# QUASI-SEPARATION OF THE BIHARMONIC PARTIAL DIFFERENTIAL EQUATION

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*This paper is dedicated to*
*Lawrence Markus*
*Emeritus Regents Professor of Mathematics*
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*for his many and manifold contributions to mathematics*

Abstract. In this paper we consider analytical and numerical solutions to the Dirichlet boundary value problem for the biharmonic partial differential equation, on a disk of finite radius in the plane. The physical interpretation of these solutions is that of the harmonic oscillations of a thin, clamped plate.

For the linear, fourth-order, biharmonic partial differential equation in the plane, it is well known that the solution method of separation in polar co-ordinates, is not possible, in general. However, in this paper, for circular domains in the plane, it is shown that a method, here called quasi-separation of variables, does lead to solutions of the partial differential equation. These solutions are products of solutions of two ordinary linear differential equations; a fourth-order radial equation, and a second-order angular differential equation.

To be expected, without complete separation of the polar variables, there is some restriction on the range of these solutions in comparison with the corresponding separated solutions of the second-order harmonic differential equation in the plane. Notwithstanding these restrictions the quasi-separation method leads to solutions of the Dirichlet boundary value problem on a disk with centre at the origin, with boundary conditions determined by the solution and its inward drawn normal taking the values zero on the edge of the disk.

One significant feature for these biharmonic boundary value problems, in general, follows from the form of the biharmonic differential expression when represented in polar co-ordinates. In this form the differential expression has a singularity at the origin, in the radial variable. This singularity translates to a singularity at the origin of the fourth-order radial separated equation; this singularity necessitates the application of a third boundary condition in order to determine a self-adjoint solution to the Dirichlet boundary value problem.

The penultimate section of the paper reports on numerical solutions to the Dirichlet boundary value problem; these results are also presented graphically. Two specific cases are studied in detail and numerical values of the eigenvalues are compared with results obtained in earlier studies.

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Contents



## 1. Introduction

This paper is concerned with the Dirichlet problem for the biharmonic partial differential equations, on a bounded region of the plane $\mathbb{R}^2$. In particular, we study this problem when the biharmonic differential expression is represented in polar co-ordinates $(r, \theta)$ with $r \in (0, \infty)$ and $\theta \in [0, 2\pi]$.

In polar co-ordinates the Laplace differential expression $\Delta$ is

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}. \tag{1.1}$$

From this result it follows, in the same co-ordinate system, that the biharmonic differential expression $\Delta^2$ has the form

$$\Delta^2 v = \frac{\partial^4 v}{\partial r^4} + \frac{2}{r^2}\frac{\partial^4 v}{\partial r^2 \partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 v}{\partial \theta^4} + \frac{2}{r}\frac{\partial^3 v}{\partial r^3} - \frac{2}{r^3}\frac{\partial^3 v}{\partial r \partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 v}{\partial r^2} + \frac{4}{r^4}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^3}\frac{\partial v}{\partial r}. \tag{1.2}$$

For the Laplace expression $\Delta$ the associated Helmholtz partial differential equation is, where $\Lambda$ is a spectral parameter,

$$-(\Delta v)(r, \theta) = \Lambda v(r, \theta) \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0, 2\pi). \tag{1.3}$$

For the biharmonic expression the corresponding partial differential equation is

$$(\Delta^2 v)(r, \theta) = \Lambda v(r, \theta) \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0, 2\pi). \tag{1.4}$$

Here, in both cases, $\Lambda \in \mathbb{C}$ is a parameter, which is to play the role of spectral parameter for operators in the Hilbert function space of integrable-square functions $\mathbf{L}^2(\mathbb{R}^2)$, or a subspace thereof.

In particular, we are concerned with the Dirichlet boundary value problem for the differential equation (1.2), when the region concerned is the open disk $D_R$, of the plane $\mathbb{R}^2$, defined by, for some given $R \in (0, \infty)$,

$$D_R := \{(r, \theta) : r \in (0, R) \text{ and } \theta \in [0, 2\pi]\}. \tag{1.5}$$

In this case the Dirichlet problem is determined by requiring a solution of (1.4) to satisfy the boundary conditions

$$v(R, \theta) = \frac{\partial}{\partial \mathbf{n}} v(R, \theta) = 0 \text{ for all } \theta \in [0, 2\pi), \tag{1.6}$$

where $\mathbf{n}$ denotes the inward drawn unit normal to the boundary $\partial D_R$ of the disk $D_R$.

We make two essential points concerning the form of our solution of the Dirichlet boundary value problem and the analytical methods adopted in this paper:.

(1) In this paper a complete solution of this biharmonic Dirichlet problem consists of using the symmetric partial differential expression (1.2), the differential equation (1.4) and the boundary conditions (1.6), to construct a self-adjoint differential operator, say $\mathbf{T}$, in the Hilbert function space $\mathbf{L}^2(D_R)$, or some subspace thereof. The properties of the operator $\mathbf{T}$ are then considered as the solution of the Dirichlet problem.

For the biharmonic Dirichlet boundary value problem we give two complete examples of such self-adjoint operators in Section 8, and the resulting regularity properties of the boundary value problems in Section 9.

To determine such an operator $\mathbf{T}$ we require not only the outer boundary conditions (1.6) but also a boundary condition at the origin of the disk $D_R$; this requirement is due to the presence of the singular factors $r^{-2}, r^{-3}, r^{-4} in$ the differential expression (1.2); see [6, Chapter 5, Section 5.5] for a remark on the abstract determination of the Friedrichs extension.

(2) The methods employed in this paper are essentially the application of classical analysis to construct such self-adjoint representations in Hilbert function spaces.

Abstract methods applied to construct self-adjoint representations are given in [6, Chapter 5, Section 5.5], and in [13, Section 4, Page 49].

As to the contents of this paper in Section 2 we give some essential properties of the Laplace and biharmonic differential expressions; in Section 3 we state and prove a critical result which is essential to the introduction of the methods of quasi-separation. In Section 4 we give the form of quasi-separated solutions of the biharmonic equation (1.4); in Section 5 we introduce the fourth-order Bessel-type ordinary differential equations; in Section 6 we define the Friedrichs and Comparison ordinary differential operators and detail their spectral properties; in Section 7 we connect these spectral properties with Bessel functions of order 2. These results lead to Section 8 where we define two self-adjoint operators, out of a continuum of such operators, which on subspaces of $\mathbf{L}^2(D_R)$ provide solutions to the biharmonic Dirichlet boundary value problem on the disk $D_R$. In Section 9 we comment on some aspects of these solutions in terms of the boundary conditions on the boundary

$\partial D_R$ of the disk, and on the requirement to introduce a boundary condition at the centre 0 of the disk; also in this section we discuss the regularity of our solutions to the Dirichlet boundary value problem. In Section 10 we present some numerical results for the solutions of the Dirichlet boundary value problem; these results are based on the explicit representation of the eigenvalues and eigenfunctions, of the biharmonic operator, in terms of the classical Bessel functions of order 2; also in this section we relate our results to certain numerical results in the application references quoted at the end of the paper. Finally, in Section 11 we indicate how the methods used in this paper can be applied to consideration of the Dirichlet boundary value problem for other domains of the plane $\mathbb{R}^2$; all of these domains are required to have angular symmetry.

## 2. Differential expressions

It is known that for the Laplace expression (1.1) the differential equation (1.3) is separable into a product of two factors. These factors give rise to two ordinary differential equations with solutions $R(\cdot)$ and $\Theta(\cdot)$ depending, respectively, on the radial variable $r$ and the angular variable $\theta$; the product $R(\cdot)\Theta(\cdot)$ is then a solution of the partial differential equation (1.3). These separated differential equations take the form, writing $\Lambda = \lambda^2$,

$$-(rR'(r))' + \frac{K}{r}R(r) = \lambda^2 rR(r) \text{ for all } r \in (0, \infty) \tag{2.1}$$

and

$$-\Theta''(\theta) = K\Theta(\theta) \text{ for all } \theta \in [0, 2\pi], \tag{2.2}$$

where $K$ is a separation parameter. Both the differential equations (2.1) and (2.2) are examples of Sturm-Liouville equations; the equation (2.1) is a form of the classical Bessel differential equation. Using the solutions of the two differential equations (2.1) and (2.2) it is possible to construct self-adjoint representations of solutions to the Dirichlet boundary value problem for the Laplacian on the disk $D_R$; for an extensive study of these Laplacian self-adjoint representations see [13, Section 5.2]; in particular, the results in [13, Section 5.1, pages 82 to 96] where the classical properties of these self-adjoints representations are linked to the corresponding abstract study in the $W^2(D_R)$ Sobolev spaces.

Conversely, it is known that for the biharmonic expression (1.2) the differential equation (1.4) is not separable in this polar co-ordinate system. However, we shall show that a quasi-form of separation of the polar biharmonic equation is possible and that this process leads to solutions, for all values of the spectral parameter $\Lambda$, which are products of Bessel and trigonometrical functions.

The genesis of the results in this paper stems from the introduction, see the paper [10], of the structured Bessel-type linear ordinary differential equations of order $2n$ where $n \in \mathbb{N}$; the case $n = 1$ gives the classical Bessel equation. In particular the case when $n = 2$ introduced the fourth-order Bessel-type differential equation which has been the subject of significant study in recent years; see the references [5], [7], [15], [16], [19] and [20].

The fourth-order case of these Bessel-type differential equations takes the form

$$((ru''(r))'' - ((9r^{-1} + \gamma r)u'(r))' = \lambda^2(\lambda^2 + \gamma)ru(r) \text{ for all } r \in (0, \infty) \tag{2.3}$$

where the real parameter $\gamma$ satisfies $\gamma \geq 0$, and $\lambda \in \mathbb{C}$ is a complex-valued parameter. This differential equations (2.3) has been extensively studied for $\gamma > 0$ (but with the notation $\gamma = 8M^{-1}$ where $M \in (0, \infty]$); see the original definition in [10], the survey paper [8] and the technical details in [5].

