

THE LEFT-DEFINITE SPECTRAL THEORY FOR THE CLASSICAL HERMITE DIFFERENTIAL EQUATION

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Dedicated to Professor A. M. Krall upon his retirement

ABSTRACT. In this paper, we develop the left-definite spectral theory associated with the self-adjoint operator A in $L^2((-\infty, \infty); \exp(-t^2))$, generated from the classic second-order Hermite differential equation

$$\ell_H[y](t) = -y'' + 2ty' + ky = \lambda y \quad (t \in (-\infty, \infty)),$$

that has the Hermite polynomials $\{H_m(t)\}_{m=0}^\infty$ as eigenfunctions. More specifically, for each $n \in \mathbb{N}$, we explicitly determine the unique left-definite Hilbert-Sobolev space W_n and associated inner product $(\cdot, \cdot)_n$, which is generated from the n^{th} integral power $\ell_H^n[\cdot]$ of $\ell_H[\cdot]$. Moreover, for each $n \in \mathbb{N}$, we determine the corresponding unique left-definite self-adjoint operator A_n in W_n and characterize its domain in terms of another left-definite space. As a consequence of this, we explicitly determine the domain of each integral power of A and, in particular, we obtain a new characterization of the domain of the classical right-definite operator A .

1. INTRODUCTION

When A is an unbounded self-adjoint operator in a Hilbert space $(H, (\cdot, \cdot))$ that is bounded below by a positive multiple of the identity operator, the authors in [4] show that there is a continuum of unique Hilbert spaces $\{(W_r, (\cdot, \cdot)_r)\}_{r>0}$ and, for each $r > 0$, a unique self-adjoint restriction A_r of A in W_r . The Hilbert space W_r is called the r^{th} *left-definite Hilbert space* associated with the pair (H, A) and the operator A_r is called the r^{th} *left-definite operator* associated with (H, A) .

Left-definite spectral theory has its roots in the classic treatise of Weyl [12] on the theory of formally symmetric second-order differential expressions. We remark, however, that even though our motivation for the general left-definite theory developed in [4] arose through our interest in certain self-adjoint differential operators, the theory developed in [4] can be applied to an arbitrary self-adjoint operator that is bounded below.

The terminology *left-definite* is due to Schäfke and Schneider (who used the German *Links-definit*) [9] in 1965 and describes one of the Hilbert space settings in which certain formally symmetric differential expressions can be analyzed. For example, consider the differential equation

$$(1.1) \quad S[y](t) = \lambda w(t)y(t) \quad (t \in I; \lambda \in \mathbb{C}),$$

where $I = (a, b)$ is an open interval of the real line \mathbb{R} , w is Lebesgue measurable, locally integrable and positive almost everywhere on I , and where $S[\cdot]$ is the formally symmetric differential expression

$$S[y](t) = \sum_{j=0}^n (-1)^j (b_j(t)y^{(j)}(t))^{(j)} \quad (t \in I),$$

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with *non-negative*, infinitely differentiable coefficients $b_j(t)$ ($j = 0, 1, \dots, n$) on I (as in all known cases when (1.1) has a sequence of orthogonal polynomial solutions). The classical Glazman-Krein-Naimark theory (see [6]) applies to (1.1) and describes all self-adjoint extensions of the minimal operator T_{\min} , generated by $w^{-1}S[\cdot]$, in the weighted Hilbert space $L_w^2(I)$ of all Lebesgue measurable functions $f : I \rightarrow \mathbb{C}$ with inner product

$$(f, f) = \int_I |f(t)|^2 w(t) dt < \infty.$$

Due to the appearance of w on the right-hand side of (1.1), the space $L_w^2(I)$ is called the *right-definite Hilbert space* for $w^{-1}S[\cdot]$. On the other hand, spectral properties of the differential expression $w^{-1}S[\cdot]$ can also be studied in a Hilbert space W generated by the Sobolev inner product

$$(f, g)_W = \sum_{j=0}^n b_j(t) f^{(j)}(t) \overline{g^{(j)}(t)} \quad (f, g \in W),$$

called the *Dirichlet inner product*, which arises naturally in connection to Green's formula for the expression $S[\cdot]$. Since this inner product is generated by the left-hand side of (1.1), we call the spectral study of $w^{-1}S[\cdot]$ in W a *left-definite spectral setting* and call W a *left-definite Hilbert space*.

In this paper, we apply this left-definite theory to the self-adjoint Hermite differential operator A , generated by the classical second-order formally symmetric Hermite differential expression

$$(1.2) \quad \begin{aligned} \ell_H[y](t) &:= -y''(t) + 2ty'(t) + ky(t) \\ &= \exp(t^2) \left(-(\exp(-t^2)y'(t))' + k \exp(-t^2)y(t) \right) \quad (t \in \mathbb{R} = (-\infty, \infty)), \end{aligned}$$

and having the Hermite polynomials as eigenfunctions. Here, k is a fixed, positive constant. The right-definite setting in this case is the Hilbert space $L^2((-\infty, \infty); \exp(-t^2))$ of Lebesgue measurable functions $f : (-\infty, \infty) \rightarrow \mathbb{C}$ satisfying $\|f\| < \infty$, where $\|\cdot\|$ is the norm generated by the inner product

$$(1.3) \quad (f, g) := \int_{-\infty}^{\infty} f(t) \overline{g(t)} \exp(-t^2) dt \quad (f, g \in H),$$

Even though the theory developed in [4] guarantees the existence of a continuum of spaces $\{W_r\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$ (they are all *differential* operators), we can only effectively determine the left-definite spaces, their inner products, and the domains of the left-definite operators when r is a positive integer; reasons for this will be made clear in the analysis below.

The contents of this paper are as follows. In Section 2, we state some of the main results developed in [4]. In Section 3, we review some of the properties of the Hermite differential equation, the Hermite polynomials and the right-definite self-adjoint operator A , generated by the second-order Hermite expression (1.2), having the Hermite polynomials as eigenfunctions. Also in this section, we obtain the formally symmetric form of each integral power of the second-order Hermite expression; as we shall see, these higher order expressions are key to determining the various left-definite inner products. Interestingly, these powers involve the Stirling numbers of the second kind. Lastly, in Section 4, we establish the left-definite theory for the Hermite expression (1.2). Specifically, we determine explicitly

- (a) the sequence $\{W_n\}_{n=1}^{\infty}$ of left-definite spaces associated with the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$,
- (b) the sequence of left-definite self-adjoint operators $\{A_n\}_{n=1}^{\infty}$, and their domains $\{\mathcal{D}(A_n)\}_{n=1}^{\infty}$, associated with $(L^2((-\infty, \infty); \exp(-t^2)), A)$, and
- (c) the domains $\mathcal{D}(A^n)$ of each integral power A^n of A .

These results culminate in Theorem 4.4. An application of this theorem yields a new result (see Corollary 4.1) concerning the characterization of functions in the domain of the right-definite operator A .

