ON THE RIGHT-DEFINITE AND LEFT-DEFINITE SPECTRAL THEORY OF THE LEGENDRE POLYNOMIALS

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Abstract. In this paper, we further develop the left-definite and right-definite spectral theory associated with the self-adjoint differential operator A in $L^2(-1,1)$, generated from the classical second-order Legendre differential equation, having the sequence of Legendre polynomials as eigenfunctions. Specifically, we determine the first *three* left-definite spaces associated with the pair $(L^2(-1,1), A)$. As a consequence of these results, we determine the explicit domain of both the associated left-definite operator A_1, first observed by Everitt, and the self-adjoint operator $A^{1/2}$. In addition, we give a new characterization of the domain $\mathcal{D}(A)$ of A and, as a corollary, we present a new proof of the Everitt-Marić result which gives optimal global smoothness of functions in $\mathcal{D}(A)$.

1. Introduction

One of the prime examples to illustrate the Glazman-Krein-Naimark (GKN) (see [1, Chapter VII and Appendix I] and [10, Chapters IV and V]) theory of self-adjoint operator extensions of formally symmetric, singular differential equations is the classical Legendre differential equation, defined by

$$\begin{aligned}\ell[y](t) :&= -(1-t^2)y''(t) + 2ty'(t) + ky(t) \quad (t \in (-1,1)) \\ &= -\left((1-t^2)y'(t)\right)' + ky(t) \\ &= \lambda y(t),\end{aligned} \tag{1.1}$$

where k is a fixed, positive real number; see the operator-theoretical account of this example in, for example, [1, Appendix II, pages 206-210], [4] and [7]. This differential equation, which was first analyzed by Titchmarsh (see [17, Sections 4.3-4.7]) in the right-definite setting $L^2(-1,1)$, is important in many areas of mathematical analysis, physics, applied mathematics, and engineering, including signal processing and sampling theory. Certainly, part of the reason for the importance of equation (1.1) stems from the fact that the m^{th} Legendre polynomial $P_m(t)$ $(m \in \mathbb{N}_0)$ is a solution of this equation when $\lambda = \lambda_m := m(m+1) + k$.

The classical self-adjoint Legendre operator A in the Hilbert space $L^2(-1,1)$, generated from $\ell[\cdot]$, which has the Legendre polynomials as eigenfunctions is well known to be a strictly positive operator. Consequently, due to a recent general left-definite theory developed for such operators by Littlejohn and Wellman (see [8]), there are a continuum of *left-definite Hilbert spaces* $\{H_r\}_{r>0}$ and *left-definite operators* $\{A_r\}_{r>0}$ associated with the pair $(L^2(-1,1), A)$; see Section 4 below for specific details. To date, the literature that deals with the left-definite functional analytic aspects of the Legendre expression (1.1) is restricted to the *first* left-definite space H_1; for example, see the contributions [4], [7], [9], [11], [12], and [13]. The space H_1 is the setting for some interesting

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properties of (1.1); indeed, it was Pleijel in [13, p. 398] who first observed that the Legendre equation changes from being limit-circle at $t = \pm 1$ in $L^2(-1,1)$ to being limit-point at $t = \pm 1$ in H_1. Moreover, Everitt [4] first considered a spectral resolution A_1, which he called the left-definite operator, of (1.1) in H_1 having the Legendre polynomials as eigenfunctions. In [5] and [7], the authors show that A_1 is a restriction of the differential operator A (considered as an operator in H_1) and they also obtain a characterization of the domain of A_1.

In this paper, we further discuss the left-definite theory - and as a consequence, the classical right-definite theory - associated with (1.1). We will see that, based on results in [8], it is the case that $H_1 = \mathcal{D}(A^{1/2})$ and $\mathcal{D}(A_1) = H_3$, the third left-definite space associated with $(L^2(-1,1), A)$; in fact, we explicitly determine these two function sets. Moreover, we show that $\mathcal{D}(A) = H_2$; this is a new characterization of the classical right-definite domain $\mathcal{D}(A)$. As an application of this new characterization of $\mathcal{D}(A)$, we give another proof of the Everitt-Marić result:

$$f \in \mathcal{D}(A) \Rightarrow \; f' \in L^2(-1,1).$$

The original intent of the Littlejohn-Wellman paper [8] was to obtain a general left-definite theory associated with a given self-adjoint, strictly positive operator B in a Hilbert space H; that is to say, a theory that would provide an umbrella for the many examples which appear in the literature (see [8] for further references). To the surprise of both authors, this general theory also provided much new information about the original operator B and its powers B^r $(r > 0)$; see Theorem 4.1 and Corollary 4.1 in Section 4.

