\tfrac{1}{2}z) + \gamma] J_1(z) + 2 \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k-1}}{(k-1)!k!} h_k \\ &= -z^{-1} - \frac{z}{4} \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k}}{(k+1)!(k+1)!} + \\ &\quad + \frac{z}{2} \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \frac{z^{2k}}{k!(k+1)!} [\log(\tfrac{1}{2}z) - \psi(k+1)] \end{aligned}$$

and

(1.15)

$$\begin{aligned}
I_1(z) &= I'_0(z) = \frac{z}{2} {}_0F_1\left(-; 2; \frac{1}{4}z^2\right) = \frac{z}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k}}{k!(k+1)!} \\
K_1(z) &= -K'_0(z) = z^{-1}I_0(z) + \left[\log\left(\frac{1}{2}z\right) + \gamma\right] I_1(z) - 2 \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k-1}}{(k-1)!k!} h_k \\
&= z^{-1} - \frac{z}{4} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k}}{(k+1)!(k+1)!} + \\
&\quad + \frac{z}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \frac{z^{2k}}{k!(k+1)!} \left[\log\left(\frac{1}{2}z\right) - \psi(k+1)\right]
\end{aligned}$$

The log terms in $Y_\nu(z), K_\nu(z)$ for $\nu = 0, 1$ may be replaced by

$$(1.16) \quad \log(z/2) = \frac{1}{2} \log(z^2/4) \text{ for } \arg(z) \in (-\pi/2, \pi/2)$$

to indicate that the four Bessel-type functions stated in (1.5) to (1.8) are actually functions of the parameters λ^2 and $c_\lambda^2 := \lambda^2 + 8M^{-1}$. Moreover we note that in view of (1.9) to (1.12) and then (1.14) and (1.15), the Bessel-type functions $J_\lambda^{0,M}(x)$ and $I_\lambda^{0,M}(x)$ are regular solutions at the left endpoint of the domain $(0, \infty)$, whilst the other two solutions $Y_\lambda^{0,M}(x)$ and $K_\lambda^{0,M}(x)$ are singular at this endpoint (see Corollary 2.1 below).

The Bessel-type function $J_\lambda^{0,M}$ was introduced in [7], and the solution $Y_\lambda^{0,M}$ was defined in [2]. The two additional solutions $I_\lambda^{0,M}$ and $K_\lambda^{0,M}$, however, were found only recently from the results of van Hoeij, see [15], using the computer algebra program MAPLE; more details of this approach are given in [10].

In [7, (1.11) and (1.12)] we showed that the Bessel-type function $J_\lambda^{0,M}(x)$, and the solution $J_\lambda^0(x) := J_0(\lambda x)$ of the classical Bessel equation (1.2), are linked to each other by two linear second-order differential expressions

$$(1.17) \quad A_x^M := -\frac{1}{4}MD_x^2 + \frac{1}{4}Mx^{-1}D_x + 1 \text{ and } B_x^M := -\frac{1}{4}MD_x^2 - \frac{3}{4}Mx^{-1}D_x + 1,$$

where $D_x := d/dx$ and $D_x^2 := d^2/dx^2$. In fact, we have

$$(1.18) \quad J_\lambda^{0,M}(x) = A_x^M [J_\lambda^0](x) \text{ and } B_x^M [J_\lambda^{0,M}](x) = a_\lambda^2 J_\lambda^0(x) \text{ for all } x \in (0, \infty).$$

A combination of these two relations immediately yields the fourth-order differential equation

$$(1.19) \quad (A_x^M \circ B_x^M) [J_\lambda^{0,M}](x) := A_x^M [B_x^M J_\lambda^{0,M}](x) = a_\lambda^2 A_x^M [J_\lambda^0](x) = a_\lambda^2 J_\lambda^{0,M}(x).$$

This suggests that the differential expression of the the Bessel-type equation (1.1) may be factorized in order to gain information about all its solutions from solutions of the classical Bessel equation.

The purpose of this present paper is to show, see Section 2 below, that such a factorization of the Bessel-type equation exists very generally. As it turns out, we can use the same differential expression A_x^M in order to trace back the solutions of the fourth-order equation (1.1) to those of a second-order Bessel equation. For the J -solution this is the classical equation (1.2) in view of (1.18), and it turns out the same relationship also for the Y -solution. Concerning the I - and the K solutions, however, we have to start from a different

Bessel equation which is obtained from (1.2) by replacing the eigenvalue parameter λ^2 by $-c_\lambda^2 = -(\lambda^2 + 8M^{-1})$. Notice that the product yields, up to a negative sign, the eigenvalue parameter (1.3) of the fourth-order Bessel equation. In the end this kind of duality is responsible for the formally similar representations of the four Bessel-type functions (1.5) to (1.8).

Diagram 1.1 on page 6 illustrates our construction of these functions.

The details of our approach are carried out in Section 2. In particular, we give an analytic proof that the four functions (1.5) to (1.8) are solutions of the differential equation (1.1). As a corollary we then deduce explicit series expansions, which are of (generalized) hypergeometric type in the two regular case solutions. In particular, the four Bessel-type functions are normalized in such a way that at the origin 0 the regular solutions satisfy

$$(1.20) \quad J_\lambda^{0,M}(0) = I_\lambda^{0,M}(0) = 1,$$

and that the the two singular solutions satisfy

$$(1.21) \quad \lim_{x \rightarrow 0^+} \left\{ x^2 Y_\lambda^{0,M}(x) \right\} = - \lim_{x \rightarrow 0^+} \left\{ x^2 K_\lambda^{0,M}(x) \right\} = \frac{1}{2}M.$$

These results suggest that it is possible to determine, by taking appropriate linear combinations of the Bessel-type functions, four linearly independent solutions of the Bessel-type equation (1.1), which behave like the powers x^r as $x \rightarrow 0^+$, where $r \in \{-2, 0, 2, 4\}$. These four exponents are the Frobenius indicial roots of the regular singularity, at the origin 0 in \mathbb{C} , of the Bessel-type differential equation; see the corresponding discussion in [2, Section 3]. These four ‘‘Frobenius’’ solutions, which are labelled according to the four indices $y_r(x, \lambda, M)$ for $r \in \{-2, 0, 2, 4\}$, are presented in Section 3 below.

In Section 4 we discuss the behaviour of the solutions of the Bessel-type equation (1.1) with spectral parameter (1.3) for M tending to $+\infty$, namely

$$(1.22) \quad [xy''(x)]'' - [9x^{-1}y'(x)]' = \lambda^4 xy(x) \text{ for all } x \in (0, \infty).$$

This differential equation deserves particular interest since it may be interpreted as the radial part of the biharmonic partial differential equation, being obtained by applying a certain quasi-separation method, for details see the the recent account in [4]. As it turns out, the basis of solutions of the equation (1.22) is given in terms of the classical Bessel functions with the parameter $\nu = 2$.