The particular case of (2.3) when $\gamma = 0$, *i.e.*

$$((ru''(r))'' - ((9r^{-1}r)u'(r))' = \lambda^4 ru(r) \text{ for all } r \in (0, \infty) \tag{2.4}$$

has received special study recently, see [16], of which details are given below in Section 5.

We show below in Section 3 that the Bessel-type ordinary differential equation (2.4) leads to a restricted form of separation, quasi-separation, of the biharmonic partial differential equation (1.4); this result compares with the role played by the radial Bessel equation (2.1) in the separation of the second-order partial differential equation (1.3). However the cost of attempting to implement separation of the fourth-order partial differential equation (1.4), using the radial equation (2.4), leads to a restricted form of the corresponding angular differential equation; to be specific this angular equation is

$$-\Theta''(\theta) = 4\Theta(\theta) \text{ for all } \theta \in [0, 2\pi], \tag{2.5}$$

with general solution

$$\Theta(\theta) = A\cos(2\theta) + B\sin(2\theta) \text{ for all } \theta \in [0, 2\pi]. \tag{2.6}$$

It is to be noted that in comparison with the free separation parameter $K$ in (2.2), the quasi-separation parameter 4 on the right-hand side of (2.5) is fixed; this result is again the cost of the quasi-separation in the biharmonic case.

Notwithstanding this restricted form of quasi-separation we show below in Sections 4 and 5 that this method yields interesting, and possibly new, solutions to the biharmonic differential equation (1.4).

As noted above one of the interesting possibilities of studying the Dirichlet problem for differential equations is to seek a self-adjoint realisation of the problem in the Hilbert function space concerned; in the case above this is the space $\mathbf{L}^2(D_R)$. The quasi-separation method does lead to such a realisation but only on a subspace of $\mathbf{L}^2(D_R)$; nevertheless the solution of the problem is defined on the whole of the disk.

## 3. The Plum lemma

The results in this paper depend critically on a result obtained by Michael Plum, University of Karlsruhe in Germany, on a visit to the University of Birmingham in 2003; details are given in the manuscript [19], but the essential features there of are given in this section. This Plum lemma resulted from discussions based on the results in the manuscript [19]; see also the manuscripts [15], [20], [21] and [22].

**Lemma 3.1.** *Let the biharmonic differential expression $\Delta^2$ be given in polar form, as in* (1.2).

*Let the function $u : (0, \infty) \to \mathbb{C}$ satisfy the differentiability conditions*

$$u^{(s)} \in AC_{loc}(0, \infty) \textit{ for } s = 0, 1, 2, 3. \tag{3.1}$$

*Let the Plum factor* $w_{A,B}(\cdot)$ *be defined by*

$$w_{A,B}(\theta) := A\cos(2\theta) + B\sin(2\theta) \text{ for all } \theta \in [0, 2\pi), \tag{3.2}$$

*where* $A, B \in \mathbb{C}$ *and* $|A|^2 + |B|^2 > 0$.

*Then, for almost all (Lebesgue)* $r \in (0, \infty)$ *and all* $\theta \in [0, 2\pi)$

$$\Delta^2 \left[u(r)w_{A,B}(\theta)\right] = \frac{1}{r}\left[(ru''(r))'' - (9r^{-1}u'(r))'\right] w_{A,B}(\theta). \tag{3.3}$$

*Conversely, if the identity* (3.3) *is satisfied for some factor* $w : [0, 2\pi] \to \mathbb{C}$ *with* $w \in C^{(4)}[0, 2\pi]$ *and for almost* $r \in (0, \infty)$, *then* $w$ *has to take the form* (3.2) *for some* $A, B \in \mathbb{C}$.

*Proof.* Firstly, since $u^{(3)} \in AC_{\mathrm{loc}}(0, \infty)$ it follows that the derivative $u^{(4)}$ is defined almost everywhere on $(0, \infty)$, and thus we can apply the biharmonic differential expression to $u$. Since $u$ depends only on the radial co-ordinate $r$ a calculation shows, using (1.2), that

$$\Delta^2 u(r) = \frac{1}{r}\left[(ru''(r))'' - (r^{-1}u'(r))'\right]. \tag{3.4}$$

Secondly, the terms in (1.2) that contain only differentiation with respect to the variable $\theta$ cancel out due to the choice of the factor $w_{A,B}$, *i.e.*

$$\frac{1}{r^4}\frac{\partial^4 w_{A,B}}{\partial\theta^4} + \frac{4}{r^4}\frac{\partial^2 w_{A,B}}{\partial\theta^2} = 0. \tag{3.5}$$

Thirdly, for the terms in (1.2) that contain differentiation with respect to both variables $r$ and $\theta$ we have

$$\left(\frac{2}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3}{\partial r\partial\theta^2}\right)\left[u(r)w_{A,B}(\theta)\right] = -\frac{1}{r}\left[(8r^{-1}u'(r)'\right]. \tag{3.6}$$

Finally combining (1.2) with (3.4), (3.5) and (3.6) we obtain (3.3).

The converse result follows from reversing the argument which gives (3.3). □

## 4. Solutions to the biharmonic equation

The quasi-separation result of Section 3 suggests that solutions of the biharmonic differential equation (1.4) can to be found in the form

$$v(r, \theta) \equiv u(r)w_{A,B}(\theta) \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0, 2\pi). \tag{4.1}$$

Inserting (4.1) into (1.2) gives, for all $r \in (0, \infty)$ and $\theta \in [0, 2\pi)$,

$$\Delta^2[v(r,\theta)] - \Lambda v(r,\theta) = \frac{1}{r}\left[(ru''(r))'' - (9r^{-1}u'(r))'\right] w_{A,B}(\theta) - \Lambda u(r)w_{A,B}(\theta). \tag{4.2}$$

Thus we have

**Lemma 4.1.** *The biharmonic partial differential equation* (1.4), *i.e.*

$$(\Delta^2 v)(r,\theta) - \Lambda v(r,\theta) = 0 \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0, 2\pi) \tag{4.3}$$

*has a solution of the form* $v(r, \theta)$ *given by* (4.1) *i.e.*

$$v(r, \theta) \equiv u(r)w_{A,B}(\theta) \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0, 2\pi), \tag{4.4}$$

*for any* $A, B \in \mathbb{C}$, *if and only if the radial factor* $u(\cdot)$ *satisfies the linear ordinary differential equation*

$$\frac{1}{r}\left[(ru''(r))'' - (9r^{-1}u'(r))'\right] = \Lambda u(r) \text{ for all } r \in (0,\infty). \tag{4.5}$$

**Remark 4.1.** For any choice of $A, B \in \mathbb{C}$ the angular factor $w_{A,B}(\cdot)$ has at most a finite number of zeros in the interval $[0, 2\pi)$; at such points the solution $v(r,\theta)$ is null for all $r \in (0,\infty)$ and for all $\Lambda \in \mathbb{C}$.

**Remark 4.2.** In the existing literature biharmonic eigenfunction theory on a disk has been developed mainly by factorising the biharmonic partial differential expression (1.2) into two second-order expressions, see for example [4, Chapter 5, Section 6]; we have commented on this method in Section 5.

In this paper we study the complete fourth-order differential ordinary differential equation given by (4.5).

## 5. Bessel-type differential equations

Writing the ordinary equation (4.5) in the form, replacing $\Lambda$ by $\lambda^4$,

$$((ru''(r))'' - (9r^{-1}u'(r))' = \lambda^4 ru(r) \text{ for all } r \in (0,\infty), \tag{5.1}$$

it follows that this equation is a special case of the family of fourth-order Bessel-type differential equations

$$((ru''(r))'' - ((9r^{-1} + \gamma r)u'(r))' = \lambda^2(\lambda^2 + \gamma)ru(r) \text{ for all } r \in (0,\infty) \tag{5.2}$$

where the real parameter $\gamma$ satisfies $\gamma \geq 0$, and $\lambda \in \mathbb{C}$ is a complex-valued parameter.

The differential equations (5.2) has been extensively studied for $\gamma > 0$ (but with the notation $\gamma = 8M^{-1}$ where $M \in (0,\infty]$); see the original definition in [10], the survey paper [8] and the technical details in [5].

Considered in the complex plane, *i.e.* with $r \in \mathbb{C}$, both the differential equations (5.1) and (5.2) have a regular singularity at the origin 0 and an irregularity singularity at infinity. The Frobenius indicial roots at 0, for both equations, are $\{4, 2, 0, -2\}$; these indices are independent of the parameter $\lambda$.

When $\gamma > 0$ the establishment of a linearly independent basis of solutions for the equation (5.2) is given in the survey paper [8]. From these studies it may be shown, see [16] and [9, Section 4], that a solution basis of the differential equation (5.1), is, for all $r \in (0,\infty)$ and all $\lambda \in \mathbb{C}$,

$$\{J_2(\lambda r), I_2(\lambda r), Y_2(\lambda r), K_2(\lambda r)\}, \tag{5.3}$$

involving the classical Bessel functions of the first and second kind, as well as the corresponding modified Bessel functions; see [23, Chapter III] for the properties of these functions.