The contents of this paper are as follows. In Section 2, we review some well-known properties of the Legendre polynomials and, more generally, the Gegenbauer polynomials; these properties are essential to establish our results. Indeed, it is the important and remarkable derivative formula of the (orthonormal) Jacobi polynomials $\{P_m^{(\alpha,\beta)}(t)\}_{m=0}^{\infty}$, namely

$$\frac{d}{dt} P_m^{(\alpha,\beta)}(t) = (m(m+\alpha+\beta+1))^{1/2} P_{m-1}^{(\alpha+1,\beta+1)}(t), \tag{1.2}$$

that allows us to obtain many of our results. In Section 3, we briefly discuss the self-adjoint operator A, generated by (1.1), that has the Legendre polynomials as eigenfunctions. Section 4 summarizes the general left-definite theory established in [8]. In Section 5, we apply these results to obtain the *first* left-definite space H_1 associated with the Legendre self-adjoint operator A. This space had previously been first found and studied by Everitt [4], and later by Loveland in [9] and Onyango-Otieno in [11]. In Section 6, we determine explicitly the *second* and *third* left-definite spaces associated with A. As a consequence of these results, we obtain our new characterization of $\mathcal{D}(A)$ and an alternative proof of the Everitt-Marić result; these results are presented in Section 7. Lastly, in Section 8, we obtain an explicit characterization of the domain of the first left-definite operator A_1. In a personal communication with Professor W. N. Everitt, these authors have learned that Everitt and V. Marić had earlier obtained this same characterization and their result is included in [5].

For the remainder of this paper, we adopt the usual mathematical notation. For example, $\mathbb{R}$ and $\mathbb{C}$ represent, respectively, the real and complex number fields, $\mathbb{N}$ the set of positive integers $\{1, 2, 3, \ldots\}$, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of non-negative integers. The generic phrase "$(x \in K)$", for example, means "for all $x \in K$". Lastly, the set $\mathcal{P}$ denotes the space of all polynomials $p(t)$, with complex coefficients, of the real variable t.

2. Some Properties of the Legendre and Ultraspherical Polynomials

For fixed $\alpha, \beta > -1$, the Jacobi polynomials $\{P_m^{(\alpha,\beta)}(\cdot)\}_{m=0}^{\infty}$, defined (see [16, page 68, formula 4.3.2]) by

$$P_m^{(\alpha,\beta)}(t) = \frac{k_m(\alpha,\beta)}{2^m} \sum_{r=0}^{m} \binom{m+\alpha}{m-r} \binom{m+\beta}{r} (t-1)^r (t+1)^{m-r} \quad (m \in \mathbb{N}_0), \tag{2.1}$$

where

$$k_m(\alpha,\beta) := \frac{(2m+\alpha+\beta+1)^{1/2}(\Gamma(m+\alpha+\beta+1))^{1/2}(m!)^{1/2}}{2^{(\alpha+\beta+1)/2}(\Gamma(m+\alpha+1))^{1/2}(\Gamma(m+\beta+1))^{1/2}},$$

form a complete *orthonormal* set in the Hilbert space

$$L^2_{\alpha,\beta}(-1,1) := L^2((-1,1);(1-t)^{\alpha}(1+t)^{\beta}) \tag{2.2}$$

of Lebesgue measurable functions $f : (-1,1) \to \mathbb{C}$ satisfying $\|f\|_{\alpha,\beta} < \infty$, where $\|\cdot\|_{\alpha,\beta}$ is the norm generated from the inner product $(\cdot,\cdot)_{\alpha,\beta}$, defined by