2. LEFT-DEFINITE HILBERT SPACES AND LEFT-DEFINITE OPERATORS

Let V denote a vector space (over the complex field \mathbb{C}) and suppose that (\cdot, \cdot) is an inner product with norm $\|\cdot\|$ generated from (\cdot, \cdot) such that $H = (V, (\cdot, \cdot))$ is a Hilbert space. Suppose V_r (the subscripts will be made clear shortly) is a linear manifold of the vector space V and let $(\cdot, \cdot)_r$ and $\|\cdot\|_r$ denote an inner product (quite possibly different from (\cdot, \cdot)) and associated norm, respectively, over V_r . We denote the resulting inner product space by $W_r = (V_r, (\cdot, \cdot)_r)$.

Throughout this paper, we assume that $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint operator that is bounded below by kI , for some $k > 0$; that is,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A));$$

that is to say, A is bounded below in H by kI , where I is the identity operator. It follows that A^r , for each $r > 0$, is a self-adjoint operator that is bounded below in H by $k^r I$.

We now make the definitions of left-definite spaces and left-definite operators.

Definition 2.1. *Let $r > 0$ and suppose V_r is a linear manifold of the Hilbert space $H = (H, (\cdot, \cdot))$ and $(\cdot, \cdot)_r$ is an inner product on V_r . Let $W_r = (V_r, (\cdot, \cdot)_r)$. We say that W_r is an r^{th} **left-definite space** associated with the pair (H, A) if each of the following conditions hold:*

- (1) W_r is a Hilbert space,
- (2) $\mathcal{D}(A^r)$ is a linear manifold of V_r ,
- (3) $\mathcal{D}(A^r)$ is dense in W_r ,
- (4) $(x, x)_r \geq k^r (x, x)$ ($x \in V_r$), and
- (5) $(x, y)_r = (A^r x, y)$ ($x \in \mathcal{D}(A^r)$, $y \in V_r$).

It is not clear, from the definition, if such a self-adjoint operator A generates a left-definite space for a given $r > 0$. However, in [4], the authors prove the following theorem; the Hilbert space spectral theorem plays a prominent role in establishing this result.

Theorem 2.1. (see [4, Theorem 3.1]) *Suppose $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint operator that is bounded below by kI , for some $k > 0$. Let $r > 0$. Define $W_r = (V_r, (\cdot, \cdot)_r)$ by*

$$(2.1) \quad V_r = \mathcal{D}(A^{r/2}),$$

and

$$(x, y)_r = (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r).$$

Then W_r is a left-definite space associated with the pair (H, A^r) . Moreover, suppose $W_r := (V_r, (\cdot, \cdot)_r)$ and $W'_r := (V'_r, (\cdot, \cdot)'_r)$ are r^{th} left-definite spaces associated with the pair (H, A) . Then $V_r = V'_r$ and $(x, y)_r = (x, y)'_r$ for all $x, y \in V_r = V'_r$; i.e. $W_r = W'_r$. That is to say, $W_r = (V_r, (\cdot, \cdot)_r)$ is the unique left-definite space associated with (H, A) .

Definition 2.2. *For $r > 0$, let $W_r = (V_r, (\cdot, \cdot)_r)$ denote the r^{th} left-definite space associated with (H, A) . If there exists a self-adjoint operator $A_r : \mathcal{D}(A_r) \subset W_r \rightarrow W_r$ that is a restriction of A ; that is to say:*

$$A_r f = A f \quad (f \in \mathcal{D}(A_r) \subset \mathcal{D}(A)),$$

*we call such an operator a r^{th} **left-definite operator associated with (H, A) .***

Again, it is not immediately clear that such an A_r exists; in fact, however, A_r exists and is unique.

Theorem 2.2. (see [4, Theorem 3.2]) *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI , for some $k > 0$. For any $r > 0$, let $W_r = (V_r, (\cdot, \cdot)_r)$ be the r^{th} left-definite*

space associated with (H, A) . Then there exists a unique left-definite operator A_r in W_r associated with (H, A) . Moreover,

$$\mathcal{D}(A_r) = V_{r+2}.$$

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of a self-adjoint operator A and each of its associated left-definite operators A_r ($r > 0$) are identical. We recall (see [3, Chapter 7]) that:

- (i) the *point spectrum* $\sigma_p(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $R_\lambda(A) := (A - \lambda I)^{-1}$ does not exist;
- (ii) the *continuous spectrum* $\sigma_c(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $R_\lambda(A)$ exists with a dense domain but is an unbounded operator;
- (iii) the *resolvent set* $\rho(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $R_\lambda(A)$ exists with a dense domain and is a bounded operator;

moreover, for a self-adjoint operator A , we remark that \mathbb{C} is the disjoint union of $\sigma_p(A)$, $\sigma_c(A)$, and $\rho(A)$.

Theorem 2.3. (see [4, Theorem 3.6]) *For each $r > 0$, let A_r denote the r^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by kI , where $k > 0$. Then*

- (a) *the point spectra of A and A_r coincide; i.e. $\sigma_p(A_r) = \sigma_p(A)$;*
- (b) *the continuous spectra of A and A_r coincide; i.e. $\sigma_c(A_r) = \sigma_c(A)$;*
- (c) *the resolvent sets of A and A_r are equal; i.e. $\rho(A_r) = \rho(A)$.*

We refer the reader to [4] for other results established on left-definite theory for self-adjoint operators A that are bounded below.

3. PRELIMINARY RESULTS ON THE HERMITE DIFFERENTIAL EQUATION

When $\lambda = 2m+k$, where $m \in \mathbb{N}_0$, the Hermite equation $\ell_H[y](t) = (\lambda+k)y(t)$, where $\ell_H[\cdot]$ is defined in (1.2) has a polynomial solution $H_m(t)$ of degree m ; the sequence of polynomials $\{H_m(t)\}_{m=0}^\infty$ is called the *Hermite* polynomials. These polynomials form a complete orthogonal set in the Hilbert space $L^2((-\infty, \infty); \exp(-t^2))$ of Lebesgue measurable functions $f : (-\infty, \infty) \rightarrow \mathbb{C}$ satisfying $\|f\| < \infty$, where $\|\cdot\|$ is the norm generated from the inner product (\cdot, \cdot) , defined by

$$(3.1) \quad (f, g) := \int_{-\infty}^{\infty} f(t)\bar{g}(t) \exp(-t^2) dt \quad (f, g \in L^2_H(-\infty, \infty)).$$

In fact, with the m^{th} Hermite polynomial defined by

$$H_m(t) = \frac{(m!)^{1/2} 2^{m/2}}{\pi^{1/4}} \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(-1)^j}{2^{2j} (m-2j)! j!} t^{m-2j} \quad (m \in \mathbb{N}_0),$$

it is the case that the sequence $\{H_m(t)\}_{m=0}^\infty$ is orthonormal in $L^2((-\infty, \infty); \exp(-t^2))$; that is,

$$(3.2) \quad (H_m, H_r) = \delta_{m,r} \quad (m, r \in \mathbb{N}_0),$$

where $\delta_{m,r}$ is the Kronecker delta function. We refer the reader to [7, Chapter 12] or [10, Chapter V] for various properties of the Hermite polynomials. The derivatives of these polynomials satisfy the identity

$$(3.3) \quad \frac{d^j(H_m(t))}{dt^j} = 2^{j/2} (P(m, j))^{1/2} H_{m-j}(t) \quad (m, j \in \mathbb{N}_0),$$

where

$$(3.4) \quad P(m, j) := m(m-1) \dots (m-j+1) \quad (m, j \in \mathbb{N}_0; j \leq m).$$