Finally, Section 5 contains some concluding remarks.

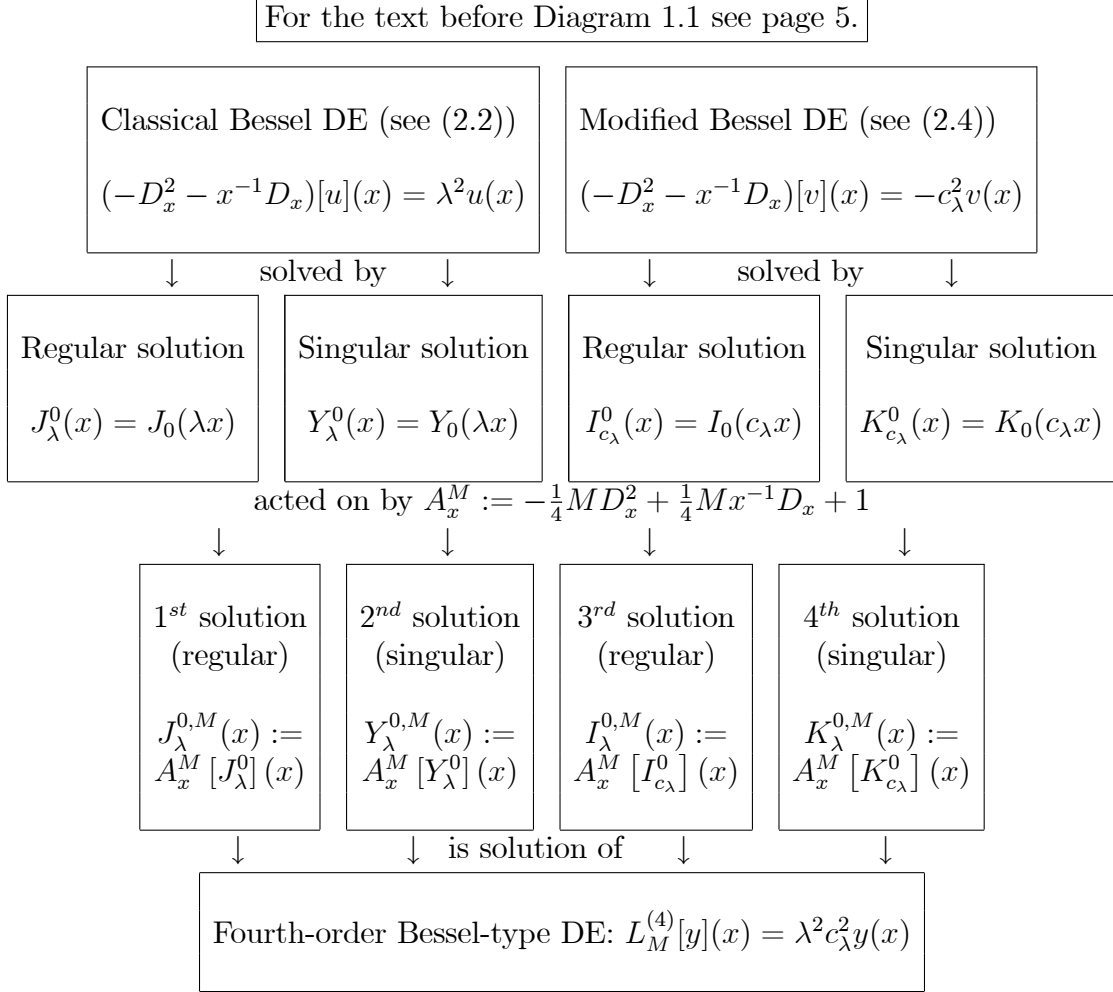


Diagram 1.1

2. FACTORIZATION OF THE FOURTH-ORDER BESSEL-TYPE DIFFERENTIAL EQUATION

The Frobenius form of the Bessel-type differential equation (1.1) is given, in operational notation, by

$$(2.1) \quad \begin{aligned} L_M^{(4)}[y](x) &:= (D_x^4 + 2x^{-1}D_x^3 - \{9x^{-2} + 8M^{-1}\}D_x^2 + \{9x^{-3} - 8M^{-1}x^{-1}\}D_x)[y](x) \\ &= \lambda^2 c_\lambda^2 y(x) \end{aligned}$$

for all $x \in (0, \infty)$. Similarly, the second-order Bessel equation (1.2) may be written in the form

$$(2.2) \quad L^{(2)}[u](x) := (-D_x^2 - x^{-1}D_x)[u](x) = \lambda^2 u(x) \text{ for all } x \in (0, \infty)$$

so that, recalling (1.4),

$$(2.3) \quad (L^{(2)} + 8M^{-1})[u](x) := (-D_x^2 - x^{-1}D_x + 8M^{-1})[u](x) = c_\lambda^2 u(x).$$

Moreover, we introduce another equation of the form (2.2), where the eigenvalue parameter λ^2 is replaced by $-c_\lambda^2$. Denoting the new dependent variable by $v(x)$ we have

$$(2.4) \quad L^{(2)}[v](x) := (-D_x^2 - x^{-1}D_x)[v](x) = -c_\lambda^2 v(x) \text{ for all } x \in (0, \infty)$$

and hence

$$(2.5) \quad (L^{(2)} + 8M^{-1})[v](x) := (-D_x^2 - x^{-1}D_x + 8M^{-1})[v](x) = -\lambda^2 v(x).$$

Lemma 2.1. *For $M \in (0, \infty)$ and $x \in (0, \infty)$ let the differential expressions A_x^M, B_x^M be defined as in (1.17). Then, for sufficiently smooth functions $y(x)$, the following two identities hold:*

$$(2.6) \quad (A_x^M \circ B_x^M)[y](x) \equiv \left(\frac{M}{4}\right)^2 L_M^{(4)}[y](x) + y(x),$$

$$(2.7) \quad (B_x^M \circ A_x^M)[y](x) \equiv \left(\frac{M}{4}\right)^2 ([L^{(2)} + 8M^{-1}] \circ L^{(2)})[y](x) + y(x).$$