$$(f,g)_{\alpha,\beta} := \int_{-1}^{+1} f(t)\overline{g(t)}(1-t)^{\alpha}(1+t)^{\beta}dt \quad (f,g \in L^2_{\alpha,\beta}(-1,1)); \tag{2.3}$$

in short, $\{P_m^{(\alpha,\beta)}(t)\}_{m=0}^{\infty}$ satisfy

$$\int_{-1}^{+1} P_m^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta}dt = \delta_{n,m} \quad (n,m \in \mathbb{N}_0), \tag{2.4}$$

where $\delta_{n,m}$ is the Kronecker delta symbol. When $\alpha = \beta = 0$, we denote the inner product $(\cdot,\cdot)_{0,0}$ by $(\cdot,\cdot)$ and the corresponding space $L^2_{0,0}(-1,1)$ by the more familiar $L^2(-1,1)$. When $\alpha = \beta = 0$, these polynomials are called the *Legendre* polynomials and it is customary to write, in this case, $P_m^{(0,0)}(t) = P_m(t)$. Moreover, throughout this paper, we shall make use of the *Gegenbauer* polynomials $\{P_m^{(j,j)}(t)\}_{m=0}^{\infty}$ for $j \in \mathbb{N}_0$. In particular, from (1.2), it follows that the Legendre sequence $\{P_m(t)\}_{m=0}^{\infty}$ satisfies the derivative formula

$$\frac{d^j}{dt^j}P_m(t) = c(m,j)P_{m-j}^{(j,j)}(t) \quad (m \in \mathbb{N}_0; j = 0,1,\dots m), \tag{2.5}$$

where

$$c(m,j) := \sqrt{\frac{(m+j)!}{(m-j)!}} \quad (j = 0,1,\dots m). \tag{2.6}$$

For various properties of the Jacobi (and, in particular, Legendre and Gegenbauer) polynomials, we refer the reader to the classical treatise of Szegö (see [16, Chapter IV]).

3. Right-definite theory of the Legendre polynomials

We now briefly discuss the right-definite operator-theoretic aspects of the Legendre differential expression (1.1). For a thorough analytic and functional analytic discussion of this expression, see [4], [7], [9], and [11]. Another excellent account of the right-definite operator theoretic results concerning the Legendre equation can be found in [1, Appendix II, pages 206-210].

In general, if the minimal operator $\mathcal{L}_0$, associated with a formally symmetric differential expression $m[\cdot]$ defined on a real interval $I = (a,b)$ has equal deficiency indices in $L^2(I)$, then the domain of each self-adjoint extension of $\mathcal{L}_0$ in $L^2(I)$ is obtained by restricting the domain $\mathcal{D}(\mathcal{L})$ of the maximal operator $\mathcal{L} = \mathcal{L}_0^*$ to those functions that satisfy certain boundary conditions. The number of boundary conditions is determined by the deficiency index of the minimal operator and the nature of the boundary conditions is explicitly described in the so-called Glazman-Krein-Naimark (GKN) theory (see Theorem 4 in [10, Chapter V, Section 18.1]). For the Legendre expression $\ell[\cdot]$, given in (1.1), we note that it is limit-circle at both endpoints $t = \pm 1$ in the space $L^2(-1,1)$, which is called the *right-definite* setting for $\ell[\cdot]$.

The *maximal domain* Δ in $L^2(-1,1)$ for the Legendre differential expression $\ell[\cdot]$ is defined to be

$$\Delta := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell[f] \in L^2(-1,1)\},$$

and the *maximal operator* $\mathcal{L}$ associated with $\ell[\cdot]$ is given by

$$\mathcal{L}[f] = \ell[f] \quad (f \in \mathcal{D}(\mathcal{L}) := \Delta).$$

Observe that, for each $m \in \mathbb{N}_0$, the m^{th} Legendre polynomial $P_m(\cdot) \in \Delta$. For $f, g \in \Delta$, we have *Green's formula*

$$\int_{-1}^{+1} \ell[f](t)\overline{g(t)}dt = [f,g](1) - [f,g](-1) + \int_{-1}^{+1} f(t)\overline{\ell[g](t)}dt, \tag{3.1}$$

where $[\cdot,\cdot](\cdot) : \Delta \times \Delta \times (-1,1) \to \mathbb{C}$ is the sesquilinear form defined by

$$[f,g](t) = -(1-t^2)\left[f'(t)\overline{g(t)} - f(t)\overline{g'(t)}\right] \quad (f, g \in \Delta;\ t \in (-1,1)).$$