From (3.2) and (3.3), we see that

$$(3.5) \quad \int_{-\infty}^{\infty} \frac{d^j(H_m(t))}{dt^j} \frac{d^j(H_r(t))}{dt^j} \exp(-t^2) dt = 2^j P(m, j) \delta_{m,r} \quad (m, r, j \in \mathbb{N}_0).$$

The maximal domain Δ of $\exp(-t^2)\ell_H[\cdot]$ in $L^2((-\infty, \infty); \exp(-t^2))$ is defined to be

$$(3.6) \quad \Delta = \{f \in L^2((-\infty, \infty); \exp(-t^2)) \mid f, f' \in AC_{loc}(-\infty, \infty); \ell_H[f] \in L^2((-\infty, \infty); \exp(-t^2))\}.$$

For functions $f, g \in \Delta$ and $[a, b] \subset \mathbb{R}$, we have Dirichlet's formula

$$(3.7) \quad \int_a^b \ell_H[f](t) \bar{g}(t) \exp(-t^2) dt = -\exp(-t^2) f'(t) \bar{g}(t) \Big|_a^b \\ + \int_a^b [f'(t) \bar{g}'(t) \exp(-t^2) + k f(t) \bar{g}(t) \exp(-t^2)] dt.$$

It is well-known (for example, see [5] and [11]) that $\exp(-t^2)\ell_H[\cdot]$ is strong limit point at $t = \pm\infty$:

$$(3.8) \quad \lim_{t \rightarrow \pm\infty} \exp(-t^2) f'(t) \bar{g}(t) = 0 \quad (f, g \in \Delta),$$

and Dirichlet at $t = \pm\infty$:

$$(3.9) \quad \int_0^{\infty} |f'(t)|^2 \exp(-t^2) dt, \quad \int_{-\infty}^0 |f'(t)|^2 \exp(-t^2) dt < \infty \quad (f \in \Delta).$$

From the Glazman-Krein-Naimark theory (see [6, Theorem 4, pages 79-80]), it follows that the operator (in the terminology of this paper, the right-definite operator)

$$A : L^2((-\infty, \infty); \exp(-t^2)) \rightarrow L^2((-\infty, \infty); \exp(-t^2)),$$

defined by

$$(3.10) \quad (Af)(t) = \ell_H[f](t) \quad (f \in \mathcal{D}(A); \text{ a.e. } t \in (-\infty, \infty)),$$

with domain

$$(3.11) \quad \mathcal{D}(A) = \Delta,$$

is self-adjoint and has the Hermite polynomials as a complete set of eigenfunctions (see also [2, Appendix II, pages 210-211] and [11]); moreover, the spectrum of A is given by

$$(3.12) \quad \sigma(A) = \{2m + k \mid m \in \mathbb{N}_0\}.$$

From (3.7), (3.8), and (3.9), it follows that

$$(3.13) \quad (Af, f) = \int_{-\infty}^{\infty} [|f'(t)|^2 \exp(-t^2) + k |f(t)|^2 \exp(-t^2)] dt \geq k(f, f) \quad (f \in \mathcal{D}(A));$$

that is, A is bounded below in $L^2((-\infty, \infty); \exp(-t^2))$ by kI . It is this inequality that explains the importance of the term $ky(t)$ in (1.2). Consequently, we can apply Theorems 2.1, 2.2, and 2.3. Notice that $(\cdot, \cdot)_1$, defined by

$$(f, g)_1 = \int_{-\infty}^{\infty} [f'(t) \bar{g}'(t) \exp(-t^2) + k f(t) \bar{g}(t) \exp(-t^2)] dt \quad (f, g \in \mathcal{D}(A)),$$

is an inner product; in fact, it is the inner product for the first left-definite space associated with the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$. Moreover, the closure of $\mathcal{D}(A)$ in the topology generated from this inner product is the first left-definite space W_1 associated with the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$.

We now turn our attention to the explicit construction of the sequence of left-definite inner products $(\cdot, \cdot)_n$ ($n \in \mathbb{N}$) associated with $(L^2((-\infty, \infty); \exp(-t^2)), A)$. As we shall see, these are generated from the integral powers $\ell_H^n[\cdot]$ ($n \in \mathbb{N}$) of the Hermite expression $\ell_H[\cdot]$, inductively given by

$$\ell_H^1[y] = \ell_H[y], \ell_H^2[y] = \ell_H(\ell_H[y]), \dots, \ell_H^n[y] = \ell_H(\ell_H^{n-1}[y]) \quad (n \in \mathbb{N}).$$

A key to the explicit determination of these powers are certain numbers $\{b_j(n, k)\}_{j=0}^n$ which we now define.

Definition 3.1. For $n \in \mathbb{N}$ and $j \in \{0, 1, \dots, n\}$, let

$$(3.14) \quad b_j(n, k) := \sum_{i=0}^j \frac{(-1)^{i+j}}{j!} \binom{j}{i} (k+i)^n.$$

If we expand the term $(k+i)^n$ in (3.14) and switch the order of summation, we find that

$$(3.15) \quad \begin{aligned} b_j(n, k) &= \sum_{m=0}^n \left(\sum_{i=0}^j \frac{(-1)^{i+j}}{j!} \binom{j}{i} i^{n-m} \right) \binom{n}{m} k^m \\ &= \sum_{m=0}^n \binom{n}{m} S_{n-m}^{(j)} k^m, \end{aligned}$$

where

$$(3.16) \quad S_n^{(j)} = \sum_{i=0}^j \frac{(-1)^{i+j}}{j!} \binom{j}{i} i^n \quad (n, j \in \mathbb{N}_0)$$

is the Stirling number of the second kind. By definition, $S_n^{(j)}$ is the number of ways of partitioning n elements into j non-empty subsets (in particular, $S_0^j = 0$ for any $j \in \mathbb{N}$); we refer the reader to [1, pp. 824-825] for various properties of these numbers. Consequently, we see that

$$(3.17) \quad b_0(n, k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases}$$

and, for $j \in \{1, 2, \dots, n\}$,

$$(3.18) \quad b_j(n, k) = \begin{cases} S_n^{(j)} & \text{if } k = 0 \\ \sum_{m=0}^{n-1} \binom{n}{m} S_{n-m}^{(j)} k^m & \text{if } k > 0. \end{cases}$$

In particular, observe that when $k > 0$, each $b_j(n, k)$ is positive. In [4], Littlejohn and Wellman show that these numbers $\{b_j(n, k)\}_{j=0}^n$ appear in the symmetric form of the n^{th} power of the classical Laguerre differential expression $\ell_\alpha[\cdot]$, defined by

$$\begin{aligned} \ell_\alpha[y](t) &= -ty'' + (t-1-\alpha)y' + ky \quad (t \in (0, \infty)) \\ &= t^{-\alpha} \exp(t) \left[- (t^{\alpha+1} \exp(-t)y'(t))' + kt^\alpha \exp(-t)y(t) \right]; \end{aligned}$$

indeed, they prove that

$$t^\alpha e^{-t} \ell_\alpha^n[y](t) = \sum_{j=0}^n (-1)^j \left(b_j(n, k) t^{\alpha+j} e^{-t} y^{(j)}(t) \right)^{(j)} \quad (n \in \mathbb{N}).$$

Moreover, in [4], the authors prove the following result concerning the numbers $\{b_j(n, k)\}_{j=0}^n$, which is important in our discussion on the Hermite differential expression.