Proof. Applying the two expressions B_x^M and A_x^M , in this order, to the left-hand side of identity (2.6) we obtain

$$\begin{aligned} & \left[-\frac{M}{4}D_x^2 + \frac{M}{4}x^{-1}D_x + 1\right] \left[-\frac{M}{4}y''(x) - \frac{3M}{4}x^{-1}y'(x) + y(x)\right] \\ &= \left(\frac{M}{4}\right)^2 \left\{ y^{(4)}(x) + [3x^{-1}y'(x)]'' - \frac{4}{M}y''(x) - x^{-1}y'''(x) - 3x^{-1}[x^{-1}y'(x)]' + \frac{4}{M}x^{-1}y'(x) - \right. \\ & \quad \left. - \frac{4}{M}y''(x) - \frac{12}{M}x^{-1}y'(x) \right\} + y(x) \\ &= \left(\frac{M}{4}\right)^2 \left\{ y^{(4)}(x) + 2x^{-1}y^{(3)}(x) - [9x^{-2} + \frac{8}{M}]y''(x) - [9x^{-3} - \frac{8}{M}x^{-1}]y'(x) \right\} + y(x) \\ &= \left(\frac{M}{4}\right)^2 L_M^{(4)}[y](x) + y(x). \end{aligned}$$

Concerning identity (2.7), we just apply the differential expressions on both sides of the identity and compare the results. \square

Theorem 2.1. *Let $M \in (0, \infty)$ and let A_x^M, B_x^M be as in (1.17).*

- (i) *Given any solution $u_\lambda(x)$ of the Bessel equation (2.2), the image function $U_\lambda(x) := A_x^M[u_\lambda](x)$ solves the fourth-order Bessel-type equation (2.1).*
- (ii) *Given any solution $v_\lambda(x)$ of the Bessel equation (2.4) with modified eigenvalue parameter, the image function $V_\lambda(x) := A_x^M[v_\lambda](x)$ also solves the fourth-order Bessel-type equation (2.1).*

Proof. (i) By definition of $U_\lambda(x)$, identity (2.7) and the assumption that $u = u_\lambda$ satisfies the differential equations (2.2) and (2.3) we obtain

$$(2.8) \quad \left\{ \begin{aligned} B_x^M[U_\lambda](x) &= (B_x^M \circ A_x^M)[u_\lambda](x) \\ &= \left(\frac{M}{4}\right)^2 ([L^{(2)} + 8M^{-1}] \circ L^{(2)})[u_\lambda](x) + u_\lambda(x) \\ &= -\left(\frac{M}{4}\right)^2 (L^{(2)} + 8M^{-1})[\lambda^2 u_\lambda](x) + u_\lambda(x) \\ &= \left\{ \left(\frac{M}{4}\right)^2 c_\lambda^2 \lambda^2 + 1 \right\} u_\lambda(x) \\ &= a_\lambda^2 u_\lambda(x). \end{aligned} \right.$$

Now apply the expression A_x^M to both sides of (2.8) and use identity (2.6) to get

$$(2.9) \quad \left\{ \begin{aligned} \left(\frac{M}{4}\right)^2 L_M^{(4)}[U_\lambda](x) &= (A_x^M \circ B_x^M)[U_\lambda](x) - U_\lambda(x) \\ &= (a_\lambda^2 - 1)U_\lambda(x) \\ &= \frac{M}{4}\lambda^2 \left(2 + \frac{M}{4}\lambda^2\right) U_\lambda(x) \\ &= \left(\frac{M}{4}\right)^2 \lambda^2 c_\lambda^2 U_\lambda(x). \end{aligned} \right.$$

Hence $U_\lambda(x)$ solves equation (2.1).

(ii) Here, identity (2.7) and the assumptions (2.4) and (2.5) give

$$(2.10) \quad \left\{ \begin{aligned} B_x^M[V_\lambda](x) &= (B_x^M \circ A_x^M)[v_\lambda](x) \\ &= \left(\frac{M}{4}\right)^2 ([L^{(2)} + 8M^{-1}] \circ L^{(2)})[v_\lambda](x) + v_\lambda(x) \\ &= -\left(\frac{M}{4}\right)^2 (L^{(2)} + 8M^{-1})[c_\lambda^2 v_\lambda](x) + v_\lambda(x) \\ &= \left\{ \left(\frac{M}{4}\right)^2 c_\lambda^2 \lambda^2 + 1 \right\} v_\lambda(x) \\ &= a_\lambda^2 v_\lambda(x). \end{aligned} \right.$$

As in item (i), the assertion then follows by applying A_x^M to both sides of (2.10) and using (2.6), *i.e.*

$$(2.11) \quad \left\{ \begin{aligned} \left(\frac{M}{4}\right)^2 L_M^{(4)}[V_\lambda](x) &= (A_x^M \circ B_x^M)[V_\lambda](x) - V_\lambda(x) \\ &= (a_\lambda^2 - 1)V_\lambda(x) \\ &= \left(\frac{M}{4}\right)^2 \lambda^2 c_\lambda^2 V_\lambda(x). \end{aligned} \right.$$

□

Theorem 2.2. *Let $M \in (0, \infty)$ and let A_x^M, B_x^M be as in (1.17).*

(i) *There are four linearly independent solutions of the Bessel-type differential equation (2.1) given by*

$$(2.12) \quad \left\{ \begin{aligned} J_\lambda^{0,M}(x) &:= A_x^M[J_\lambda^0](x) && \text{where } J_\lambda^0(x) := J_0(\lambda x) \\ Y_\lambda^{0,M}(x) &:= A_x^M[Y_\lambda^0](x) && \text{where } Y_\lambda^0(x) := Y_0(\lambda x) \\ I_\lambda^{0,M}(x) &:= A_x^M[I_{c_\lambda}^0](x) && \text{where } I_{c_\lambda}^0(x) := I_0(c_\lambda x) \\ K_\lambda^{0,M}(x) &:= A_x^M[K_{c_\lambda}^0](x) && \text{where } K_{c_\lambda}^0(x) := K_0(c_\lambda x). \end{aligned} \right.$$

(ii) *The four Bessel-type functions defined in item (i) are related to their classical counterparts by the inversion formulae*

$$(2.13) \quad B_x^M[J_\lambda^{0,M}](x) = \left(1 + \frac{1}{4}M\lambda^2\right)^2 J_\lambda^0(x), \quad B_x^M[Y_\lambda^{0,M}](x) = \left(1 + \frac{1}{4}M\lambda^2\right)^2 Y_\lambda^0(x)$$

$$(2.14) \quad B_x^M[I_\lambda^{0,M}](x) = \left(1 - \frac{1}{4}Mc_\lambda^2\right)^2 I_{c_\lambda}^0(x), \quad B_x^M[K_\lambda^{0,M}](x) = \left(1 - \frac{1}{4}Mc_\lambda^2\right)^2 K_{c_\lambda}^0(x).$$

Observing that $1 + \frac{1}{4}M\lambda^2 = -1 + \frac{1}{4}Mc_\lambda^2$ it follows that the two factors $\left(1 + \frac{1}{4}M\lambda^2\right)^2$ and $\left(1 - \frac{1}{4}Mc_\lambda^2\right)^2$ actually match.