By definition of Δ, the limits

$$[f,g](\pm 1) = \lim_{t \to \pm 1} [f,g](t);$$

exist, and are finite, for each $f, g \in \Delta$. In particular, we note that $1 \in \Delta$ and

$$[f,1](\pm 1) := \lim_{t \to \pm 1} -(1-t^2)f'(t).$$

The *minimal operator* $\mathcal{L}_0$ in $L^2(-1,1)$ associated with $\ell[\cdot]$ is defined by

$$\mathcal{L}_0 f = \ell[f] \quad (f \in \mathcal{D}(\mathcal{L}_0) := \{f \in \Delta \mid [f,g](\pm 1) = 0 \text{ for all } g \in \Delta\}).$$

It is well known (see [10]) that $\mathcal{L}_0$ is a closed, symmetric operator in $L^2(-1,1)$ with $\mathcal{L}_0^* = \mathcal{L}$ and $\mathcal{L}^* = \mathcal{L}_0$.

From the GKN theory (see [10]), the operator $A : \mathcal{D}(A) \subset L^2(-1,1) \to L^2(-1,1)$ defined by

$$(Af)(t) = \ell[f](t) \quad (\text{a.e. } t \in (-1,1)) \tag{3.2}$$

$$f \in \mathcal{D}(A) := \{f \in \Delta \mid [f,1](\pm 1) = 0\} \tag{3.3}$$

is self-adjoint. Moreover, the spectrum of A is given by $\sigma(A) = \{m(m+1) + k \mid m \in \mathbb{N}_0\}$ and the Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ are eigenfunctions of A. In fact, for each $m \in \mathbb{N}_0$,

$$(AP_m)(t) = (m(m+1) + k)P_m(t) \quad (t \in (-1,1)).$$

Furthermore, A is the *Friedrich's extension* of the minimal operator $\mathcal{L}_0$ (see, for example, [6]). In particular, this means that the limits

$$\lim_{t \to \pm 1} f(t) \quad (f \in \mathcal{D}(A)) \tag{3.4}$$

exist and are finite (see [4, Section 3]). In fact, it is well known that the condition (3.4) is *equivalent* to the boundary condition $[f,1](\pm 1) = 0$ ($f \in \mathcal{D}(A)$) (see [4]). As a consequence of this equivalence, the following important facts concerning $\mathcal{D}(A)$ can be easily established.

Proposition 3.1. *Let $f, g \in \mathcal{D}(A)$. Then*

(i) $f', g' \in L^2_{1,1}(-1,1)$;

(ii)

$$(Af, g) = \int_{-1}^{+1} (Af)(t)\overline{g(t)}dt = \int_{-1}^{+1} \left[(1-t^2)f'(t)\overline{g'(t)} + kf(t)\overline{g(t)}\right] dt. \tag{3.5}$$

Proof. Let $f, g \in \mathcal{D}(A)$; fix $0 < T < 1$. After one integration by parts, we see that

$$\int_{-T}^{T} \left\{(1-t^2)f'(t)\overline{g'(t)} + kf(t)\overline{g(t)}\right\} dt = (1-t^2)f'(t)\overline{g(t)} \mid_{-T}^{T} + \int_{-T}^{T} (Af)(t)\overline{g(t)}dt. \tag{3.6}$$

By definition of $\mathcal{D}(A)$, the integral on the right-hand side of (3.6) is finite as $T \to 1$; moreover, from (3.4) and the (equivalent) boundary condition $\lim_{t\to\pm1}(1-t^2)f'(t) = 0$, we see that

$$(1-T^2)f'(T)\overline{g(T)} \to 0 \text{ as } T \to \pm 1.$$

Hence, the limit on the left-hand side of (3.5) exists and is finite and, consequently, both (i) and (ii) follow. □

The identity in (3.5) is known as *Dirichlet's formula.* Notice, since $(1-t^2) > 0$ on $(-1, 1)$, we have

$$(Af, f) \geq k(f, f) \quad (f \in \mathcal{D}(A)); \tag{3.7}$$

that is to say, A is bounded below by kI, where I is the identity operator in $L^2(-1, 1)$.