Lemma 3.1. For each $n \in \mathbb{N}$, the numbers $b_j = b_j(n, k)$, defined in (3.14), are the unique solutions to the equations

$$(3.19) \quad (m+k)^n = \sum_{j=0}^n P(m, j) b_j \quad (m \in \mathbb{N}_0),$$

where $P(m, j)$ is defined in (3.4).

With \mathcal{P} denoting the space of all (possibly complex-valued) polynomials, we are now in position to prove the following theorem.

Theorem 3.1. Let $n \in \mathbb{N}$ and let $\ell_H[\cdot]$ denote the Hermite differential expression defined in (1.2). Then

(a)

$$(3.20) \quad \int_{-\infty}^{\infty} \ell_H^n[p](t) \bar{q}(t) \exp(-t^2) dt = \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} p^{(j)}(t) \bar{q}^{(j)}(t) \exp(-t^2) dt \quad (p, q \in \mathcal{P}),$$

where

$$(3.21) \quad c_0(n, k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases}$$

and, for $j = 1, 2, \dots, n$,

$$(3.22) \quad c_j(n, k) = \begin{cases} 2^{n-j} S_n^{(j)} & \text{if } k = 0 \\ 2^{n-j} \sum_{m=0}^{n-1} \binom{n}{m} S_{n-m}^{(j)} \left(\frac{k}{2}\right)^m & \text{if } k > 0. \end{cases}$$

(b) For $k > 0$, each $c_j(n, k)$ is positive ($j = 0, 1, \dots, n$).

(c) For each $n \in \mathbb{N}$, the n^{th} power $\ell_H^n[\cdot]$ of the Hermite expression $\ell_H[\cdot]$ is Lagrangian symmetrizable with symmetry factor $w(t) = \exp(-t^2)$ and the Lagrangian symmetric form of $\exp(-t^2)\ell_H^n[\cdot]$ is given by

$$(3.23) \quad \exp(-t^2)\ell_H^n[y](t) = \sum_{j=0}^n (-1)^j \left(c_j(n, k) \exp(-t^2) y^{(j)}(t) \right)^{(j)}.$$

Proof. The fact that the numbers $\{c_j(n, k)\}_{j=0}^n$ are positive follows from the positivity of the numbers $\{b_j(n, k)\}_{j=0}^n$ for $k > 0$. Since the Hermite polynomials $\{H_m(t)\}_{m=0}^{\infty}$ form a basis for \mathcal{P} , it suffices to show (3.20) is valid for $p = H_m(t)$ and $q = H_r(t)$, where $m, r \in \mathbb{N}_0$ are arbitrary. From the identity

$$\ell_H^n[H_m](t) = (2m+k)^n H_m(t) \quad (m \in \mathbb{N}_0),$$

it follows, with this particular choice of p and q , that the left-hand side of (3.20) reduces to $(2m+k)^n \delta_{m,r}$. On the other hand, from (3.3), the right-hand side of (3.20) yields

$$(3.24) \quad \begin{aligned} & \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} \frac{d^j(H_m(t))}{dt^j} \frac{d^j(H_r(t))}{dt^j} \exp(-t^2) dt \\ &= \sum_{j=0}^n 2^j P(m, j) c_j(n, k) \delta_{m,r} \text{ by (3.3).} \end{aligned}$$

Consequently, the identity in (3.20) holds if and only if

$$(3.25) \quad (2m+k)^n = \sum_{j=0}^n 2^j P(m, j) c_j(n, k),$$

or, after rearranging terms,

$$\left(m + \frac{k}{2}\right)^n = \sum_{j=0}^n b_j(n, k) P(m, j),$$

where

$$b_j(n, k) = 2^{j-n} c_j(n, k).$$

From Lemma 3.1, (3.17) and (3.18), it follows that the numbers $\{c_j(n, k)\}_{j=0}^n$ are given as in (3.21) and (3.22), establishing (3.20).

To prove (3.23), define the differential expression

$$(3.26) \quad m_H[y](t) := \exp(t^2) \sum_{j=0}^n (-1)^j \left(c_j(n, k) \exp(-t^2) y^{(j)}(t) \right)^{(j)},$$

where the numbers $c_j(n, k)$ ($j = 0, 1, \dots, n$) are as above. For $p, q \in \mathcal{P}$ and $[a, b] \subset (-\infty, \infty)$, we apply integration by parts to obtain

$$\begin{aligned} & \int_a^b m_H[p](t) \bar{q}(t) \exp(-t^2) dt \\ &= \sum_{j=0}^n (-1)^j c_j(n, k) \sum_{r=1}^j (-1)^{r+1} \left(p^{(j)}(t) \exp(-t^2) \right)^{(j-r)} \bar{q}^{(r-1)}(t) \Big|_a^b \\ & \quad + \sum_{j=0}^n c_j(n, k) \int_a^b p^{(j)}(t) \bar{q}^{(j)}(t) \exp(-t^2) dt. \end{aligned}$$

Now, for any $p \in \mathcal{P}$, $(p^{(j)}(t) \exp(-t^2))^{(j-r)} = p_{j,r}(t) \exp(-t^2)$ for some $p_{j,r} \in \mathcal{P}$; in particular,

$$\lim_{t \rightarrow \pm\infty} \left(p^{(j)}(t) \exp(-t^2) \right)^{(j-r)} \bar{q}^{(r-1)}(t) = 0 \quad (p, q \in \mathcal{P}; r, j \in \mathbb{N}, r \leq j).$$

Consequently, as $a \rightarrow -\infty$ and $b \rightarrow \infty$, we see that

$$(3.27) \quad \int_{-\infty}^{\infty} m_H[p](t) \bar{q}(t) \exp(-t^2) dt = \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} p^{(j)}(t) \bar{q}^{(j)}(t) \exp(-t^2) dt \quad (p, q \in \mathcal{P}).$$

Consequently, from (3.27) and (3.20), we see that for all polynomials p and q , we have

$$(\ell_H^n[p] - m_H[p], q) = 0.$$

From the density of polynomials in $L^2((-\infty, \infty); \exp(-t^2))$, it follows that

$$(3.28) \quad \ell_H^n[p](t) = m_H[p](t) \quad (t \in (-\infty, \infty))$$

for all polynomials p . This latter identity implies that the expression $\ell_H^n[\cdot]$ has the form given in (3.23). \square

For example, we see from this theorem that

$$\exp(-t^2)\ell_H^2[y](t) = (\exp(-t^2)y'')'' - ((2k+2)\exp(-t^2)y')' + k^2\exp(-t^2)y,$$

and

$$\begin{aligned} \exp(-t^2)\ell_H^3[y](t) &= -(\exp(-t^2)y''')''' + ((3k+6)\exp(-t^2)y'')'' \\ &\quad - ((3k^2+6k+4)\exp(-t^2)y')' + k^3\exp(-t^2)y \end{aligned}$$

The following corollary lists some additional properties of $\ell_H^n[\cdot]$.