(iii) *The Bessel-type functions defined in (2.12) possess the representations (1.5) to (1.8).*

Proof. (i) Since $J_\lambda^0(x)$ and $Y_\lambda^0(x)$ are two solutions of the Bessel equation (2.2). it follows from item (i) of Theorem 2.1 that their images under the expression A_x^M , i.e. $J_\lambda^{0,M}(x)$ and $Y_\lambda^{0,M}(x)$, solve the fourth-order Bessel-type equation (2.1). By Theorem 2.1, item (ii), the same holds true for the two functions $I_\lambda^{0,M}(x)$ and $K_\lambda^{0,M}(x)$, since $I_{c_\lambda}^0(x)$ and $K_{c_\lambda}^0(x)$ solve the Bessel equation (2.4) with eigenvalue parameter $-c_\lambda^2$. All four solutions are linearly independent by construction and by comparison with explicit representations, see Corollaries 2.1 and 2.2 below.

(ii) The two identities in (2.13) readily follow by (2.8), whilst (2.10) yields the two identities in (2.14).

(iii) In view of (2.2) and (1.14) the Bessel function $J_\lambda^0(x) = J_0(\lambda x)$ satisfies

$$\left(-\frac{M}{4}D_x^2 - \frac{M}{4}x^{-1}D_x\right)[J_0(\lambda\cdot)](x) = \frac{M}{4}\lambda^2 J_0(\lambda x)$$

and

$$\frac{M}{2}x^{-1}D_x[J_0(\lambda\cdot)](x) = -\frac{M}{2}x^{-1}\lambda J_1(\lambda x).$$

A combination of these two formulae then implies the representation (1.5), since

$$(2.15) \quad \left\{ \begin{aligned} J_\lambda^{0,M}(x) &:= A_x^M[J_0(\lambda\cdot)](x) := \left(-\frac{M}{4}D_x^2 - \frac{M}{4}x^{-1}D_x + 1 + \frac{M}{2}x^{-1}D_x\right)[J_0(\lambda\cdot)](x) \\ &= \left(1 + \frac{M}{4}\lambda^2\right)J_0(\lambda x) - \frac{M}{2}\lambda^2(\lambda x)^{-1}J_1(\lambda x). \end{aligned} \right.$$

The same argument applies to $Y_\lambda^0(x) = Y_0(\lambda x)$ which proves formula (1.6).

Concerning (1.7) we note that, in view of (2.4) and (1.15),

$$\left(-\frac{M}{4}D_x^2 - \frac{M}{4}x^{-1}D_x\right)[I_0(c_\lambda\cdot)](x) = -\frac{M}{4}c_\lambda^2 I_0(c_\lambda x)$$

and

$$\frac{M}{2}x^{-1}D_x[I_0(c_\lambda\cdot)](x) = \frac{M}{2}x^{-1}c_\lambda I_1(c_\lambda x).$$

Hence

$$(2.16) \quad \left\{ \begin{aligned} I_\lambda^{0,M}(x) &:= A_x^M[I_0(c_\lambda\cdot)](x) := \left(-\frac{M}{4}D_x^2 - \frac{M}{4}x^{-1}D_x + 1 + \frac{M}{2}x^{-1}D_x\right)[I_0(c_\lambda\cdot)](x) \\ &= \left(1 - \frac{M}{4}c_\lambda^2\right)I_0(c_\lambda x) + \frac{M}{2}c_\lambda x^{-1}I_1(c_\lambda x) \\ &= -a_\lambda I_0(c_\lambda x) + \frac{M}{2}c_\lambda^2(c_\lambda x)^{-1}I_1(c_\lambda x). \end{aligned} \right.$$

Analogously, $K_0(c_\lambda x)$ yields

$$(2.17) \quad \left\{ \begin{aligned} K_\lambda^{0,M}(x) &:= A_x^M[K_0(c_\lambda\cdot)](x) := \left(-\frac{M}{4}D_x^2 - \frac{M}{4}x^{-1}D_x + 1 + \frac{M}{2}x^{-1}D_x\right)[K_0(c_\lambda\cdot)](x) \\ &= \left(1 - \frac{M}{4}c_\lambda^2\right)K_0(c_\lambda x) - \frac{M}{2}c_\lambda x^{-1}K_1(c_\lambda x) \\ &= -a_\lambda K_0(c_\lambda x) - \frac{M}{2}c_\lambda^2(c_\lambda x)^{-1}K_1(c_\lambda x). \end{aligned} \right.$$

□

This concludes the proof of Theorem 2.2.

Corollary 2.1. *For $M \in (0, \infty)$ and $\lambda \in \mathbb{C}$ with $\arg(\lambda) \in (-\pi/2, \pi/2)$ let $a_\lambda := 1 + \frac{1}{4}M\lambda^2$ and $c_\lambda^2 := \lambda^2 + 8M^{-1}$ as in (1.4). On the interval $x \in (0, \infty)$ the four Bessel-type functions (1.5) to (1.8) possess the following the following series expansions:*

$$(2.18) \quad J_\lambda^{0,M}(x) = \sum_{k=0}^{\infty} \{1 + ka_\lambda\} \left(-\frac{1}{4}\lambda^2\right)^k \frac{x^{2k}}{k!(k+1)!} = {}_1F_2(1 + a_\lambda^{-1}; a_\lambda^{-1}, 2; -\frac{1}{4}(\lambda x)^2),$$

$$(2.19) \quad \begin{cases} Y_\lambda^{0,M}(x) &= \frac{M}{2}x^{-2} + \frac{M}{8}\lambda^2 \sum_{k=0}^{\infty} \left(-\frac{1}{4}\lambda^2\right)^k \frac{x^{2k}}{(k+1)!(k+1)!} \\ &+ \sum_{k=0}^{\infty} \{1 + ka_\lambda\} \left(-\frac{1}{4}\lambda^2\right)^k \frac{x^{2k}}{k!(k+1)!} \left[\frac{1}{2} \log\left(\frac{1}{4}\lambda^2 x^2\right) - \psi(k+1)\right], \end{cases}$$

$$(2.20) \quad I_\lambda^{0,M}(x) = \sum_{k=0}^{\infty} \{1 - ka_\lambda\} \left(\frac{1}{4}c_\lambda^2\right)^k \frac{x^{2k}}{k!(k+1)!} = {}_1F_2(1 - a_\lambda^{-1}; -a_\lambda^{-1}, 2; \frac{1}{4}c_\lambda^2 x^2),$$

$$(2.21) \quad \begin{cases} K_\lambda^{0,M}(x) &= -\frac{M}{2}x^{-2} + \frac{M}{8}c_\lambda^2 \sum_{k=0}^{\infty} \left(\frac{1}{4}c_\lambda^2\right)^k \frac{x^{2k}}{(k+1)!(k+1)!} \\ &- \sum_{k=0}^{\infty} \{1 - ka_\lambda\} \left(\frac{1}{4}c_\lambda^2\right)^k \frac{x^{2k}}{k!(k+1)!} \left[\frac{1}{2} \log\left(\frac{1}{4}c_\lambda^2 x^2\right) - \psi(k+1)\right]. \end{cases}$$

Proof. The formulae (2.18) to (2.21) follow by inserting the definitions of the classical Bessel functions of order 0 and 1 given in Section 1 into the representations (1.5) to (1.8) of the four Bessel-type functions.