We remark that Everitt [4, page 97] obtains several equivalent characterizations of functions in the domain $\mathcal{D}(A)$ of A. Likewise, the authors in [6, Lemma 2, page 774] derive further characterizations for functions in the domain of the Friedrich's extensions for a large class (which includes the Legendre equation) of Sturm-Liouville second-order differential expressions on (a, b) which are limit-circle at both endpoints a and b. We note that we will obtain a new characterization of $\mathcal{D}(A)$ in Section 7 (see Theorem 7.1 and Corollary 7.1).

4. General Left-definite Theory

Suppose B is a self-adjoint operator in a Hilbert space H, with inner product $(\cdot,\cdot)_H$, that is bounded below by cI for some $c > 0$; that is,

$$(Bx, x)_H \geq c(x, x)_H \quad (x \in \mathcal{D}(B)).$$

In [8], the authors define the notion of an r^{th} left-definite space associated with the pair (H, B).

Definition 4.1. *Let $r > 0$ and suppose V_r is a linear manifold of the Hilbert space H and $(\cdot,\cdot)_r$ is an inner product on $V_r \times V_r$. Let $H_r = (V_r, (\cdot,\cdot)_r)$ denote the resulting inner product space. We say that H_r is an r^{th} **left-definite space** associated with the pair (H, B) if each of the following conditions hold:*
(1) H_r is a Hilbert space,
(2) $\mathcal{D}(B^r)$ is a linear manifold of V_r,
(3) $\mathcal{D}(B^r)$ is dense in H_r,
(4) $(x, x)_r \geq c^r (x, x)_H \quad (x \in V_r)$, and
(5) $(x, y)_r = (B^r x, y)_H \quad (x \in \mathcal{D}(B^r),\ y \in V_r)$.

The terminology *left-definite* originates in previous work of Schäfke and Schneider [15]. It is not clear, from the definition, if such a self-adjoint operator B generates an r^{th} left-definite space for a given $r > 0$. However, in [8], the authors prove the following existence and uniqueness theorem for self-adjoint operators B that are bounded below:

Theorem 4.1. (*see* [8, Theorem 3.1]) *Suppose $B : \mathcal{D}(B) \subset H \to H$ is a self-adjoint operator that is bounded below by cI, for some $c > 0$. Let $r > 0$. Define $H_r = (V_r, (\cdot,\cdot)_r)$ by*

$$V_r = \mathcal{D}(B^{r/2}), \tag{4.1}$$

and

$$(x, y)_r = (B^{r/2}x, B^{r/2}y)_H \quad (x, y \in V_r).$$

Then H_r is a left-definite space associated with the pair (H, B^r). Moreover, suppose $H_r := (V_r, (\cdot,\cdot)_r)$ and $H_r' := (V_r', (\cdot,\cdot)_r')$ are r^{th} left-definite spaces associated with the pair (H, B). Then $V_r = V_r'$ and

$(x,y)_r = (x,y)'_r$ *for all* $x,y \in V_r = V'_r$; *i.e.* $H_r = H'_r$. *That is to say,* $H_r = (V_r, (\cdot,\cdot)_r)$ *is the unique left-definite space associated with* (H,B). *Furthermore,*

$$V_r \subset V_s \quad (0 < s < r). \tag{4.2}$$

The following corollary follows immediately from this existence and uniqueness theorem.

Corollary 4.1. *Under the conditions of the above theorem, we have* $\mathcal{D}(B^r) = V_{2r}$ *for all* $r > 0$; *in particular,* $\mathcal{D}(B^{1/2}) = V_1$ *and* $\mathcal{D}(B) = V_2$.