This form of the solutions of the differential equation (5.1) suggests that this fourth-order equation

$$\frac{1}{r}\left((ru''(r))'' - \left(\frac{9}{r}u'(r)\right)'\right) - \lambda^4 u(r) = 0 \text{ for all } r \in (0,\infty) \tag{5.4}$$

can be factorised and written as a product of the two second-order Bessel differential equations

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{2^2}{r^2}+\lambda^2\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{2^2}{r^2}-\lambda^2\right)u(r)=0 \text{ for all } r\in(0,\infty). \tag{5.5}$$

This result holds good; it can be checked by hand, but use of the Maple program is recommended. Each of these second-order differential equations can be used in the study of the fourth–order differential equation; see ([4, Chapter 5, Section 6]) and Example 8.9 in [24, Chapter 8, Section 5].

## 6. Bessel-type differential operators

6.1. **Definition.** In order to define a self-adjoint operator in a Hilbert function space that represents a solution of the polar biharmonic boundary value problem given in (1.4) and (1.6), we first introduce a self-adjoint operator generated by the Bessel-type differential equation (5.1) and its spectral parameter $\Lambda$.This operator is defined in the weighted Hilbert function space $L^2((0,R);r)$, where $R$ is the given radius of the disk $D_R$ as defined in Section 1, and the variable $r\in[0,R]$ acts as the weight. Thus the space $L^2((0,R);r)$ is the collection of all Lebesgue measurable functions $f:(0,R)\to\mathbb{C}$ with norm and inner-product

$$\|f\|_R^2 := \int_0^R |f(r)|^2 \ rdr \text{ and } (f,g)_R := \int_0^R f(r)\overline{g}(r)\ rdr. \tag{6.1}$$

In this account we follow the details given in the paper [5, Sections 4 to 10] for results and proofs, note that these results hold for the differential equation (5.2); nevertheless all these also hold for the case when $\gamma=0$ and so apply to the equation (5.1).

We note that the differential equation (5.1), as considered on the interval $(0,R]$, has a regular endpoint at $R$ but a singular endpoint at $0^+$.

To define the required differential operator in this space we first define the differential expression $N[\cdot]$, see (5.1), with domain $D(N)$ as follows:

$$D(N) := \{f:(0,R]\to\mathbb{C} : f^{(s)}\in AC_{\mathrm{loc}}(0,R] \text{ for } s=0,1,2,3\}, \tag{6.2}$$

and then for all $f\in D(N)$

$$N[f](r) := (f''(r))'' - (9r^{-1}f'(r))' \text{ for all } r\in(0,R]; \tag{6.3}$$

it follows that

$$N : D(N)\to L^1_{\mathrm{loc}}(0,R].$$

The Green's formula for $N$ on any compact interval $[\alpha,R]\subset(0,R]$ takes the form, for all $f,g\in D(N)$,

$$\int_\alpha^R \left\{\overline{g}(r)N[f](r) - f(r)\overline{N[g]}(r)\right\}\ dr = [f,g]_N(r)|_\alpha^R \tag{6.4}$$

where the symplectic form $[\cdot,\cdot]_N(\cdot) : D(N)\times D(N)\times(0,R] \to \mathbb{C}$ is given by, for all $r \in (0,R]$,

$$\begin{aligned}[f,g]_N(r) = \overline{g}(r)(rf''(r))' - (r\overline{g}''(r))'f(r) \\ - r\left(\overline{g}'(r)f''(r) - \overline{g}''(r)f'(r)\right) \\ - 9r^{-1}\left(\overline{g}(r)f'(r) - \overline{g}'(r)f(r)\right).\end{aligned} \tag{6.5}$$

The maximal operator $T_1$ generated by the differential expression $N$ in $L^2((0,R);r)$ is defined by

$$D(T_1) := \{f \in D(N) : f, r^{-1}N(f) \in L^2((0,R);r)\} \tag{6.6}$$

and

$$T_1 f := r^{-1}N(f) \text{ for all } f \in D(T_1). \tag{6.7}$$

**Lemma 6.1.** *From the Green's formula* (6.4) *it follows that at the singular endpoint* $0^+$ *the symplectic form* (6.5) *has a finite limit*

$$\lim_{r\to 0+}[f,g]_N(r) \in \mathbb{C} \textit{ for all } f,g \in D(T_1). \tag{6.8}$$

*From* [5, Section 8] *we note that all the elements* $f$ *of* $D(T_1)$ *satisfy, notwithstanding the singularity for* $N$ *at the singular endpoint* $0^+$,

$$f \in AC[0,R] \qquad f' \in AC[0,R] \textit{ and } f'(0) = 0 \qquad f'' \in AC_{loc}(0,R] \textit{ and } f'' \in C[0,R]. \tag{6.9}$$

*From* [5, Lemma 9.1] *the functions* $1, r^2$ *on* $[0,R]$ *belong to* $D(T_1)$ *and satisfy*

$$[1,1](0^+) = [r^2,r^2](0^+) = 0 \textit{ and } [1,r^2](0^+) = 16. \tag{6.10}$$

*From* [5, Lemma 9.2] *the following identities hold for all* $f \in D(T_1)$

$$[f,1](0^+) = -8f''(0) \textit{ and } [f,r^2](0^+) = 16f(0). \tag{6.11}$$

The minimal operator $T_0$ generated by the differential expression $N[\cdot]$ is defined by, see [18, Chapter V, Section 17.5, VI] and [5, Section 5, (5.4) and (5.5), and Section 9, Corollary 9.1],

$$D(T_0) := \{f \in D(T_1) : f(0) = f''(0) = 0 \text{ and } f^{(s)}(R) = 0 \text{ for } s = 0,1,2,3\}, \tag{6.12}$$

and

$$T_0 f := r^{-1}N[f] \text{ for all } f \in D(T_0). \tag{6.13}$$

The operators $T_0$ and $T_1$ have the adjoint properties, see [18, Chapter V, Section 17],

$$T_0 \subseteq T_1 \qquad T_0^* = T_1 \qquad T_1^* = T_0. \tag{6.14}$$

The operator $T_0$ is closed and symmetric; the operator $T_1$ is closed in the Hilbert space $L^2((0,R);r)$. If $T$ is a self-adjoint operator generated by $N[\cdot]$ in this space then

$$T_0 \subseteq T \subseteq T_1. \tag{6.15}$$

With this information to hand we can define self-adjoint operators $T$ in $L^2((0,R);r)$ which are restrictions of the maximal operator $T_1$. These operators are determined by placing boundary conditions on the elements of $D(T_1)$, according to the general theory developed in the texts [18, Chapter V, Section 18.1, Theorem 4], [12, Chapter V, Section 4, in particular pages 105 to 107] and [11, Setion 4, Theorem 1], to give the domain of $T$.

The more detailed information in [5, Section 6 and Section 10] gives the classification of the differential expression $N$ in $L^2((0,R);r)$ as $(3,3)$ for the singular endpoint $0^+$ and $(4,4)$ for the regular endpoint $R$. With this information we can apply the general theory of differential operators as given in [18, Section 18.1, Theorem 4] and [12, Chapter V, Section 4, Example 2, case order $4, d=3$] to see that we require three independent symmetric boundary conditions to determine any self-adjoint restriction $T$ of the maximal operator $T_1$. Here in this paper we take the case of separated conditions with one condition at the singular endpoint $0^+$, and two independent conditions at the regular endpoint $R$.

These boundary conditions are determined by a set of three elements $\{\varphi_0, \varphi_1, \varphi_2)$ taken from the maximal domain $D(T_1)$, where $\varphi_0$ has non-null local support at $0^+$, and $\varphi_1, \varphi_2$ have linearly independent local support at $R_-$; together they satisfy the symmetric conditions

$$[\varphi_0, \varphi_0](0^+) = 0 \text{ and } [\varphi_s, \varphi_t](R) = 0 \text{ for } s,t = 1,2. \tag{6.16}$$

The self-adjoint operator $T$ is then determined by

$$D(T) = \{f \in D(T_1) : [f, \varphi_0](0^+) = 0 \text{ and } [f, \varphi_s](R) = 0 \text{ for } s = 1,2\} \tag{6.17}$$

and

$$(Tf)(r) := r^{-1} N[f](r) \text{ for all } r \in (0,R] \text{ and all } f \in D(T). \tag{6.18}$$

To determine the particular self-adjoint operators required for the biharmonic boundary value problem we choose the boundary conditions as follows:

(1) For the singular endpoint $0^+$ we impose either the **Friedrichs** boundary condition (here we invoke the result in [14] and the application thereof as in [5, Section 15] to justify the use of the term Friedrichs) to determine the self-adjoint operator $T_F$; or the **Comparison** boundary condition to determine the self-adjoint operator $T_C$.