For the two regular solutions (1.5) and (1.7) we have

$$J_\lambda^{0,M}(x) = \sum_{k=0}^{\infty} g_\lambda^M(k) \left(-\frac{1}{4}\lambda^2\right)^k \frac{x^{2k}}{k!(k+1)!} \quad \text{and} \quad I_\lambda^{0,M}(x) = \sum_{k=0}^{\infty} h_\lambda^M(k) \left(\frac{1}{4}c_\lambda^2\right)^k \frac{x^{2k}}{k!(k+1)!}$$

where

$$g_\lambda^M(k) = \left(1 + \frac{M}{4}\lambda^2\right) (k+1) - \frac{M}{4}\lambda^2 = 1 + \left(1 + \frac{M}{4}\lambda^2\right) k,$$

$$h_\lambda^M(k) = -\left(1 + \frac{M}{4}\lambda^2\right) (k+1) + \frac{M}{4}c_\lambda^2 = 1 - \left(1 + \frac{M}{4}\lambda^2\right) k.$$

Concerning the two singular solutions (1.6) and (1.8), it turns out that the two coefficients $g_\lambda^M(k)$ and $h_\lambda^M(k)$ also arise in the second sums of their representations, respectively. \square

Corollary 2.2. *In the limit $x \rightarrow 0^+$, the Bessel-type functions behave asymptotically in the form*

$$(2.22) \quad \left\{ \begin{array}{l} J_\lambda^{0,M}(x) = 1 - \frac{1}{4}(1 + \frac{1}{8}M\lambda^2)\lambda^2x^2 + \frac{1}{192}(3 + \frac{1}{2}M\lambda^2)\lambda^4x^4 + \mathcal{O}(x^6) \\ Y_\lambda^{0,M}(x) = \frac{1}{2}Mx^{-2} + \frac{1}{8}M\lambda^2 + \gamma + \frac{1}{2}\log(\frac{1}{4}\lambda^2x^2) - \left\{ \frac{M}{128}\lambda^4 + \frac{1}{4}(\gamma - 1)(1 + \frac{M}{8}\lambda^2)\lambda^2 \right\} x^2 \\ \quad - \frac{1}{8}(1 + \frac{1}{8}M\lambda^2)\lambda^2x^2 \log(\frac{1}{4}\lambda^2x^2) + \mathcal{O}(x^4 \log(x)) \\ I_\lambda^{0,M}(x) = 1 - \frac{1}{4}(1 + \frac{1}{8}M\lambda^2)\lambda^2x^2 + \frac{1}{192}(1 + \frac{1}{2}M\lambda^2)(\lambda^2 + 8M^{-1})^2x^4 + \mathcal{O}(x^6) \\ K_\lambda^{0,M}(x) = -\frac{1}{2}Mx^{-2} + 1 + \frac{1}{8}M\lambda^2 - \gamma - \frac{1}{2}\log(\frac{1}{4}c_\lambda^2x^2) + \left\{ \frac{1}{2M}(1 + \frac{1}{8}M\lambda^2)^2 \right. \\ \quad \left. - \frac{\gamma-1}{4}(1 + \frac{1}{8}M\lambda^2)\lambda^2 \right\} x^2 + \frac{1}{8}(1 + \frac{1}{8}M\lambda^2)\lambda^2x^2 \log(\frac{1}{4}c_\lambda^2x^2) + \mathcal{O}(x^4 \log(x)). \end{array} \right.$$

3. FOUR SOLUTIONS OF FROBENIUS TYPE

The purpose of this section is to derive four solutions of the Bessel-type equation (2.1) which behave at the origin in the form

$$(3.1) \quad y_r(x, \lambda, M) \cong x^r \text{ as } x \rightarrow 0^+ \text{ for } r = -2, 0, 2, 4.$$

Clearly, we may choose $J_\lambda^{0,M}(x)$ with $x \in [0, \infty)$, for $y_0(x, \lambda, M)$. Moreover, the function

$$y_{-2}(x, \lambda, M) := 2M^{-1}Y_\lambda^{0,M}(x) \text{ for all } x \in (0, \infty)$$

satisfies (3.1) for $r = -2$.

In order to determine two further solutions of equation (2.1) having zeros of order 2 and 4 at the origin, respectively, we require appropriate linear combinations of all the four Bessel-type functions, given in (2.18) through to (2.21).

Corollary 3.1. *(i) A unique solution of the Bessel-type equation (2.1) satisfying property (3.1) for $r = 4$, is given by*

$$(3.2) \quad y_4(x, \lambda, M) := 3M^2a_\lambda^{-3} \left[J_\lambda^{0,M}(x) - I_\lambda^{0,M}(x) \right] = x^4 + \mathcal{O}(x^6) \text{ as } x \rightarrow 0^+.$$

(ii) The sum of the two singular Bessel-type functions is a regular solution of equation (2.1) with asymptotic behaviour

$$(3.3) \quad \begin{aligned} Y_\lambda^{0,M}(x) + K_\lambda^{0,M}(x) &= \left[a_\lambda + \frac{1}{2}\log(\lambda^2c_\lambda^{-2}) \right] + \frac{1}{2M} \left[a_\lambda + \frac{1}{2}(1 - a_\lambda^2)\log(\lambda^2c_\lambda^{-2}) \right] x^2 \\ &\quad + \mathcal{O}(x^4 \log(x)) \text{ as } x \rightarrow 0^+. \end{aligned}$$

(iii) A solution of equation (2.1) satisfying property (3.1) for $r = 2$, is given by

$$(3.4) \quad \begin{aligned} y_2(x, \lambda, M) &:= 2Ma_\lambda^{-3} \left\{ Y_\lambda^{0,M}(x) + K_\lambda^{0,M}(x) - \left[a_\lambda + \frac{1}{2}\log(\lambda^2c_\lambda^{-2}) \right] J_\lambda^{0,M}(x) \right\} \\ &= x^2 + \mathcal{O}(x^4 \log(x)) \text{ as } x \rightarrow 0^+. \end{aligned}$$