Corollary 3.1. *Let $n \in \mathbb{N}$. Then*

(a) *the n^{th} power of the classical Hermite differential expression*

$$\mathcal{L}_H[y](t) := -y''(t) + 2ty'(t)$$

is symmetrizable with symmetry factor $w(t) = \exp(-t^2)$ and has the Lagrangian symmetric form

$$\exp(-t^2)\mathcal{L}_H^n[y](t) := \sum_{j=1}^n (-1)^j \left(S_n^{(j)} 2^{n-j} \exp(-t^2)y^{(j)}(t) \right)^{(j)},$$

where $S_n^{(j)}$ is the Stirling number of the second kind defined in (3.16);

(b) *the bilinear form $(\cdot, \cdot)_n$ defined on $\mathcal{P} \times \mathcal{P}$ by*

$$(3.29) \quad (p, q)_n := \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} p^{(j)}(t) \bar{q}^{(j)}(t) \exp(-t^2) dt \quad (p, q \in \mathcal{P})$$

is an inner product when $k > 0$ and satisfies

$$(3.30) \quad (\ell_H^n[p], q) = (p, q)_n \quad (p, q \in \mathcal{P});$$

(c) *the Hermite polynomials $\{H_m(t)\}_{m=0}^{\infty}$ are orthogonal with respect to the inner product $(\cdot, \cdot)_n$; in fact,*

$$(3.31) \quad (H_m, H_r)_n = \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} \frac{d^j(H_m(t))}{dt^j} \frac{d^j(H_r(t))}{dt^j} \exp(-t^2) dt = (2m+k)^n \delta_{m,r}.$$

Proof. The proof of (i) follows immediately from Theorem 3.1 and identities (3.21) and (3.22). The proof of (ii) is clear since all the numbers $\{c_j(n, k)\}_{j=0}^n$ are positive when $k > 0$. The identity in (3.30) follows from (3.27) and (3.28). Lastly, (3.31) is a restatement of (3.24), using (3.25). \square

4. THE LEFT-DEFINITE THEORY FOR THE HERMITE EQUATION

For results that follow in this section, it is convenient to use the following notation. For $n \in \mathbb{N}$, let

$$AC_{\text{loc}}^{(n-1)}(-\infty, \infty) := \{f : (-\infty, \infty) \rightarrow \mathbb{C} \mid f, f', \dots, f^{(n-1)} \in AC_{\text{loc}}(-\infty, \infty)\}.$$

For the rest of this section, we assume that $k > 0$.

Definition 4.1. *For each $n \in \mathbb{N}$, define*

(4.1)

$$V_n := \{f : (-\infty, \infty) \rightarrow \mathbb{C} \mid f \in AC_{\text{loc}}^{(n-1)}(-\infty, \infty); f^{(j)} \in L^2((-\infty, \infty); \exp(-t^2)) \ (j = 0, 1, \dots, n)\}$$

and let $(\cdot, \cdot)_n$ and $\|\cdot\|_n$ denote, respectively, the inner product

$$(4.2) \quad (f, g)_n = \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} f^{(j)}(t) \overline{g^{(j)}(t)} \exp(-t^2) dt \quad (f, g \in V_n),$$

(see (3.29) and (3.30)) and the norm $\|f\|_n = (f, f)_n^{1/2}$, where the numbers $c_j(n, k)$ are defined in (3.21) and (3.22).

The inner product $(\cdot, \cdot)_n$, defined in (4.2), is a Sobolev inner product and is more commonly called the *Dirichlet inner product* associated with the symmetric differential expression $\exp(-t^2)\ell_H^n[\cdot]$.

We remark that, for each $r > 0$, the r^{th} left-definite inner product $(\cdot, \cdot)_r$ is abstractly given by

$$(f, g)_r = \int_{\mathbb{R}} \lambda^r dE_{f, g} \quad (f, g \in V_r := \mathcal{D}(A^{r/2})),$$

where E is the spectral resolution of the identity for A ; see [4]. However, we are able to determine this inner product in terms of the differential expression $\ell_H^r[\cdot]$ only when $r \in \mathbb{N}$.

We aim to show (see Theorem 4.4) that

$$W_n := (V_n, (\cdot, \cdot)_n)$$

is the n^{th} left-definite space associated with the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$, where A is defined in (3.10) and (3.11). We begin by showing that W_n is a complete inner product space.

Theorem 4.1. *For each $n \in \mathbb{N}$, W_n is a Hilbert space.*

Proof. Let $n \in \mathbb{N}$. Suppose $\{f_m\}_{m=1}^{\infty}$ is Cauchy in W_n . Since each of the numbers $c_j(n, k)$ is positive, we see that $\{f_m^{(n)}\}_{m=1}^{\infty}$ is Cauchy in $L^2((-\infty, \infty); \exp(-t^2))$ and hence there exists $g_{n+1} \in L^2((-\infty, \infty); \exp(-t^2))$ such that

$$f_m^{(n)} \rightarrow g_{n+1} \text{ in } L^2((-\infty, \infty); \exp(-t^2)).$$

Fix $t, t_0 \in \mathbb{R}$ (t_0 will be chosen shortly) and assume $t_0 \leq t$. From Hölder's inequality,

$$\begin{aligned} \int_{t_0}^t |f_m^{(n)}(t) - g_{n+1}(t)| dt &= \int_{t_0}^t |f_m^{(n)}(t) - g_{n+1}(t)| \exp(-t^2/2) \exp(t^2/2) dt \\ &\leq \left(\int_{t_0}^t |f_m^{(n)}(t) - g_{n+1}(t)|^2 \exp(-t^2) dt \right)^{1/2} \cdot \left(\int_{t_0}^t \exp(t^2) dt \right)^{1/2} \\ &= M(t_0, t) \left(\int_{t_0}^t |f_m^{(n)}(t) - g_{n+1}(t)|^2 \exp(-t^2) dt \right)^{1/2} \rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Moreover, since $f_m^{(n-1)} \in AC_{\text{loc}}(-\infty, \infty)$, we see that

$$(4.3) \quad f_m^{(n-1)}(t) - f_m^{(n-1)}(t_0) = \int_{t_0}^t f_m^{(n)}(t) dt \rightarrow \int_{t_0}^t g_{n+1}(t) dt,$$

and, in particular, $g_{n+1} \in L^1_{\text{loc}}(-\infty, \infty)$. Furthermore, from the definition of $(\cdot, \cdot)_n$, we see that $\{f_m^{(n-1)}\}_{m=0}^{\infty}$ is Cauchy in $L^2((-\infty, \infty); \exp(-t^2))$; hence, there exists $g_n \in L^2((-\infty, \infty); \exp(-t^2))$ such that

$$f_m^{(n-1)} \rightarrow g_n \text{ in } L^2((-\infty, \infty); \exp(-t^2)).$$

Repeating the above argument, we see that $g_n \in L^1_{\text{loc}}(-\infty, \infty)$ and, for any $t, t_1 \in \mathbb{R}$,

$$(4.4) \quad f_m^{(n-2)}(t) - f_m^{(n-2)}(t_1) = \int_{t_1}^t f_m^{(n-1)}(t) dt \rightarrow \int_{t_1}^t g_n(t) dt.$$