($i$) For the Friedrichs boundary condition to give $T_F$ we define

$$\varphi_{0,F}(r) := \frac{r^2}{16} \tag{6.19}$$

in some neighbourhood of $0^+$, which gives from (6.11) the required condition on elements of $D(T_1)$

$$[f, \varphi_{0,F}](0^+) = f(0^+) = 0. \tag{6.20}$$

($ii$) For the Comparison boundary condition to give $T_C$ we define

$$\varphi_{0,C}(r) := -\frac{1}{8} \tag{6.21}$$

in some neighbourhood of $0^+$, which gives, again from (6.11), the required condition on elements of $D(T_1)$

$$[f, \varphi_{0,C}](0^+) = f''(0^+) = 0. \tag{6.22}$$

(2) At the regular endpoint $R$, for both the Friedrichs and the Comparison operators, the boundary conditions are determined by taking $\varphi_1, \varphi_2$ as real solutions of the differential equation $N[\varphi] = 0$ with initial conditions at $R$

$$\begin{cases} \varphi_1(R) = 0 & \varphi_1'(R) = 0 & \varphi_1''(R) = 0 & \varphi_1'''(R) = -R^{-1} \\ \varphi_2(R) = 0 & \varphi_2'(R) = 0 & \varphi_2''(R) = R^{-1} & \varphi_2'''(R) = -R^{-2} \end{cases} \tag{6.23}$$

to give two independent boundary conditions, satisfying the symmetry condition

$$[\varphi_s, \varphi_t](R) = 0 \ for \ s, t = 1, 2. \tag{6.24}$$

This choice then gives the required conditions on the elements of $D(T_1)$

$$[f, \varphi_1](R) \equiv f(R) = 0 \text{ and } [f, \varphi_2](R) \equiv f'(R) \ = 0. \tag{6.25}$$

Taken together these conditions give the explicit form of the boundary conditions, see (6.11) and (6.18),

($i$) as the boundary conditions to determine the domain $D(T_F)$ of the **Friedrichs** self-adjoint operator $T_F$

$$f(0^+) = 0 \quad \text{and} \quad f(R) = f'(R) = 0 \tag{6.26}$$

($ii$) as the boundary conditions to determine the domain $D(T_K)$ of the **Comparison** self-adjoint operator $T_C$

$$f''(0^+) = 0 \quad \text{and} \quad f(R) = f'(R) = 0. \tag{6.27}$$

**Lemma 6.2.** *We have the Dirichlet identities for $T_F$ and $T_C$*

$$\begin{aligned}(T_F f, f)_R &= \int_0^R (T_F f)(r)\overline{f}(r) r \ dr \\ &= \int_0^R \{r\,|f''(r)|^2 + 9r^{-1}\,|f'(r)|^2\}\ dr \ \textit{for all } f \in D(T_F)\end{aligned} \tag{6.28}$$

*and*

$$\begin{aligned}(T_C f, f)_R &= \int_0^R (T_C f)(r)\overline{f}(r) r \ dr \\ &= \int_0^R \{r\,|f''(r)|^2 + 9r^{-1}\,|f'(r)|^2\}\ dr \ \textit{for all } f \in D(T_C).\end{aligned} \tag{6.29}$$

*Proof.* See [5, Section 4, (4.7) and (4.9)]. □

6.2. **Properties.** We now consider the spectral properties of the operators $T_F$ and $T_C$ in the complex plane $\mathbb{C}$, with complex parameter $\Lambda$.

We state the properties of the Friedrichs differential operator $T_F$ in the following theorem; the results for the operator $T_C$ are similar.

**Theorem 6.1.** *Let the differential operator $T_F$, with domain $D(T_F)$ in the Hilbert function space $L^2((0,R);r)$, be defined by* (6.17) *with boundary conditions* (6.18) (*equivalently by the boundary conditions in explicit form* (6.26)); *then:*

(1) *$T_F$ is self-adjoint in $L^2((0,R);r)$, is real, bounded below and has a positive lower bound in this space.*

(2) *$T_F$ has an empty essential spectrum and a discrete spectrum of eigenvalues $\{\Lambda_n^F : n \in \mathbb{N}_0\}$ with* (*allowing for multiplicity*)

$$0 < \Lambda_0^F \leq \Lambda_1^F \leq \cdots \Lambda_n^F \leq \Lambda_{n+1}^F \cdots, \ \textit{and} \ \lim_{n\to+\infty} \Lambda_n^F = +\infty; \tag{6.30}$$

*each eigenvalue has finite multiplicity.*

(3) *$T_F$ has a corresponding set of orthogonal eigenvectors $\{\psi_n^F : n \in \mathbb{N}_0\}$ which are complete in $L^2((0,R);r)$. The eigenvectors $\{\psi_n^F : n \in \mathbb{N}_0\}$ all belong to the domain $D(T_1)$ and satisfy*

$$T_F\psi_n^F = \Lambda_n^F r\psi_n^F \text{ for all } n \in \mathbb{N}_0. \tag{6.31}$$

(4) *$T_F$ is bounded below in $L^2((0,R);r)$, i.e.*

$$(T_F f, f)_R \geq \Lambda_0^F (f,f)_R \text{ for all } f \in D(T_F), \tag{6.32}$$

*where $\Lambda_0^F > 0$.*

*Proof.* (1) For the self-adjoint property see Section 6.1 above; from the Dirichlet identity (6.28) we obtain

$$(T_F f, f)_R \geq 0 \text{ for all } f \in D(T_F). \tag{6.33}$$

(2) The proof of this result follows from the analysis in the paper [5]; see the definition of the operator $T_L$ in [5, Section 13, (13.5)], and then the subsequent proof [5, Section 13, (13.9)] that the essential spectrum of the operator $T_L$ is empty. These methods apply to the operator $T_F$ to give the essential spectrum is empty; the general theory of linear operators in Hilbert space given in [18, Chapter IV, Section 12.5] then implies that the spectrum of $T$ is discrete and consists only of separated eigenvalues; the countable number of the set of eigenvalues follows from the dimension of the Hilbert space $L^2((0,R);r)$. The spectrum of $T_F$, as a differential operator, is unbounded above.

The order of the differential expression $N[\cdot]$ implies the spectral multiplicity of any eigenvalue cannot exceed 4; see below when the identification of the eigenvectors $\{\psi_n^F : n \in \mathbb{N}_0\}$ is made as solutions of the differential equation (5.1).

(3) The completeness of the set of eigenvectors in $L^2((0,R);r)$ follows from the given spectral properties of the self-adjoint operator $T_F$. The results (6.31) follows from the construction of the sequence $\{\psi_n^F : n \in \mathbb{N}_0\}$.

(4) The spectral properties of $T_F$ imply this operator is bounded below in $L^2((0,R);r)$, and that the lower bound is as given in (6.32) involving the first eigenvalue $\Lambda_0^F$; here $\Lambda_0^F > 0$ follows from the Dirichlet identity (6.28) on inserting $f = \psi_0^F$.

□

There is a similar theorem and proof for the Comparison operator $T_C$, with notations $D(T_C)$, $\Lambda_n^C$ and $\psi_n^C$.

## 7. Connections

In this section we bring together the results from the Bessel-type differential equation as given in Section 5, and the results from the Bessel-type differential operator in Section 6.

We state this theorem for the Friedrichs operator $T_F$; there is a similar theorem for the Comparison operator $T_C$.

**Theorem 7.1.** (1) *The eigenvectors* $\{\psi_n^F : n \in \mathbb{N}_0\}$ *of item* 3 *of Theorem 6.1 are the eigenfunctions of the differential boundary value problem, see* (5.1),

$$N[u](r) = \lambda^4 r u(r) \text{ for all } r \in (0, R] \tag{7.1}$$

*with boundary conditions*

$$u(0^+) = 0 \text{ and } u(R) = u'(R) = 0. \tag{7.2}$$

(2) *The eigenvalues* $\{\lambda_n^F : n \in \mathbb{N}_0\}$ *of this boundary value problem are positive and related to the eigenvalues* $\{\Lambda_n^F : n \in \mathbb{N}_0\}$, *of the operator* $T_F$, *by* $\lambda_n^F$ *being the positive* $4^{th}$ *root of the positive number* $\Lambda_n^F$, *i.e.*

$$\left(\lambda_n^F\right)^4 = \Lambda_n^F. \tag{7.3}$$

(*Note that we refer to both the sets* $\{\Lambda_n : n \in \mathbb{N}_0\}$ *and* $\{\lambda_n : n \in \mathbb{N}_0\}$ *as eigenvalues associated with the operator and differential boundary value problems.*)

(3) *The eigenfunctions* $\{\psi_n^F : n \in \mathbb{N}_0\}$, *as solutions of* (7.1), *are dependent only upon the two Bessel solutions* $\{J_2(\lambda_n^F r), I_2(\lambda_n^F r)\}$, *taken from the general basis, see* (5.3), *with* $\lambda = \lambda_n^F$,

$$\{J_2(\lambda r), I_2(\lambda r), Y_2(\lambda r), K_2(\lambda r)\}.$$

(4) *The parameter* $\lambda$ *is an eigenvalue from the set* $\{\lambda_n^F : n \in \mathbb{N}_0\}$ *if and only if* $\lambda$ *is a positive zero of the complex entire function defined on the complex plane* $\mathbb{C}$ *by*

$$\lambda \longmapsto J_2(\lambda R)\left(\frac{d}{dr} I_2(\lambda r)\right)(R) - \left(\frac{d}{dr} J_2(\lambda r)\right)(R) I_2(\lambda R). \tag{7.4}$$

(5) *The multiplicity of any eigenvalue* $\Lambda_n^F$ *of the operator* $T_F$ *does not exceed the number* 2.

*Proof.* (1) This result follows from the definition of the self-adjoint operator $T_F$ given by (6.7), and the form of the boundary conditions (6.26).

(2) It is also clear from the definition (6.18) of $T_F$ and the differential equation (7.1) that we can identify the eigenvalues $\Lambda_n^F$ with $\lambda_n^F$, for all $n \in \mathbb{N}_0$; see again (7.3).

(3) For any $n \in \mathbb{N}_0$ the eigenvalue $\lambda_n^F$ of the operator $T_F$ gives the corresponding eigenvector $\psi_n^F$ as an eigenfunction of the differential equation (7.1), and so can be represented as linearly dependent upon the basis of solutions $\{J_2(\lambda r), I_2(\lambda r), Y_2(\lambda r), K_2(\lambda r)\}$. If $\psi_n^F$ is to satisfy the singular Frobenius boundary condition (6.20) at the origin, *i.e.* $\psi_n^F(0^+) = 0$, then it is clear from the properties of the Bessel functions $Y_2(\lambda r)$ and $K_2(\lambda r)$ that these two solutions, nor any linear combination of them, cannot be involved in this representation. Thus we can write

$$\psi_n^F(r) = \alpha_n J_2(\lambda_n^F r) + \beta_n I_2(\lambda_n^F r) \text{ for all } r \in (0, R], \tag{7.5}$$

where $\alpha_n, \beta_n$ are numbers, not both zero, dependent upon the integer $n \in \mathbb{N}_0$.