Proof. (i) Employing the series representations (2.18) and (2.20) we have

$$(3.5) \quad J_\lambda^{0,M}(x) - I_\lambda^{0,M}(x) = \sum_{k=0}^{\infty} \frac{\delta_k}{k!(k+1)!} x^{2k}$$

where for all $k \in \mathbb{N}_0$,

$$\begin{aligned} \delta_k &= (1 + ka_\lambda) \left(-\frac{1}{4}\lambda^2\right)^k - (1 - ka_\lambda) \left(\frac{1}{4}c_\lambda^2\right)^k \\ &= M^{-k} \left\{ (1 + ka_\lambda)(1 - a_\lambda)^k - (1 - ka_\lambda)(1 + a_\lambda)^k \right\}. \end{aligned}$$

The first three coefficients simplify to $\delta_0 = \delta_1 = 0$ and $\delta_2 = (2/M)^2 a_\lambda^3$. So on multiplying equation (3.5) by $2!3!\delta_2^{-1}$ we obtain (3.2). The uniqueness of the solution $y_4(x, \lambda, M)$ becomes clear when considering the items (ii) and (iii).

(ii) In view of (2.19) and (2.21) we have

$$(3.6) \quad \tilde{y}_0(x, \lambda, M) := Y_\lambda^{0,M}(x) + K_\lambda^{0,M}(x) = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{(k+1)!(k+1)!} x^{2k} + \sum_{k=0}^{\infty} \frac{\zeta_k}{k!(k+1)!} x^{2k}$$

where the coefficients of the first sum yield, for all $k \in \mathbb{N}_0$,

$$\varepsilon_k = \frac{M}{2} \left[\left(\frac{1}{4}c_\lambda^2\right)^{k+1} - \left(-\frac{1}{4}\lambda^2\right)^{k+1} \right] = \frac{1}{2}M^{-k} \left[(1 + a_\lambda)^{k+1} - (1 - a_\lambda)^{k+1} \right].$$

Hence $\varepsilon_0 = a_\lambda$ and $\varepsilon_1 = \frac{2}{M}a_\lambda$. Concerning the second sum we obtain

$$\begin{aligned} \zeta_k &= (1 + ka_\lambda) \left(-\frac{1}{4}\lambda^2\right)^k \left[\frac{1}{2} \log\left(\frac{1}{4}\lambda^2 x^2\right) - \psi(k+1) \right] \\ &\quad - (1 - ka_\lambda) \left(\frac{1}{4}c_\lambda^2\right)^k \left[\frac{1}{2} \log\left(\frac{1}{4}c_\lambda^2 x^2\right) - \psi(k+1) \right]. \end{aligned}$$

so that

$$\zeta_0 = \frac{1}{2} \log\left(\frac{1}{4}\lambda^2 x^2\right) - \frac{1}{2} \log\left(\frac{1}{4}c_\lambda^2 x^2\right) = \frac{1}{2} \log(\lambda^2 c_\lambda^{-2})$$

$$\begin{aligned} \zeta_1 &= \frac{1}{M}(1 - a_\lambda^2) \left[\frac{1}{2} \log\left(\frac{1}{4}\lambda^2 x^2\right) - \psi(2) \right] - \frac{1}{M}(1 - a_\lambda^2) \left[\frac{1}{2} \log\left(\frac{1}{4}c_\lambda^2 x^2\right) - \psi(2) \right] \\ &= \frac{1}{2M}(1 - a_\lambda^2) \log(\lambda^2 c_\lambda^{-2}). \end{aligned}$$

Combining the first two terms in both sums on the right-hand side of (3.6) then gives (3.3).

(iii) By (2.18) we find as $x \rightarrow 0^+$,

$$\begin{aligned} \left[a_\lambda + \frac{1}{2} \log(\lambda^2 c_\lambda^{-2}) \right] J_\lambda^{0,M}(x) &= \left[a_\lambda + \frac{1}{2} \log(\lambda^2 c_\lambda^{-2}) \right] \left[1 + \frac{1}{2M}(1 - a_\lambda^2)x^2 + O(x^4) \right] \\ &= \left[a_\lambda + \frac{1}{2} \log(\lambda^2 c_\lambda^{-2}) \right] + \frac{1}{2M}(1 - a_\lambda^2)x^2 + \frac{1}{4M}(1 - a_\lambda^2) \log(\lambda^2 c_\lambda^{-2})x^2 + \mathcal{O}(x^4). \end{aligned}$$

Subtracting this last expression from the right-hand side of (3.3) gives, as $x \rightarrow 0^+$,

$$Y_\lambda^{0,M}(x) + K_\lambda^{0,M}(x) - \left[a_\lambda + \frac{1}{2} \log(\lambda^2 c_\lambda^{-2}) \right] J_\lambda^{0,M}(x) = \frac{1}{2M}a_\lambda^3 x^2 + \mathcal{O}(x^4 \log(x)).$$

Now multiply both sides by $2Ma_\lambda^{-3}$ to arrive at (3.4). □

4. THE BESSEL-TYPE EQUATION AND ITS SOLUTIONS IN THE LIMIT $M \rightarrow +\infty$

Let us recall from (1.22) that in the limit as $M \rightarrow \infty$ the Bessel-type equation tends to the equation

$$(4.1) \quad [xy''(x)]'' - [9x^{-1}y'(x)]' = \lambda^4 xy(x) \text{ for all } x \in (0, \infty).$$

In view of (2.1), its Frobenius form is given by

$$(4.2) \quad (D_x^4 + 2x^{-1}D_x^3 - 9x^{-2}D_x^2 + 9x^{-3}D_x) [y](x) = \lambda^4 y(x).$$

The aim of this section is to derive four linearly independent solutions of this equation by employing the solution basis (1.5) to (1.8) of the Bessel type equation (1.1). Let J_2, Y_2 and I_2, K_2 denote the Bessel and modified Bessel functions of order $\nu = 2$.