Moreover, from [8, Theorem 3.12], there exists a subsequence $\{f_{m_k, n-1}^{(n-1)}\}$ of $\{f_m^{(n-1)}\}_{m=1}^\infty$ such that

$$f_{m_k, n-1}^{(n-1)}(t) \rightarrow g_n(t) \text{ a.e. } t \in \mathbb{R}$$

Choose $t_0 \in \mathbb{R}$ in (4.3) such that $f_{m_k, n-1}^{(n-1)}(t_0) \rightarrow g_n(t_0)$ and then pass through this subsequence in (4.3) to obtain

$$g_n(t) - g_n(t_0) = \int_{t_0}^t g_{n+1}(t) dt \quad (\text{a.e. } t \in \mathbb{R}).$$

That is to say,

$$(4.5) \quad g_n \in AC_{\text{loc}}(-\infty, \infty) \text{ and } g_n'(t) = g_{n+1}(t) \text{ a.e. } t \in \mathbb{R}.$$

Again, from the definition of $(\cdot, \cdot)_n$, we see that $\{f_m^{(n-2)}\}_{m=1}^\infty$ is Cauchy in $L^2((-\infty, \infty); \exp(-t^2))$; consequently, there exists $g_{n-1} \in L^2((-\infty, \infty); \exp(-t^2))$ such that

$$f_m^{(n-2)} \rightarrow g_{n-1} \text{ in } L^2((-\infty, \infty); \exp(-t^2)).$$

As above, we find that $g_{n-1} \in L_{\text{loc}}^1(-\infty, \infty)$; moreover, for any $t, t_2 \in \mathbb{R}$

$$f_m^{(n-3)}(t) - f_m^{(n-3)}(t_2) = \int_{t_2}^t f_m^{(n-2)}(t) dt \rightarrow \int_{t_2}^t g_{n-1}(t) dt,$$

and there exists a subsequence $\{f_{m_k, n-2}^{(n-2)}\}$ of $\{f_m^{(n-2)}\}$ such that

$$f_{m_k, n-2}^{(n-2)}(t) \rightarrow g_{n-1}(t) \text{ a.e. } t \in \mathbb{R}.$$

In (4.4), choose $t_1 \in \mathbb{R}$ such that $f_{m_k, n-2}^{(n-2)}(t_1) \rightarrow g_{n-1}(t_1)$ and pass through the subsequence $\{f_{m_k, n-2}^{(n-2)}\}$ in (4.4) to obtain

$$g_{n-1}(t) - g_{n-1}(t_1) = \int_{t_1}^t g_n(t) dt \quad (\text{a.e. } t \in \mathbb{R}).$$

Consequently, $g_{n-1} \in AC_{\text{loc}}^{(1)}(-\infty, \infty)$ and $g_{n-1}''(t) = g_n'(t) = g_{n+1}(t)$ a.e. $t \in \mathbb{R}$. Continuing in this fashion, we obtain $n+1$ functions $g_{n-j+1} \in L^2((-\infty, \infty); \exp(-t^2)) \cap L_{\text{loc}}^1(-\infty, \infty)$ ($j = 0, 1, \dots, n$) such that

- (i) $f_m^{(n-j)} \rightarrow g_{n-j+1}$ in $L^2((-\infty, \infty); \exp(-t^2))$ ($j = 0, 1, \dots, n$),
- (ii) $g_1 \in AC_{\text{loc}}^{(n-1)}(-\infty, \infty)$; $g_2 \in AC_{\text{loc}}^{(n-2)}(-\infty, \infty)$, \dots , $g_n \in AC_{\text{loc}}(-\infty, \infty)$,
- (iii) $g_{n-j}'(t) = g_{n-j+1}(t)$ a.e. $t \in \mathbb{R}$ ($j = 0, 1, \dots, n-1$),
- (iv) $g_1^{(j)} = g_{j+1}$ ($j = 0, 1, \dots, n$).

In particular, we see that $f_m^{(j)} \rightarrow g_1^{(j)}$ in $L^2((-\infty, \infty); \exp(-t^2))$ for $j = 0, 1, \dots, n$ and $g_1 \in V_n$. Hence, we see that

$$\begin{aligned} \|f_m - g_1\|_n^2 &= \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} |f_m^{(j)}(t) - g_1^{(j)}(t)|^2 \exp(-t^2) dt \\ &\rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Hence W_n is complete. \square

We now show that \mathcal{P} is dense in W_n ; consequently, $\{H_m(t)\}_{m=0}^\infty$ is a complete orthogonal set in W_n .

Theorem 4.2. *The Hermite polynomials $\{H_m(t)\}_{m=0}^\infty$ form a complete orthogonal set in the space W_n . In particular, the space \mathcal{P} of polynomials is dense in W_n .*

Proof. Let $f \in W_n$; in particular, $f^{(n)} \in L^2((-\infty, \infty); \exp(-t^2))$. Consequently, from the completeness and orthonormality of $\{H_m(t)\}_{m=0}^\infty$ in $L^2((-\infty, \infty); \exp(-t^2))$, it follows that

$$\sum_{m=0}^r c_{m,n} H_m \rightarrow f^{(n)} \text{ as } r \rightarrow \infty \text{ in } L^2((-\infty, \infty); \exp(-t^2)),$$

where the numbers $\{c_{m,n}\}_{m=0}^\infty \subset \ell^2$ are the Fourier coefficients of $f^{(n)}$ defined by

$$c_{m,n} = \int_{-\infty}^{\infty} f^{(n)}(t) H_m(t) \exp(-t^2) dt \quad (m \in \mathbb{N}_0).$$

For $r \geq n$, define the polynomials

$$(4.6) \quad p_r(t) = \sum_{m=n}^r \frac{c_{m-n,n}}{2^{n/2} (P(m,n))^{1/2}} H_m(t)$$

(see (3.3)). Then, using the derivative formula (3.3) for the Hermite polynomials, we see that

$$(4.7) \quad p_r^{(j)}(t) = \sum_{m=n}^r \frac{c_{m-n,n} 2^{j/2} (P(m,j))^{1/2}}{2^{n/2} (P(m,n))^{1/2}} H_{m-j}(t) \quad (j = 1, 2, \dots),$$

and, in particular, as $r \rightarrow \infty$,

$$p_r^{(n)} = \sum_{m=n}^r c_{m-n,n} H_{m-n} \rightarrow f^{(n)} \text{ in } L^2((-\infty, \infty); \exp(-t^2)).$$

Furthermore, from [8, Theorem 3.12], there exists a subsequence $\{p_{r_j}^{(n)}\}$ of $\{p_r^{(n)}\}$ such that

$$(4.8) \quad p_{r_j}^{(n)}(t) \rightarrow f^{(n)}(t) \text{ a.e. } t \in \mathbb{R}.$$