(4) Given the eigenvalue $\lambda_n^F$ then with the representation (7.5) of $\psi_n^F$ we seek the consequences from the property that $\psi_n^F$ satisfies the two regular boundary conditions at the endpoint $R$, *i.e.*

$$\psi_n^F(R) = \left(\frac{d}{dr}\psi_n^F\right)(R) = 0. \tag{7.6}$$

Substituting from (7.5) into (7.6) we obtain the two linear equations to determine the pair $\{\alpha_n, \beta_n\}$

$$\alpha_n J_2(\lambda_n^F R) + \beta_n I_2(\lambda_n^F R) = 0 \tag{7.7}$$

$$\alpha_n \left( \frac{d}{dr} J_2(\lambda_n^F r) \right)(R) + \beta_n \left( \frac{d}{dr} I_2(\lambda_n^F r) \right)(R) = 0 \tag{7.8}$$

This determination thus depends upon consideration of the matrix $D_n$, as dependent upon the integer $n \in \mathbb{N}_0$, defined by

$$D_n := \begin{bmatrix} J_2(\lambda_n^F R) & I_2(\lambda_n^F R) \\ \left( \frac{d}{dr} J_2(\lambda_2^F r) \right)(R) & \left( \frac{d}{dr} I_2(\lambda_n^F r) \right)(R) \end{bmatrix}. \tag{7.9}$$

Since we are given the existence of the non-null pair $\{\alpha_n, \beta_n\}$ it follows that

$$\operatorname{rank}(D_n) \leq 1; \tag{7.10}$$

this result gives the required property (7.4).

Conversely, given $\lambda$ as a positive zero of the analytic function (7.4) then there exists a non-null pair $\{\alpha, \beta\}$ such that the solution

$$\psi(r) = \alpha J_2(\lambda r) + \beta I_2(\lambda r) \text{ for all } r \in (0, R]$$

satisfies the boundary conditions

$$\psi(R) = \psi'(R) = 0.$$

The boundary condition $\psi(0^+) = 0$ is automatically satisfied from the properties of the solutions $J_2$ and $I_2$ at the singular endpoint $0^+$. Thus $\psi$ is an eigenfunction of the boundary value problem and so, in view of the completeness of the set $\{\psi_n^F : n \in \mathbb{N}_0\}$ of eigenfunctions in the space $L^2((0, R]; r)$, as in Theorem (6.1), the given $\lambda$ must belong to the set of eigenvalues $\{\lambda_n^F : n \in \mathbb{N}_0\}$.

(5) It is clear from the two equations (7.7) and (7.8) that the number of independent solutions to give the pair $\{\alpha_n, \beta_n\}$ to determine the geometric multiplicity of the eigenvalue $\lambda_n^F$, depends upon rank $D_n$ as follows:

(7.11)

| Geometric multiplicity of $\lambda_n$ | Rank of matrix $D_n$ |
|---|---|
| 1 | 1 |
| 2 | 0 |

This establishes the geometric multiplicity result for $\Lambda_n^F$, and so for the differential operator $T_F$.

□

We point out that the property (7.4) is also used in [4] and in [24]. It is evident that (7.4) also has a zero at the point $0 \in \mathbb{C}$; however when $\lambda = 0$ it may be seen that a solution base for (5.1) is $\{1, r^2, r^4, r^{-2}\}$ and so 0 is not an eigenvalue for either of the self-adjoint operators $T_F$ and $T_C$.

There is a similar theorem for the Comparison operator but with the following changes; we show only the changes that are essential, otherwise the results and their proofs follow as for the Friedrichs case.

($i$) In the statement of the theorem part of item 3 is changed to:
For any $n \in \mathbb{N}_0$ the eigenvalue $\lambda_n^C$ of the operator $T_C$ gives the corresponding eigenvector $\psi_n^C$ as an eigenfunction of the differential equation (7.1), and so can represented as linearly dependent upon the basis of solutions $\{J_2(\lambda r), I_2(\lambda r), Y_2(\lambda r), K_2(\lambda r)\}$. If the eigenfunction $\psi_n^C$ is to satisfy the singular Comparison boundary condition at the origin, *i.e.*

$$\left(\frac{d^2}{dr^2}\psi_n^C(r)\right)(0^+) = 0,$$

then it is clear from the solution properties of the four Bessel functions *given* in (5.3) that only the two solutions $J_2 - I_2$ and $Y_2 + \dfrac{2}{\pi}K_2$ can be used to represent $\psi_n^C$, *i.e.*

$$\psi_n^C(r) = \alpha_n\left(J_2 - I_2\right)(\lambda_n^C r) + \beta_n\left(Y_2 + \frac{2}{\pi}K_2\right)(\lambda_n^C r) \text{ for all } r \in (0, R],$$

for some non-null pair $\{\alpha_n, \beta_n\}$.

($ii$) In the statement of the theorem part of item 4 is changed to:
The parameter $\lambda$ is an eigenvalue from the set $\{\lambda_n^C : n \in \mathbb{N}_0\}$ if and only if $\lambda$ is a positive zero of the complex function defined on the complex plane $\mathbb{C}\backslash(\infty, 0]$ by

$$\begin{aligned}\lambda \longmapsto &\left(\left(J_2(\lambda r) - I_2(\lambda r)\right)\right)(R)\left(\frac{d}{dr}\left(Y_2(\lambda r) + \frac{2}{\pi}K_2(\lambda r)\right)\right)(R)\\ &-\left(\frac{d}{dr}\left(J_2(\lambda r) - I_2(\lambda r)\right)\right)(R)\left(Y_2(\lambda r) + \frac{2}{\pi}K_2(\lambda r)\right)(R).\end{aligned} \tag{7.12}$$

The need for the replacement of $\mathbb{C}$ by $\mathbb{C}\backslash(-\infty, 0]$ in this statement is due to the presence of logarithmic terms in the solutions $Y_2(\lambda r)$ and $K_2(\lambda r)$. With this change made the analytic function (7.12) is regular in the open set $\mathbb{C}\backslash(-\infty, 0]$.

## 8. Biharmonic operators

Let $\mathbf{L}^2(D_R)$ denote the Lebesgue function space on $D_R$, *i.e.*

($i$) the set of $f : D_R \to \mathbb{C}$ with $f$ Lebesgue measurable on $D_R$

($ii$) $f$ satisfies

$$\int_{D_R} |f(r,\theta)|^2 \; r dr d\theta < \infty$$

($iii$) with the inner-product

$$(f, g) := \int_{D_R} f(r,\theta)\overline{g}(r,\theta) \; r dr d\theta = \int_0^R \int_0^{2\pi} f(r,\theta)\overline{g}(r,\theta) \; r dr d\theta, \tag{8.1}$$

where $\overline{g}$ denotes complex conjugation.

Then $\mathbf{L}^2(D_R)$ is a Hilbert function space.

We now define a self-adjoint partial differential operators in a subspace of the Lebesgue function space $\mathbf{L}^2(D_R)$, which represents solutions to the biharmonic polar boundary value problem as stated in Section 1.

Define the Plum factor $w_{A,B} : [0, 2\pi] \to \mathbb{C}$, where the parameters $A, B \in \mathbb{C}$ with $|A|^2 + |B|^2 > 0$, by

$$w_{A,B}(\theta) := A\cos(2\theta) + B\sin(2\theta) \text{ for all } \theta \in [0, 2\pi]. \tag{8.2}$$

We define the space $\mathbf{L}^2_{A,B}(D_R)$ with elements $F \equiv (F, f)$, where $f \in L^2((0, R); r)$,

$$F(r, \theta) := f(r)w_{A,B}(\theta) \text{ for all } r \in (0, R) \text{ and all } \theta \in [0, 2\pi]; \tag{8.3}$$

similarly for elements $(G, g)$ and $(H, h)$. Further define the inner-product

$$(F, G)_{A,B} := \int_0^R \int_0^{2\pi} f(r)w_{A,B}(\theta)\overline{g}(r)\overline{w}_{A,B}(\theta)\ rdrd\theta, \tag{8.4}$$

and norm

$$\|F\|^2_{A,B} := \int_0^R \int_0^{2\pi} |f(r)|^2 |w_{A,B}(\theta)|^2\ rdrd\theta \tag{8.5}$$

From these definitions it follows that $F \in \mathbf{L}^2(D_R)$, and that $\mathbf{L}^2_{A,B}(D_R)$ is a closed subspace of $\mathbf{L}^2(D_R)$.

8.1. **The Friedrichs case.** We now define the Friedrichs operator in the space $\mathbf{L}^2_{A,B}(D_R)$

$$\mathbf{T}_{A,B} : D(\mathbf{T}_{A,B}) \subset \mathbf{L}^2_{A,B}(D_R) \to \mathbf{L}^2_{A,B}(D_R)$$

by

$$D(\mathbf{T}_{A,B}) := \{(F, f) \in \mathbf{L}^2_{A,B}(D_R) : f \in D(T_F)\} \tag{8.6}$$

where the domain $D(T)$ is defined in (6.17) of Section 6.1, and then

$$(\mathbf{T}_{A,B}F)(r, \theta) := (T_F f)(r)w_{A,B}(\theta) \text{ for all } r \in (0, R] \text{ and all } \theta \in [0, 2\pi], \tag{8.7}$$

with $F$ as in (8.3).