Theorem 4.1. *A basis of solutions of the equation (4.1) is given by*

$$(4.3) \quad \begin{cases} J_\lambda^{0,\infty}(x) := J_2(\lambda x) \text{ and } I_\lambda^{0,\infty}(x) := I_2(\lambda x) \text{ for all } x \in (0, \infty) \text{ and } \lambda \in \mathbb{C} \\ Y_\lambda^{0,\infty}(x) := Y_2(\lambda x) \text{ and } K_\lambda^{0,\infty}(x) := K_2(\lambda x) \text{ for all } x \in (0, \infty) \text{ and } \lambda \in \mathbb{C} \setminus (-\infty, 0]. \end{cases}$$

Proof. The expression $a_\lambda = 1 + \frac{1}{4}M\lambda^2$ does not vanish if $\lambda = 0$ or if M is chosen large enough. So we may divide the Bessel-type functions (1.5) to (1.8) by $-a_\lambda$ and let $M \rightarrow \infty$. Observing that

$$(4.4) \quad \begin{cases} -J_0(z) + 2z^{-1}J_1(z) = J_2(z) & \text{and} & -Y_0(z) + 2z^{-1}Y_1(z) = Y_2(z) \\ I_0(z) - 2z^{-1}I_1(z) = I_2(z) & \text{and} & K_0(z) + 2z^{-1}K_1(z) = K_2(z) \end{cases}$$

for any $z \neq 0$, see [12, 7.2.8 and 7.11], we arrive, at least formally, at the required solutions (4.3) of equation (4.1).

To make these results rigorous, let us define the two second-order differential expressions by, see (1.17),

$$(4.5) \quad A_x^\infty := -\lim_{M \rightarrow \infty} \left[\frac{4}{M} A_x^M \right] := D_x^2 - x^{-1}D_x \quad \text{and} \quad B_x^\infty := -\lim_{M \rightarrow \infty} \left[\frac{4}{M} B_x^M \right] := D_x^2 + 3x^{-1}D_x.$$

An application of the well-known identities, see [12, 7.2.8 (52), (53) for $m = 2$],

$$(4.6) \quad \begin{cases} (z^{-1}d/dz)^2 [z^{-\nu}J_\nu(z)] = z^{-\nu-2}J_{\nu+2}(z) \\ (z^{-1}d/dz)^2 [z^\nu J_\nu(z)] = z^{\nu-2}J_{\nu-2}(z), \end{cases}$$

then yields

$$(4.7) \quad A_x^\infty [J_0(\lambda \cdot)](x) = \lambda^2 J_2(\lambda x) \text{ and } B_x^\infty [J_2(\lambda \cdot)](x) = \lambda^2 J_0(\lambda x)$$

and hence

$$(4.8) \quad (A_x^\infty \circ B_x^\infty) [J_2(\lambda \cdot)](x) = \lambda^2 A_x^\infty [J_0(\lambda \cdot)](x) = \lambda^4 J_2(\lambda x).$$

On the other hand, a direct calculation shows that for any smooth function $y(x)$,

$$(4.9) \quad \begin{cases} (A_x^\infty \circ B_x^\infty) [y](x) & := [D_x^2 - x^{-1}D_x] [D_x^2 + 3x^{-1}D_x] [y](x) \\ & = (D_x^4 + 2x^{-1}D_x^3 - 9x^{-2}D_x^2 + 9x^{-3}D_x) [y](x). \end{cases}$$

Consequently, $y(x) = J_2(\lambda x)$ is a solution of the equation (4.2).

Concerning the other three solutions stated in Theorem 4.1 we just note that the identities (4.6) to (4.8) also hold if the Bessel function J_ν is replaced by any of the three other functions Y_ν, I_ν, K_ν , see [12, 7.2.8, 7.11]. \square

Remark 4.1. In the proof of case $\alpha = 1$ in [7, Corollary 3.3] we already showed that for any smooth function $y(x)$,

$$(4.10) \quad (A_x^\infty \circ B_x^\infty) [x^2 y(x)] = x^2 [D_x^2 + 5x^{-1} D_x]^2 [y](x).$$

Choosing here $y(x) = J_\lambda^2(x) := 8(\lambda x)^{-2} J_2(\lambda x)$ and observing that $J_\lambda^2(x)$ satisfies the particular case $\nu = 2$ of the classical Bessel equation

$$[D_x^2 + (2\nu + 1)x^{-1} D_x] [J_\lambda^\nu](x) = -\lambda^2 J_\lambda^\nu(x),$$

we arrive again at (4.8):

$$\begin{aligned} (A_x^\infty \circ B_x^\infty) [J_2(\lambda \cdot)](x) &= \frac{1}{8} \lambda^2 (A_x^\infty \circ B_x^\infty) [x^2 J_\lambda^2(x)] \\ &= \frac{1}{8} \lambda^2 x^2 [D_x^2 + 5x^{-1} D_x]^2 [J_\lambda^2](x) \\ &= \frac{1}{8} (\lambda x)^2 \lambda^4 J_\lambda^2(x) \\ &= \lambda^4 J_2(\lambda x). \end{aligned}$$

In view of the series representations of the four Bessel functions of order $\nu = 2$, see [17, 2.11, 3.52, 3.7, 3.71] and [12, 7.2.1, 7.2.4, 7.2.2, 7.2.5], we readily obtain

$$(4.11) \quad \left\{ \begin{array}{l} J_\lambda^{0,\infty}(x) = \frac{1}{8} (\lambda x)^2 {}_0F_1(-; 3; -\frac{1}{4}(\lambda x)^2) \\ \quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{1}{2}\lambda x\right)^{2k+2} \\ Y_\lambda^{0,\infty}(x) = -2(\lambda x)^{-2} - \frac{1}{2} + \\ \quad \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{1}{2}\lambda x\right)^{2k+2} [\log(\frac{1}{4}\lambda^2 x^2) - \psi(k+1) - \psi(k+3)] \\ I_\lambda^{0,\infty}(x) = \frac{1}{8} (\lambda x)^2 {}_0F_1(-; 3; \frac{1}{4}(\lambda x)^2) \\ \quad = \sum_{k=0}^{\infty} \frac{1}{k!(k+2)!} \left(\frac{1}{2}\lambda x\right)^{2k+2} \\ K_\lambda^{0,\infty}(x) = 2(\lambda x)^{-2} - \frac{1}{2} - \\ \quad \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!(k+2)!} \left(\frac{1}{2}\lambda x\right)^{2k+2} [\log(\frac{1}{4}\lambda^2 x^2) - \psi(k+1) - \psi(k+3)]. \end{array} \right.$$

Corollary 4.1. For $0 < \lambda x < \infty$ we have, as $\lambda x \rightarrow 0^+$,

$$(4.12) \quad \left\{ \begin{array}{l} J_\lambda^{0,\infty}(x) - I_\lambda^{0,\infty}(x) = -\sum_{j=0}^{\infty} 2 \frac{(\frac{1}{2}\lambda x)^{4j+4}}{(2j+1)!(2j+3)!} \\ \qquad \qquad \qquad = -\frac{1}{3} (\frac{1}{2}\lambda x)^4 + \mathcal{O}((\lambda x)^8) \\ \\ Y_\lambda^{0,\infty}(x) + K_\lambda^{0,\infty}(x) = -1 - \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\lambda x)^{4j+4}}{(2j+1)!(2j+3)!} [\log(\frac{1}{4}\lambda^2 x^2) - \psi(2j+2) - \psi(2j+4)] \\ \qquad \qquad \qquad = -1 + \mathcal{O}((\lambda x)^4 \log(\frac{1}{4}\lambda^2 x^2)). \end{array} \right.$$