Returning to (4.7), observe that since $\frac{2^{j/2} (P(m,j))^{1/2}}{2^{n/2} (P(m,n))^{1/2}} \rightarrow 0$ as $m \rightarrow \infty$ for $j = 0, 1, \dots, n-1$, we see that

$$\left\{ \frac{c_{m-n,n} 2^{j/2} (P(m,j))^{1/2}}{2^{n/2} (P(m,n))^{1/2}} \right\}_{m=n}^\infty$$

is a square-summable sequence. Thus, from the completeness of the Hermite polynomials $\{H_m(t)\}_{m=0}^\infty$ in $L^2((-\infty, \infty); \exp(-t^2))$ and the Riesz-Fischer theorem (see [8, Chapter 4, Theorem 4.17]), there exists $g_j \in L^2((-\infty, \infty); \exp(-t^2))$ such that

$$(4.9) \quad p_r^{(j)} \rightarrow g_j \text{ in } L^2((-\infty, \infty); \exp(-t^2)) \text{ as } r \rightarrow \infty \quad (j = 0, 1, \dots, n-1).$$

Since, for a.e. $a, t \in (-\infty, \infty)$,

$$p_{r_j}^{(n-1)}(t) - p_{r_j}^{(n-1)}(a) = \int_a^t p_{r_j}^{(n)}(u) du \rightarrow \int_a^t f^{(n)}(u) du = f^{(n-1)}(t) - f^{(n-1)}(a) \quad (j \rightarrow \infty),$$

we see that, as $j \rightarrow \infty$,

$$(4.10) \quad p_{r_j}^{(n-1)}(t) \rightarrow f^{(n-1)}(t) + c_1 \quad (\text{a.e. } t \in (-\infty, \infty)),$$

where c_1 is some constant. From (4.9), with $j = n-1$, we deduce that

$$g_{n-1}(t) = f^{(n-1)}(t) + c_1 \quad (\text{a.e. } t \in (-\infty, \infty)).$$

Next, from (4.10) and one integration, we obtain

$$p_{r_j}^{(n-2)}(t) \rightarrow f^{(n-2)}(t) + c_1 t + c_2 \quad (j \rightarrow \infty),$$

for some constant c_2 and hence, from (4.9),

$$g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2 \quad (\text{a.e. } t \in (-\infty, \infty)).$$

We continue this process to see that, for $j = 0, 1, \dots, n-1$,

$$g_j(t) = f^{(j)}(t) + q_{n-j-1}(t) \quad (\text{a.e. } t \in (-\infty, \infty)),$$

where q_{n-j-1} is a polynomial of degree $\leq n-j-1$ satisfying

$$q'_{n-j-1}(t) = q_{n-j-2}(t).$$

Combined with (4.9), we see that, as $r \rightarrow \infty$,

$$p_r^{(j)} \rightarrow f^{(j)} + q_{n-j-1} \text{ in } L^2((-\infty, \infty); \exp(-t^2)) \quad (j = 1, 2, \dots, n).$$

For each $r \geq n$, define the polynomials

$$\pi_r(t) := p_r(t) - q_{n-1}(t),$$

and observe that

$$\begin{aligned} \pi_r^{(j)} &= p_r^{(j)} - q_{n-1}^{(j)} \\ &= p_r^{(j)} - q_{n-j-1} \\ &\rightarrow f^{(j)} \text{ in } L^2((-\infty, \infty); \exp(-t^2)). \end{aligned}$$

Hence, as $r \rightarrow \infty$,

$$\|f - \pi_r\|_n^2 = \sum_{j=0}^n c_j(n, k) \int_{-\infty}^{\infty} |f^{(j)}(t) - \pi_r^{(j)}|^2 \exp(-t^2) dt \rightarrow 0.$$

□

The next result, which gives a simpler characterization of the function space V_n , follows from ideas in the above proof of Theorem 4.2. Due to the importance of this theorem (which can be seen in the statement of Corollary 4.1), we sketch the proof; specific details are given in Theorem 4.2.

Theorem 4.3. *For each $n \in \mathbb{N}$,*

$$(4.11) \quad V_n = \{f : (-\infty, \infty) \rightarrow \mathbb{C} \mid f \in AC_{\text{loc}}^{(n-1)}(-\infty, \infty); f^{(n)} \in L^2((-\infty, \infty); \exp(-t^2))\}.$$

Proof. Let $n \in \mathbb{N}$ and recall the definition of V_n in (4.1). Define

$$V'_n = \{f : (-\infty, \infty) \rightarrow \mathbb{C} \mid f \in AC_{\text{loc}}^{(n-1)}(-\infty, \infty); f^{(n)} \in L^2((-\infty, \infty); \exp(-t^2))\}.$$

It is clear that $V_n \subset V'_n$. Conversely, suppose $f \in V'_n$ so $f^{(n)} \in L^2((-\infty, \infty); \exp(-t^2))$ and $f \in AC_{\text{loc}}^{(n-1)}(-\infty, \infty)$. As shown in Theorem 4.2, as $r \rightarrow \infty$,

$$\sum_{m=0}^r c_{m,n} H_m \rightarrow f^{(n)} \quad \text{in } L^2((-\infty, \infty); \exp(-t^2)),$$

where

$$c_{m,n} = \int_{-\infty}^{\infty} f^{(n)}(t) H_m(t) \exp(-t^2) dt \quad (m \in \mathbb{N}_0).$$

For $r \geq n$, let $p_r(t)$ be the polynomial that is defined in (4.6). Then, for any $j \in \mathbb{N}_0$, the j^{th} derivative of p_r is given in (4.7) and, as in Theorem 4.2,

$$p_r^{(n)} \rightarrow f^{(n)} \text{ as } r \rightarrow \infty \text{ in } L^2((-\infty, \infty); \exp(-t^2)),$$

and, for $j = 0, 1, \dots, n-1$, there exists polynomials q_{n-j-1} of degree $\leq n-j-1$ satisfying $q'_{n-j-1}(t) = q_{n-j-2}(t)$, such that

$$\begin{aligned} p_r^{(j)} &\rightarrow f^{(j)} + q_{n-j-1} \text{ as } r \rightarrow \infty \text{ in } L^2((-\infty, \infty); \exp(-t^2)), \\ &= f^{(j)} + q_{n-1}^{(j)}. \end{aligned}$$

Consequently, for each $j = 0, 1, \dots, n-1$, $\{p_r^{(j)} - q_{n-1}^{(j)}\}_{r=n}^{\infty}$ converges in $L^2((-\infty, \infty); \exp(-t^2))$ to $f^{(j)}$. From the completeness of $L^2((-\infty, \infty); \exp(-t^2))$, we conclude that $f^{(j)} \in L^2((-\infty, \infty); \exp(-t^2))$ for $j = 0, 1, \dots, n-1$. That is to say, $f \in V_n$. This completes the proof. \square

We are now in position to prove the main result of this section.