The essential spectral properties of the operator $\mathbf{T}_{A,B}$ are given in the following theorem:

**Theorem 8.1.** *We have*

1. $\mathbf{T}_{A,B}$ *is a symmetric operator in the Hilbert space* $\mathbf{L}^2_{A,B}(D_R)$
2. $\mathbf{T}_{A,B}$ *is bounded below in the Hilbert space* $\mathbf{L}^2_{A,B}(D_R)$
3. $\mathbf{T}_{A,B}$ *is self-adjoint in the Hilbert space* $\mathbf{L}^2_{A,B}(D_R)$
4. *The spectrum of* $\mathbf{T}_{A,B}$ *includes as eigenvalues all the eigenvalues* $\{\Lambda^F_n : n \in \mathbb{N}_0\}$ *of the ordinary differential operator* $T_F$*, see* (6.30)
5. *The eigenvectors* $\{\Psi^{(A,B)}_n : n \in \mathbb{N}_0\}$ *of* $\mathbf{T}_{A,B}$ *at the eigenvalues* $\{\Lambda^F_n : n \in \mathbb{N}_0\}$ *are given by, for all* $n \in \mathbb{N}_0$,

$$\Psi^{(A,B)}_n(r, \theta) = (\Psi^{(A,B)}_n, \psi^F_n)(r, \theta) := \psi^F_n(r)w_{A,B}(\theta) \textit{ for all } r \in (0, R] \textit{ and all } \theta \in [0, 2\pi], \tag{8.8}$$

*where* $\{\psi_n^F : n \in \mathbb{N}_0\}$ *are the eigenvectors of the operator* $T_F$*; these eigenvectors of* $\mathbf{T}_{A,B}$ *satisfy the boundary conditions, for all* $\theta \in [0, 2\pi]$ *and all* $n \in \mathbb{N}_0$*,*

$$\Psi_n^{(A,B)}(0^+,\theta) = 0, \qquad \Psi_n^{(A,B)}(R,\theta) = 0 \text{ and } \left(\frac{d}{dr}\Psi_n^{(A,B)}(r,\theta)\right)(R) = 0. \tag{8.9}$$

(6) *The eigenvectors* $\{\Psi_n^{(A,B)} : n \in \mathbb{N}_0\}$ *are orthogonal and complete in the space* $\mathbf{L}^2_{A,B}(D_R)$*.*
(7) *The essential spectrum of* $\mathbf{T}_{A,B}$ *is empty.*
(8) *The spectrum of* $\mathbf{T}_{A,B}$ *consists only of the eigenvalues* $\{\Lambda_n^F : n \in \mathbb{N}_0\}$ *and is independent of the choice of the parameters* $A$ *and* $B$ *in the Plum factor* $w_{A,B}$*.*

*Proof.* Many of these properties of $\mathbf{T}_{A,B}$ follow from the corresponding properties of the ordinary differential operator $T$ in Section 6.

(1) We have from the symmetry of $T_F$ in $L^2((0,R);r)$ the Hermitian property of $\mathbf{T}_{A,B}$

$$\begin{aligned}(\mathbf{T}_{A,B}F,G)_{A,B} &= \int_0^R \int_0^{2\pi} (T_F f)(r) w_{A,B}(\theta)\overline{g}(r)\overline{w}_{A,B}(\theta)\ r dr d\theta \\ &= \int_0^R \int_0^{2\pi} f(r) w_{A,B}(\theta)\left(\overline{T_F g}\right)(r)\overline{w}_{A,B}(\theta)\ r dr d\theta \\ &= (F, \mathbf{T}_{A,B}G)_{A,B}\end{aligned}$$

for all $F, G \in D(\mathbf{T}_{A,B})$. Further the domain $D(\mathbf{T}_{A,B})$ is dense in $\mathbf{L}^2_{A,B}(D_R)$ on using the result that $D(T_F)$ is dense in $L^2((0,R);r)$; this establishes the symmetry of $\mathbf{T}_{A,B}$.

(2) We have, on using (6.32) for the operator $T_F$,

$$\begin{aligned}(\mathbf{T}_{A,B}F,F)_{A,B} &= \int_0^R \int_0^{2\pi} (T_F f)(r) w_{A,B}(\theta)\overline{f}(r)\overline{w}_{A,B}(\theta)\ r dr d\theta \\ &= \int_0^R (T_F f)(r)\overline{f}(r)\ r dr \int_0^{2\pi} |w_{A,B}(\theta|^2\ d\theta \\ &\geq \Lambda_0^F \int_0^R |f(r)|^2\ r dr \int_0^{2\pi} |w_{A,B}(\theta|^2\ d\theta \\ &= \Lambda_0^F \|F\|^2_{A,B} \text{ for all } F \in D(\mathbf{T}_{A,B}).\end{aligned}$$

(3) Since both the symmetric operators $T_F$ and $\mathbf{T}_{A,B}$ have a positive lower bound the point $-1$ belongs to the resolvent sets of both operators; since $T_F$ is self-adjoint the set

$$\{T_F f + If : f \in D(T_F)\} = L^2((0,R);r);$$

from the definition of $\mathbf{T}_{A,B}$ it then follows that the set

$$\{\mathbf{T}_{A,B}F + \mathbf{I}F : F \in D(\mathbf{T}_{A,B})\} = \mathbf{L}^2_{A,B}(D_R);$$

this implies that $\mathbf{T}_{A,B}$ is self-adjoint in $\mathbf{L}^2_{A,B}(D_R)$.

(4) The eigenvalues of the self-adjoint operator $T_F$ in $L^2((0,R);r)$ are $\{\Lambda_n^F : n \in \mathbb{N}_0\}$, as given in (6.30); from the definition of $\mathbf{T}_{A,B}$ it follows that $\{\Lambda_n^F : n \in \mathbb{N}_0\}$ are all eigenvalues of $\mathbf{T}_{A,B}$; this implies that the eigenvalues $\{\Lambda_n^F : n \in \mathbb{N}_0\}$ are contained in the spectrum of $\mathbf{T}_{A,B}$.

(5) From the definition of $\mathbf{T}_{A,B}$ it now follows that the eigenvectors $\{\Psi_n^{(A,B)} : n \in \mathbb{N}_0\}$ of $\mathbf{T}_{A,B}$ at these eigenvalues are given by, for all $n \in \mathbb{N}_0$,

$$\Psi_n^{(A,B)}(r,\theta) = (\Psi_n^{(A,B)}, \psi_n^F)(r,\theta) := \psi_n^F(r) w_{A,B}(\theta) \text{ for all } r \in (0,R] \text{ and all } \theta \in [0,2\pi];$$

also that, on examination, these eigenvectors of $\mathbf{T}_{A,B}$ satisfy the boundary conditions, for all $\theta \in [0,2\pi]$,

$$\Psi_n^{(A,B)}(0^+,\theta) = 0, \qquad \Psi_n^{(A,B)}(R,\theta) = 0 \text{ and } \left(\frac{d}{dr}\Psi_n^{(A,B)}(r,\theta)\right)(R) = 0.$$

(6) From this definition of the eigenvectors for $\mathbf{T}_{A,B}$ it then follows from the completeness of the eigenvectors $\{\psi_n^F : n \in \mathbb{N}_0\}$ of $T_F$ that the eigenvectors $\{\Psi_n^{(A,B)} : n \in \mathbb{N}_0\}$ are orthogonal and complete in the space $\mathbf{L}^2_{A,B}(D_R)$.
(7) The completeness of the eigenvectors $\{\Psi_n^{(A,B)} : n \in \mathbb{N}_0\}$ implies that there can be no additional points in the spectrum of $\mathbf{T}_{A,B}$ other than the eigenvalues $\{\Lambda_n^F : n \in \mathbb{N}_0\}$; thus the essential spectrum of $\mathbf{T}_{A,B}$ is empty.
(8) This result follows from the proofs of items 4 and 7.

□

**Corollary 8.1.** *The eigenvectors* $\{\Psi_n^{(A,B)} : n \in \mathbb{N}_0\}$ *are orthogonal but not complete in the Hilbert space* $\mathbf{L}^2(D_R)$.

*Proof.* The orthogonality property is clear; the restriction of the $\theta$ component in the eigenvectors to the Plum factor $w_{A,B}$ implies that completeness in $\mathbf{L}^2(D_R)$ is not possible. □

**Remark 8.1.** We note that the definition of the biharmonic operator $\mathbf{T}_{A,B}$ depends upon the choice of the parameters $A, B$ in the definition of the Plum factor $w_{A,B}$ in (8.2).

**Remark 8.2.** As in Section 6, see (1) above (6.19), we invoke the result in [14] and the application thereof as in [5, Section 15], to justify the use of the term Friedrichs for the operator $\mathbf{T}_{A,B}$.

8.2. **The Comparison case.** We now define a Comparison operator in the space $\mathbf{L}^2_{A,B}(D_R)$

$$\mathbf{S}_{A,B} : D(\mathbf{S}_{A,B}) \subset \mathbf{L}^2_{A,B}(D_R) \to \mathbf{L}^2_{A,B}(D_R)$$

by

$$D(\mathbf{S}_{A,B}) := \{(F,f) \in \mathbf{L}^2_{A,B}(D_R) : f \in D(T_C)\} \tag{8.10}$$

where the domain $D(T_C)$ is defined in (6.17) of Section 6.1, and then

$$(\mathbf{S}_{A,B}F)(r,\theta) := (T_C f)(r) w_{A,B}(\theta) \text{ for all } r \in (0,R] \text{ and all } \theta \in [0,2\pi]. \tag{8.11}$$

The essential spectral properties of the operator $\mathbf{S}_{A,B}$ are given in the following theorem:

**Theorem 8.2.** *We have*
(1) $\mathbf{S}_{A,B}$ *is a symmetric operator in the Hilbert space* $\mathbf{L}^2_{A,B}(D_R)$
(2) $\mathbf{S}_{A,B}$ *is bounded below in the Hilbert space* $\mathbf{L}^2_{A,B}(D_R)$
(3) $\mathbf{S}_{A,B}$ *is self-adjoint in the Hilbert space* $\mathbf{L}^2_{A,B}(D_R)$

(4) *The spectrum of* $\mathbf{S}_{A,B}$ *includes as eigenvalues all the eigenvalues* $\{\Lambda_n^C : n \in \mathbb{N}_0\}$ *of the ordinary differential operator* $T_C$*, see* (6.30).