5. REMARKS

Remark 5.1. According to [7, Theorem 4.1] the differential expression B_x^M defined in (1.17) may be interpreted as the “weighted” formal adjoint of A_x^M in the sense that

$$B_x^M[y](x) = x^{-1} [xA_x^M]^+ [y](x) \text{ for any } y \in C^{(2)}(0, \infty).$$

Moreover, [7, Theorem 4.2] states

$$\begin{aligned} & \int_0^b J_\lambda^{0,M}(x) J_\mu^{0,M}(x) x \, dx + \frac{M}{2} J_\lambda^{0,M}(0) J_\mu^{0,M}(0) \\ &= \int_0^b A_x^M [J_\lambda^0](x) J_\mu^{0,M}(x) x \, dx + \frac{M}{2} \\ &= a_\lambda a_\mu^{-1} \int_0^b J_\lambda^0(x) B_x^M [J_\mu^{0,M}](x) x \, dx + R(b; \lambda, \mu, M) \\ &= a_\lambda a_\mu \int_0^b J_\lambda^0(x) J_\mu^0(x) x \, dx + R(b; \lambda, \mu, M) \end{aligned}$$

where

$$R(b; \lambda, \mu, M) := M J_\lambda^0(b) J_\mu^0(b) - \frac{1}{4} M^2 (J_\lambda^0)'(b) (J_\mu^0)'(b)$$

tends to 0 as $b \rightarrow \infty$.

This result is to be seen in the light of the following two distributional orthogonality relations of the classical Bessel and Bessel-type functions, respectively; see [6, Propositions 2.1 and 2.5]. Given $\mu > 0$ we have, for all $\lambda > 0$,

$$(5.1) \quad \lambda \int_0^\infty J_\lambda^0(x) J_\mu^0(x) x \, dx = \delta(\lambda - \mu),$$

$$(5.2) \quad \lambda a_\lambda^{-2} \left\{ \int_0^\infty J_\lambda^{0,M}(x) J_\mu^{0,M}(x) x \, dx + \frac{M}{2} J_\lambda^{0,M}(0) J_\mu^{0,M}(0) \right\} = \delta(\lambda - \mu)$$

in the sense of distributions on $C_0^\infty(0, \infty)$. While (5.1) is basic for the classical Hankel transform, the property (5.2) has given rise to the definition of a generalized Hankel transform, see [6].

According to the construction of the functions

$$J_\lambda^{0,\infty}(x) := J_2(\lambda x) = - \lim_{M \rightarrow \infty} \left[(a_\lambda)^{-1} J_\lambda^{0,M}(x) \right]$$

in Section 4, the distributional orthogonal relation (5.2) reduces, in the limit as $M \rightarrow \infty$, to

$$(5.3) \quad \lambda \int_0^\infty J_\lambda^{0,\infty}(x) J_\mu^{0,\infty}(x) x \, dx = \delta(\lambda - \mu).$$

Since $J_2(\lambda x) = \frac{1}{8}(\lambda x)^2 J_\lambda^2(x)$, this is equivalent to the distributional orthogonality relation of the classical Bessel functions, see [7, (1.7), case $\alpha = 2$]

$$(5.4) \quad \lambda \int_0^\infty J_2(\lambda x) J_2(\mu x) x \, dx = (2^2 \Gamma(3))^{-2} \lambda^5 \int_0^\infty J_\lambda^2(x) J_\mu^2(x) x^5 \, dx = \delta(\lambda - \mu).$$

Remark 5.2. Let us rewrite the fourth-order Bessel-type equation (2.1) by

$$(5.5) \quad \left[L_M^{(4)} + (4/M)^2 \right] [y](x) = \omega^2 y(x) \text{ for all } x \in (0, \infty)$$

for any complex parameter ω . Since by (1.4),

$$\lambda^2 c_\lambda^2 + (4/M)^2 = (4/M)^2 (a_\lambda - 1)(a_\lambda + 1) + (4/M)^2 = (4/M)^2 a_\lambda^2,$$

we may identify $\omega = (4/M)a_\lambda$. So in view of (2.18) and (2.20), two regular solutions of equation (5.5) are given by

$$(5.6) \quad \begin{cases} R_\pm(x) &= \sum_{k=0}^{\infty} \left\{ 1 \pm \frac{M}{4} \omega k \right\} \left(\frac{1}{M} \mp \frac{\omega}{4} \right)^k \frac{x^{2k}}{k!(k+1)!} \\ &= {}_1F_2 \left(1 + \frac{4}{M\omega}; \pm \frac{4}{M\omega}, 2; \left[\frac{1}{M} \mp \frac{\omega}{4} \right] x^2 \right). \end{cases}$$

Hence, these two solutions are analytic functions of the spectral parameter $\omega \in \mathbb{C}$. notice that in the case $\omega = 0$ in (5.6) both solutions reduce to the same function

$$(5.7) \quad R(x) = \sum_{k=0}^{\infty} \left(\frac{1}{M} \right)^k \frac{x^{2k}}{k!(k+1)!} = {}_0F_1 \left(-; 2; \frac{1}{M} x^2 \right) = \sqrt{M} x^{-1} I_1 \left(\frac{2}{\sqrt{M}} x \right).$$

Following (2.19) and (2.21), there are two singular solutions of equation (5.5) with representations

$$(5.8) \quad \begin{aligned} S_\pm(x) &= \frac{M}{2} x^{-2} - \frac{M}{2} \sum_{k=0}^{\infty} \left(\frac{1}{M} \mp \frac{\omega}{4} \right)^{k+1} \frac{x^{2k}}{(k+1)!(k+1)!} + \\ &+ \sum_{k=0}^{\infty} \left\{ 1 \pm \frac{M}{4} \omega k \right\} \left(\frac{1}{M} \mp \frac{\omega}{4} \right)^k \frac{x^{2k}}{k!(k+1)!} \left[\frac{1}{2} \log \left(\left[\frac{\omega}{4} \mp \frac{1}{M} \right] x^2 \right) - \psi(k+1) \right]. \end{aligned}$$

When considering the two functions $S_\pm(x)$ for any fixed $x \in (0, \infty)$, as a function of the spectral parameter ω , they are analytically defined on the complex plane \mathbb{C} cut along the negative real axis.

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