The characterization that $\mathcal{D}(B) = V_2$ is essential in our new characterization of $\mathcal{D}(A)$, where A is the self-adjoint Legendre operator defined in (3.2) and (3.3). We note that Theorem 4.1 and Corollary 4.1 are applicable to this Legendre operator A since it was established in the previous section that A is self-adjoint and is bounded below in $L^2(-1,1)$ by kI, where $k > 0$.

In [8], the authors give the following definition of a *left-definite operator.*

Definition 4.2. *Suppose* $B : \mathcal{D}(B) \subset H \to H$ *is a self-adjoint operator that is bounded below in* H *by* cI *where* $c > 0$. *Let* $r > 0$ *and suppose that* H_r *is the* r^{th} *left-definite space associated with* (H,B). *If there exists a self-adjoint operator* $B_r : \mathcal{D}(B_r) \subset H_r \to H_r$ *that is a restriction of* B, *that is to say,*

$$B_r x = Bx$$
$$x \in \mathcal{D}(B_r) \subset \mathcal{D}(B),$$

we call such an operator an r^{th} ***left-definite operator associated with*** (H,B).

In [8], the following theorem is obtained:

Theorem 4.2. (*see* [8, Theorems 3.2, 3.6, and 3.7]) *Suppose* B *is a self-adjoint operator in a Hilbert space* H *that is bounded below by* cI *for some* $c > 0$. *For each* $r > 0$, *let* $H_r = (V_r, (\cdot,\cdot)_r)$ *be the* r^{th} *left-definite space associated with* (H,B). *Define* $B_r : \mathcal{D}(B_r) \subset H_r \to H_r$ *by*

$$B_r x = Bx$$
$$x \in \mathcal{D}(B_r) := V_{r+2} = \mathcal{D}(B^{(r+2)/2}).$$

Then B_r *is a left-definite operator associated with* (H,B). *Furthermore, if there exists a self-adjoint operator* $B'_r : \mathcal{D}(B'_r) \subset H_r \to H_r$ *such that* $B'_r x = Bx$ *for all* $x \in \mathcal{D}(B'_r) \subset \mathcal{D}(B)$, *then* $B_r = B'_r$. *In addition, each* B_r *is bounded below by* cI *in* H_r *and* $\sigma(B_r) = \sigma(B)$. *Lastly, if* $\{\varphi_n\}_{n=0}^{\infty}$ *is a complete orthogonal set of eigenfunctions of* B *in* H *then, for each* $r > 0$, $\{\varphi_n\}_{n=0}^{\infty}$ *is a complete orthogonal set of eigenfunctions of* B_r *in* H_r.

5. The first left-definite space for $(L^2(-1,1), A)$

Let A denote the self-adjoint Legendre operator defined in (3.2) and (3.3). According to Corollary 4.1, the first left-definite Hilbert space H_1 associated with the pair $(L^2(-1,1), A)$ is the domain $\mathcal{D}(A^{1/2})$ of A. Everitt first defined this space H_1 in [4].

Definition 5.1. *Let*

$$V_1 := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1,1); f \in L^2(-1,1); f' \in L^2_{1,1}(-1,1)\}, \tag{5.1}$$

and define $(\cdot,\cdot)_1$ *on* $V_1 \times V_1$ *by*

$$(f,g)_1 := \int_{-1}^{+1} \left[(1-t^2)f'(t)\overline{g'(t)} + kf(t)\overline{g(t)}\right] dt. \tag{5.2}$$

Finally, let

$$H_1 := (V_1, (\cdot,\cdot)_1). \tag{5.3}$$

Note, from (3.5) that

$$(Af, g) = (f, g)_1 \quad (f, g \in \mathcal{D}(A)) \tag{5.4}$$

We omit the proof of the next theorem that says H_1 is the first left-definite space associated with the pair $(L^2(-1, 1), A)$; indeed, a close examination of the proofs in [4] and [9] show that, for $r = 1$, properties (i)-(v) in Definition 4.1 are satisfied.

Theorem 5.1. *The space $H_1 = (V_1, (\cdot, \cdot)_1)$, defined in* (5.1), (5.2), *and* (5.3), *is the first left-definite space associated with $(L^2(-1, 1), A)$, where A is the Legendre self-adjoint operator defined in* (3.2) *and* (3.3).