Theorem 4.4. *For $k > 0$, let $A : \mathcal{D}(A) \subset L^2((-\infty, \infty); \exp(-t^2)) \rightarrow L^2((-\infty, \infty); \exp(-t^2))$ denote the self-adjoint operator, defined in (3.6), (3.10), and (3.11), having the Hermite polynomials $\{H_m(t)\}_{m=0}^{\infty}$ as eigenfunctions. For each $n \in \mathbb{N}$, let V_n be given as in (4.1) or (4.11) and let $(\cdot, \cdot)_n$ denote the inner product defined in (3.29). Then $W_n = (V_n, (\cdot, \cdot)_n)$ is the n^{th} left-definite space for the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$. Moreover, the Hermite polynomials $\{H_m(t)\}_{m=0}^{\infty}$ form a complete orthogonal set in W_n satisfying the orthogonality relation (3.31). Furthermore, define*

$$A_n : \mathcal{D}(A_n) \subset W_n \rightarrow W_n$$

by

$$A_n f = \ell_H[f] \quad (f \in \mathcal{D}(A_n) := V_{n+2}),$$

where $\ell_H[\cdot]$ is the Hermite differential expression defined in (1.2). Then A_n is a self-adjoint differential operator in W_n ; more specifically, A_n is n^{th} left-definite operator associated with the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$. Furthermore, the Hermite polynomials $\{H_m(t)\}_{m=0}^{\infty}$ are eigenfunctions of A_n and the spectrum of A_n is given by

$$\sigma(A_n) = \{2m + k \mid m \in \mathbb{N}_0\}.$$

Proof. To show that W_n is the n^{th} left-definite space for the pair $(L^2((-\infty, \infty); \exp(-t^2)), A)$, we must show that the five conditions in Definition 2.1 are satisfied.

(i) W_n is complete:

The proof of (i) is given in Theorems 4.1 and 4.3.

(ii) $\mathcal{D}(A^n) \subset W_n \subset L^2((-\infty, \infty); \exp(-t^2))$:

Let $f \in \mathcal{D}(A^n)$. Since the Hermite polynomials $\{H_m(t)\}_{m=0}^{\infty}$ form a complete orthonormal set in $L^2((-\infty, \infty); \exp(-t^2))$, we see that

$$(4.12) \quad p_j \rightarrow f \text{ in } L^2((-\infty, \infty); \exp(-t^2)) \quad (j \rightarrow \infty),$$

where

$$p_j(t) := \sum_{m=0}^j c_m H_m(t),$$

and $\{c_m\}_{m=0}^{\infty}$ are the Fourier coefficients of f in $L^2((-\infty, \infty); \exp(-t^2))$ defined by

$$c_m = (f, H_m) = \int_{-\infty}^{\infty} f(t) H_m(t) \exp(-t^2) dt \quad (m \in \mathbb{N}_0).$$

Since $A^n f \in L^2((-\infty, \infty); \exp(-t^2))$, we see that

$$\sum_{m=0}^j \alpha_m H_m \rightarrow A^n f \text{ in } L^2((-\infty, \infty); \exp(-t^2)) \quad (j \rightarrow \infty),$$

where

$$\alpha_m = (A^n f, H_m) = (f, A^n H_m) = (2m)^n (f, H_m) = (2m)^n c_m;$$

that is to say,

$$A^n p_j \rightarrow A^n f \text{ in } L^2((-\infty, \infty); \exp(-t^2)) \quad (j \rightarrow \infty).$$

Moreover, from (3.30), we see that

$$\begin{aligned} \|p_j - p_r\|_n^2 &= (A^n[p_j - p_r], p_j - p_r) \\ &\rightarrow 0 \text{ as } j, r \rightarrow \infty; \end{aligned}$$

that is to say, $\{p_j\}_{j=0}^\infty$ is Cauchy in W_n . From Theorem 4.1, we see that there exists $g \in W_n \subset L^2((-\infty, \infty); \exp(-t^2))$ such that

$$p_j \rightarrow g \text{ in } W_n \quad (j \rightarrow \infty).$$

Furthermore, by definition of $(\cdot, \cdot)_n$ and the fact that $c_0(n, k) = k^n$ for $k > 0$, we see that

$$(p_j - g, p_j - g)_n \geq k^n (p_j - g, p_j - g);$$

hence

$$(4.13) \quad p_j \rightarrow g \text{ in } L^2((-\infty, \infty); \exp(-t^2)).$$

Comparing (4.12) and (4.13), we see that $f = g \in W_n$; this completes the proof of (ii).

(iii) $\mathcal{D}(A^n)$ is dense in W_n :

Since polynomials are contained in $\mathcal{D}(A^n)$ and are dense in W_n (see Theorem 4.2), it is clear that (iii) is valid. Furthermore, from Theorem 4.2, we see that $\{H_m(t)\}_{m=0}^\infty$ forms a complete orthogonal set in W_n ; see also (3.31).

(iv) $(f, f)_n \geq k^n (f, f)$ for all $f \in V_n$:

This is clear from the definition of $(\cdot, \cdot)_n$, the positivity of the coefficients $c_j(n, k)$, and the fact that $c_0(n, k) = k^n$.

(v) $(f, g)_n = (A^n f, g)$ for $f \in \mathcal{D}(A^n)$ and $g \in V_n$:

Observe that this identity is true for any $f, g \in \mathcal{P}$; indeed, this is seen in (3.30). Let $f \in \mathcal{D}(A^n) \subset W_n$ and $g \in W_n$; since polynomials are dense in both W_n and $L^2((-\infty, \infty); \exp(-t^2))$ and convergence in W_n implies convergence in $L^2((-\infty, \infty); \exp(-t^2))$, there exists sequences of polynomials $\{p_j\}_{j=0}^\infty$ and $\{q_j\}_{j=0}^\infty$ such that, as $j \rightarrow \infty$,

$$p_j \rightarrow f \text{ in } W_n, \quad A^n p_j \rightarrow A^n f \text{ in } L^2((-\infty, \infty); \exp(-t^2)) \quad (\text{see the proof of part (ii)}),$$

and

$$q_j \rightarrow g \text{ in } W_n \text{ and } L^2((-\infty, \infty); \exp(-t^2)).$$

Hence, from (3.30),

$$(A^n[f], g) = \lim_{j \rightarrow \infty} (A^n[p_j], q_j) = \lim_{j \rightarrow \infty} (p_j, q_j)_n = (f, g)_n.$$

This proves (v). The rest of the proof follows immediately from Theorems 2.2 and 2.3. \square

The following corollary follows immediately from this Theorems 4.4 and 4.3, as well as (2.1). Remarkably, it characterizes the domain of each of the integral powers of A . In particular, the characterization given below of the domain $\mathcal{D}(A)$ of the classical Hermite differential operator A having the Hermite polynomials as eigenfunctions seems to be new.

Corollary 4.1. *For each $n \in \mathbb{N}$, the domain $\mathcal{D}(A^n)$ of the n^{th} power A^n of the classical self-adjoint operator A , defined in (3.10), (3.6), and (3.11), is given by*

$$\mathcal{D}(A^n) = V_{2n} = \{f : (-\infty, \infty) \rightarrow \mathbb{C} \mid f \in AC_{\text{loc}}^{(2n-1)}(-\infty, \infty); f^{(2n)} \in L^2((-\infty, \infty); \exp(-t^2))\}.$$

In particular,

$$\mathcal{D}(A) = V_2 = \{f : (-\infty, \infty) \rightarrow \mathbb{C} \mid f \in AC_{\text{loc}}^{(1)}(-\infty, \infty); f'' \in L^2((-\infty, \infty); \exp(-t^2))\}.$$

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