(5) *The eigenvectors* $\{\Phi_n^{(A,B)} : n \in \mathbb{N}_0\}$ *of* $\mathbf{S}_{A,B}$ *at the eigenvalues* $\{\Lambda_n^C : n \in \mathbb{N}_0\}$ *are given by, for all* $n \in \mathbb{N}_0$,

(8.12)
$$\Phi_n^{(A,B)}(r,\theta) = (\Phi_n^{(A,B)}, \psi_n^C)(r,\theta) := \psi_n^C(r) w_{A,B}(\theta) \text{ for all } r \in (0,R] \text{ and all } \theta \in [0,2\pi],$$

*where* $\{\psi_n^C : n \in \mathbb{N}_0\}$ *are the eigenvectors of the of the operator* $T_C$*; these eigenvectors of* $\mathbf{S}_{A,B}$ *satisfy the boundary conditions, for all* $\theta \in [0,2\pi]$ *and all* $n \in \mathbb{N}_0$,

$$\left(\frac{d^2}{dr^2}\Phi_n^{(A,B)}(r,\theta)\right)(0^+,\theta) = 0, \qquad \Phi_n^{(A,B)}(R,\theta) = 0 \text{ and } \left(\frac{d}{dr}\Phi_n^{(A,B)}(r,\theta)\right)(R) = 0. \tag{8.13}$$

(6) *The eigenvectors* $\{\Phi_n^{(A,B)} : n \in \mathbb{N}_0\}$ *are orthogonal and complete in the space* $\mathbf{L}^2_{A,B}(D_R)$.

(7) *The essential spectrum of* $\mathbf{S}_{A,B}$ *is empty.*

(8) *The spectrum of* $\mathbf{S}_{A,B}$ *consists only of the eigenvalues* $\{\Lambda_n^C : n \in \mathbb{N}_0\}$ *and is independent of the choice of the parameters* $A$ *and* $B$ *in the Plum factor* $w_{A,B}$.

*Proof.* We omit details of this proof since it follows closely on the proof of Theorem 8.1 for the Friedrichs case. □

**Corollary 8.2.** *The eigenvectors* $\{\Phi_n^{(A,B)} : n \in \mathbb{N}_0\}$ *are orthogonal but not complete in the Hilbert space* $\mathbf{L}^2(D_R)$.

*Proof.* The orthogonality property is clear; the restriction of the $\theta$ component in the eigenvectors to the Plum factor $w_{A,B}$ implies that completeness in $\mathbf{L}^2(D_R)$ is not possible. □

**Remark 8.3.** We note that the definition of the biharmonic operator $\mathbf{S}_{A,B}$ depends upon the choice of the parameters $A, B$ in the definition of the Plum factor $w_{A,B}$ in (8.2).

## 9. Biharmonic boundary value problem

We comment in this section on some aspects of the operator theoretic solution to the biharmonic boundary value problem as considered in Section 1.

### 9.1. Boundary conditions.

($i$) At the regular endpoint the Dirichlet boundary conditions (1.6) are

$$v(R,\theta) = 0$$
$$\frac{\partial}{\partial \mathbf{n}} v(R,\theta) = 0.$$

With the solutions of the biharmonic differential equation, see (4.3), in the quasi-separation form (4.4)

$$v(r,\theta) \equiv u(r) w_{A,B}(\theta) \text{ for all } r \in (0,\infty) \text{ and } \theta \in [0,2\pi) \tag{9.1}$$

these Dirichlet boundary conditions translate to conditions on the radial component $u$ given by, see (7.2),

$$u(R) = 0, \qquad u'(R) = 0. \tag{9.2}$$

These boundary conditions are given in GKN form in (6.25) involving the two independent local boundary condition functions $\varphi_1$ and $\varphi_2$; these functions satisfy the GKN symmetric condition

$$[\varphi_1, \varphi_2](R) = 0. \tag{9.3}$$

(*ii*) At the singular endpoint $0^+$ we require just one boundary condition; there is a continuum of symmetric boundary conditions but the significant conditions are named after Friedrichs and Comparison.

(*a*) For the Friedrichs case at the singular endpoint $0^+$ we choose the GKN function to be $\varphi_0$, determined locally by $\varphi_0(r) = r^2/16$ with the symmetric property $[\varphi_0, \varphi_0](0^+) = 0$. The singular boundary condition is then $[u, \varphi_0](0^+) = 0$ which translates to the explicit form, see (7.2),

$$u(0^+) = 0 \tag{9.4}$$

when $u$ is an element of the maximal domain $D(T_1)$. In the general theory of ordinary differential operators this condition (9.4) is the classical Friedrichs boundary condition, so significant in the consideration of mathematical models for physical situations. This is further highlighted in Section 3 above, where it is noted that the eigenvalues of our Friedrichs operator are included in the eigenvalues of [4, Chapter 5, Section 6] and in [24]. Note that this singular boundary condition can only be written in the explicit form (9.4) due to the remarkable smoothness properties of the domain $D(T_1)$ of the maximal operator $T_1$ defined in $L^2((0, R); r)$, see Lemma 6.1.

(*b*) For the Comparison case at the singular endpoint $0^+$ the same remarks hold but with $\varphi_0$ determined locally by $\varphi_0(r) = -1/8$ to give

$$u''(0^+) = 0. \tag{9.5}$$

(*c*) Other choices of the boundary condition at the singular endpoint $0^+$ are possible; the general condition is

$$[u, \alpha + \beta r^2](0^+) = 0 \tag{9.6}$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 > 0$; this construction thereby gives a continuum of GKN symmetric boundary conditions at the endpoint $0^+$. The Friedrichs condition is given by the choice $\alpha = 0$ and $\beta = 1$, the Comparison condition by $\alpha = 1$ and $\beta = 0$.

9.2. **Regularity results.** (*a*) In the Friedrichs case we have represented the solutions to the biharmonic polar boundary value problem in the form of the properties of the self-adjoint operator $\mathbf{T}_{A,B}$ in the Hilbert function space $\mathbf{L}^2_{A,B}(D_R)$, see Theorem 8.1. However, these solutions can also be represented in a classical form as follows:

**Theorem 9.1.** *The polar biharmonic boundary value problem with partial differential equation*

$$(\Delta^2 v)(r, \theta) - \Lambda v(r, \theta) = 0 \textit{ for all } r \in (0, \infty) \textit{ and } \theta \in [0, 2\pi) \tag{9.7}$$

*with Dirichlet boundary conditions*

$$v(R,\theta) = 0 \tag{9.8}$$

$$\frac{\partial}{\partial \mathbf{n}} v(R,\theta) = 0 \tag{9.9}$$

*has a solution* $v$ *with the properties, see* (9.1),

$$v(r,\theta) \equiv u(r)w_{A,B}(\theta) \textit{ for all } r \in (0,\infty) \textit{ and } \theta \in [0,2\pi),$$

*and*

$$v(\cdot,\theta) \in C^{(4)}(0,R] \textit{ for all } \theta \in [0,2\pi] \tag{9.10}$$

$$v(r,\cdot) \in C^{(4)}[0,2\pi] \textit{ for all } r \in (0,R]$$

$$v(0^+,\theta) = 0 \textit{ for all } \theta \in [0,2\pi] \tag{9.11}$$

$$\left(\frac{d}{dr} v(r,\theta)\right)(0^+) = 0 \textit{ for all } \theta \in [0,2\pi]. \tag{9.12}$$

*These results hold if the parameter* $\Lambda \in \{\Lambda_n^F : n \in \mathbf{N}_0\}$ *where the numbers*

$$0 < \Lambda_0^F \leq \Lambda_1^F \leq \cdots \Lambda_n^F \leq \Lambda_{n+1}^F \cdots, \textit{ and } \lim_{n\to+\infty} \Lambda_n^F = +\infty \tag{9.13}$$

*are defined in* (6.30) *of Theorem* 6.1. *This solution* $v$ *has the form*

$$v(r,\theta) = \psi_n^F(r)w_{A,B}(\theta) \textit{ for all } r \in (0,R] \textit{ and all } \theta \in [0,2\pi]. \tag{9.14}$$

*where details of the radial component* $\psi_n^F$ *and angular component* $w_{A,B}$ *given in Theorem* 8.1.

*Proof.* See the details in the statements of the Theorems quoted. □

**Remark 9.1.** It is of interest to note that in terms of applications to the physical properties of a thin circular disk, the boundary conditions (9.8) and (9.9) imply that the disk $D_R$ is clamped on the edge $\partial D_R$. Additionally, in this the Friedrichs case, the boundary condition (9.11), together with the induced property (9.12), which is not a boundary condition, imply that the disk is also clamped at its centre 0.

($b$) In the Comparison case there are entirely similar results with $\mathbf{T}_{A,B}$ replaced by $\mathbf{S}_{A,B}$, but in this case the boundary condition (9.11) is replaced by

$$\left(\frac{d^2}{dr^2} v(r,\theta)\right)(0^+) = 0 \text{ for all } \theta \in [0,2\pi], \tag{9.15}$$

and the induced property (9.12) still holds.

**Remark 9.2.** The interest of these results is that it shows that these solutions of the type (9.14) of the biharmonic partial differential equation have classical fourth-order partial derivatives; further that all the partial derivatives used in the differential equation are all classical derivatives; that is no weak derivatives are required in this particular study of this Dirichlet boundary value problem.

## 10. Numerical results

We present the first five eigenvalues for the Friedrichs and Comparison extension in the unit disc ($R = 1$) using (7.4) and (7.12), respectively.

In order to construct these eigenvalues, i.e. to find the positive zeros of (7.4) and (7.12), we employed the Matlab routine *fzero* and this routine is based on an algorithm presented in [2]. The obtained values for the eigenvalues are rounded-off to three decimals and are presented in Table 1.

We point out that for the Friedrichs case, these eigenvalues correspond to $\lambda_2^{(j)}$, $j = 1, \ldots, 5$, of the results in [4, Ch. 5 Sect. 6]. This can be seen since the expression (7.4) is equivalent with the final formula of [4, Ch. 5 Sect 6]. However, for the Comparison extension, the corresponding eigenvalues do not seem to have been reported earlier in the literature.

In both cases equation (1.4) can be used to construct the free vibrations of a linear elastic, homogeneous and isotropic plate which has a uniform thickness small in comparison with its other dimensions. Assuming time-harmonic dependence in the free motion, $v$ in (1.4) is (apart from some physical constants) the transverse displacement of the plate, see for example [1, Sect. 11].

In the context of plate theory the term "clamped" is understood to mean that the plate is not allowed to move and not to bend at specific points. Thus the boundary conditions in Section 9.1 (i) ensures that the (circular) plate is clamped at its outer boundary. In the Friedrichs case, according to Remark 9.1, the physical consequence is that the plate is clamped also at the origin. Classically, the vibrating modes have been calculated for plates clamped only at its outer boundary, see [4, Ch. 5 Sect 6] and [1, Table 11.1]. However, some of these modes will satisfy the condition of Section 9.3 (ii)(a) at the origin and therefore these eigenvalues will be the same in the classical and the Friedrichs case, see further [3] where 3-D plots are presented for the classical clamped eigenmodes.

For the comparison case a different condition, see Section 9.1 (ii)(b), is imposed at the origin but since we were unable to locate these eigenvalues in the literature, none of the classical clamped vibration modes appear to satisfy this condition. Thus, the comparison case models another physical situation which seems not to have been previously studied. For more information on the applications (in for example fluid dynamics and plate theory) and history of the biharmonic operator, we refer to [17].

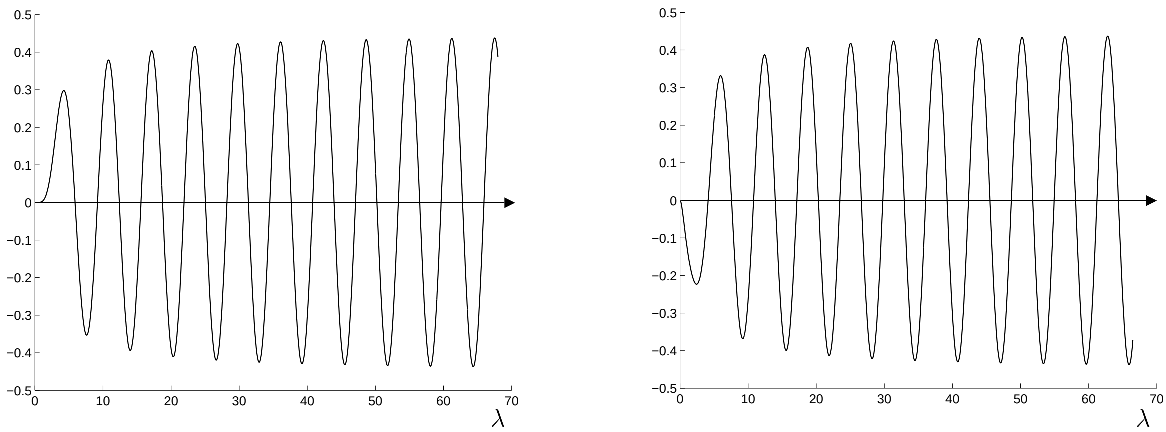
In Figure 1a) and Figure 1b), we plot the functions (7.4) and (7.12), respectively, divided by the factor $e^{\lambda}$ (we divide by this factor to ease the large oscillations present which occur due to the growth of the Bessel function $I_2$). In these figures the first 20 zeros of the Friedrichs and Comparison extension, respectively, are presented. This and (7.11) seem to indicate that the corresponding eigenvalues all are simple. Note that both the expressions (7.4) and (7.12) are equal to zero when $\lambda = 0$. However, as is seen from (6.30) in Theorem 6.1, $\lambda = 0$ is not eigenvalue for the Friedrichs extension, and also not an eigenvalue for the Comparison case.

We also point out that from the numerical simulations it is reasonable to conjecture that both for the Friedrichs and the Comparison extension, the $n$-th eigenfunction has $n$ zeros in the open interval $(0, 1)$.

TABLE 1. The first 5 eigenvalues for the Friedrich and the Comparison extension for the unit disc (results rounded off to 3 decimals)

| | Friedrichs | Comparison |
|---|---|---|
| $\lambda_0$ | 5.906 | 4.045 |
| $\lambda_1$ | 9.197 | 7.572 |
| $\lambda_2$ | 12.400 | 10.805 |
| $\lambda_3$ | 15.579 | 13.993 |
| $\lambda_4$ | 18.744 | 17.163 |

FIGURE 1. The first 20 zeros for the Friedrichs and Comparison extensions in the unit disc (functions scaled with the factor $e^{-\lambda}$)




a) Friedrichs (based on (7.4)) b) Comparison (based on (7.12))

## 11. Other domains

In this paper we consider the polar Dirichlet problem for the quasi-separated biharmonic partial differential equation, on the disk $D_R := \{(r,\theta) : r \in (0,R] \text{ and } \theta \in [0,2\pi]\}$.

Other possible domains for which the quasi-separation method yields results are:

(1) The annulus $D_{\rho,R}$, with $0 < \rho < R < +\infty$, defined by

$$D_{\rho,R} := \{(r,\theta) : r \in [\rho,R] \text{ and } \theta \in [0,2\pi]\}.$$

Here the associated Hilbert function space $\mathbf{L}^2_{A,B}(D_{\rho,R})$ with elements $(F,f)$ as before, has the inner-product, using the same Plum factor $w_{A,B}$,

$$(F,G)_{A,B} := \int_\rho^R \int_0^{2\pi} f(r)\overline{g}(r)\,|w_{A,B}(\theta)|^2\ rdrd\theta.$$

The Dirichlet problem for the radial biharmonic differential equation (5.1), requires the following boundary conditions applied to the elements of the domain of the corresponding maximal operator $T_1$ in the space $L^2((\rho,R);r)$ see (6.6) and (6.7),

$$u(\rho) = u'(\rho) = 0 \text{ and } u(R) = u'(R) = 0.$$

In this case there is a unique self-adjoint $T_\rho^R$ generated by these boundary conditions in the radial Hilbert function space $L^2((\rho,R);r)$, which leads to a unique self-adjoint operator $\mathbf{T}_{A,B}$ in the space $\mathbf{L}^2_{A,B}(D_{\rho,R})$; compare with Section 8 above. The operator $\mathbf{T}_{A,B}$ has a discrete spectrum $\{\Lambda_n^{\rho,R} : n \in \mathbb{N}_0\}$, of multiplicity not exceeding 2 and with $\Lambda_0^{\rho,R} > 0$.

(2) The exterior $D^R$ of the disk $D_R$ defined by

$$D^R := \{(r,\theta) : r \in [R,\infty) \text{ and } \theta \in [0,2\pi]\}.$$

Here the associated Hilbert function space $\mathbf{L}^2_{A,B}(D^R)$ with elements $(F,f)$ as before, has the inner-product, using the same Plum factor $w_{A,B}$,

$$(F,G)_{A,B} := \int_R^\infty \int_0^{2\pi} f(r)\overline{g}(r)\,|w_{A,B}(\theta)|^2\ rdrd\theta.$$

The Dirichlet problem for the radial biharmonic differential equation (5.1), requires the following boundary conditions applied to the elements of the domain of the corresponding maximal operator $T_1$ in the space $L^2((R,\infty);r)$, see (6.6) and (6.7),

$$u(R) = u'(R) = 0.$$

Following the analysis in [5, Section 13] there is no requirement for a GKN boundary condition at the singular end-point $+\infty$.

In this case there is a unique self-adjoint operator $T^R$ generated by these boundary conditions in the radial Hilbert function space $L^2((R,\infty);r)$, which leads to a unique self-adjoint operator $\mathbf{T}_{A,B}$ in the space $\mathbf{L}^2_{A,B}(D^R)$; compare with Section 8 above. Again from the analysis in [5, Section 13], the operator $\mathbf{T}_{A,B}$ has a continuous spectrum on $[0,\infty) \subset \mathbb{C}$ but no eigenvalues.

(3) The methods considered here lead to the definition of self-adjoint operators in the space $L^2(\mathbb{R}^2)$; here the inner product in radial terms is

$$(F,G)_{A,B} := \int_0^\infty \int_0^{2\pi} f(r)\overline{g}(r)\left|w_{A,B}(\theta)\right|^2 \ rdrd\theta,$$

taking into account the Plum factor $w_{A,B}$.

For this domain we mention the Friedrichs self-adjoint operator for which we require only one boundary condition at the origin of $\mathbb{R}^2$, viz the condition

$$u(0^+) = 0.$$

The resulting operator has no eigenvalues and a continuous spectrum on the half-line $[0.\infty)$ in $\mathbb{C}$.

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