In [4] and [9], the authors prove, using operator-theoretic ideas, that the Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ form a complete orthogonal set in H_1. We now give another, more constructive proof, of this result; this method of proof leads to a new, simpler characterization of H_1 (see Theorem 5.3).

Theorem 5.2. *The Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ form a complete orthogonal set in H_1.*

Proof. First, an elementary calculation, using (2.4) and (2.5), shows that

$$(P_m, P_n)_1 = (m(m + 1) + k)\, \delta_{m,n} \quad (m, n \in \mathbb{N}_0). \tag{5.5}$$

To show that $\{P_m(\cdot)\}_{m=0}^{\infty}$ is complete in H_1, it suffices (see [14, Theorem 4.18]) to show that $\mathcal{P}$ is dense in H_1. To this end, let $f \in H_1$; in particular, $f' \in L^2_{1,1}(-1, 1)$. Since $\{P_m^{(1,1)}(\cdot)\}_{m=0}^{\infty}$ is a complete orthonormal set in $L^2_{1,1}(-1, 1)$, we see that

$$\sum_{m=0}^{r} d_m P_m^{(1,1)} \to f' \text{ as } r \to \infty \text{ in } L^2_{1,1}(-1, 1), \tag{5.6}$$

where $\{d_m\}_{m=0}^{\infty} \in \ell^2$ (square-summable sequences of complex numbers) is the sequence of Fourier coefficients of f' in $L^2_{1,1}(-1, 1)$, given by

$$d_m = (f', P_m^{(1,1)})_{1,1} = \int_{-1}^{+1} f'(t) P_m^{(1,1)}(t)(1 - t^2)dt \quad (m \in \mathbb{N}_0).$$

For $r \geq 1$, define the polynomials

$$p_r(t) = \sum_{m=1}^{r} \frac{d_{m-1}}{c(m, 1)} P_m(t),$$

where $c(m, 1) = \sqrt{(m + 1)m}$. Then, from (2.5) and (2.6),

$$p_r^{(j)}(t) = \sum_{m=j}^{r} \frac{d_{m-1}}{c(m, 1)} c(m, j) P_{m-j}^{(j,j)}(t) \quad (j \in \mathbb{N}_0);$$

in particular, from (5.6),

$$p_r' = \sum_{m=1}^{r} d_{m-1} P_{m-1}^{(1,1)} \to f' \text{ in } L^2_{1,1}(-1, 1). \tag{5.7}$$

Furthermore, since $\left\{ \frac{d_{m-1}}{c(m,1)} \right\}_{m=1}^{\infty} \in \ell^2$, there exists $g \in L^2(-1, 1)$ such that

$$p_r \to g \text{ in } L^2(-1, 1) \tag{5.8}$$

and a subsequence $\{p_{r_j}\}_{j=1}^{\infty}$ of $\{p_r\}$ such that

$$p_{r_j}(t) \to g(t) \quad (\text{a.e. } t \in (-1, 1)). \tag{5.9}$$

Moreover, since $p_r' \to f'$ in $L^2_{1,1}(-1,1)$ and $f \in AC_{\text{loc}}(-1,1)$, we have

$$p_{r_j}(t) - p_{r_j}(t_0) = \int_{t_0}^{t} p'_{r_j}(u)du \to \int_{t_0}^{t} f'(u)du = f(t) - f(t_0) \quad (t, t_0 \in (-1,1)). \tag{5.10}$$

Choose $t_0 \in (-1,1)$ so that $p_{r_j}(t_0) \to g(t_0)$. Letting $j \to \infty$ in (5.10), we see that

$$g(t) = f(t) + c \quad (\text{a.e. } t \in (-1,1)) \tag{5.11}$$

where $c = g(t_0) - f(t_0)$. Define, for each integer $r \geq 1$, $\pi_r(t) = p_r(t) - c$. From (5.7) and (5.8), we see that

$$\pi_r \to g - c = f \text{ in } L^2(-1,1),$$

and

$$\pi_r' = p_r' \to f' \text{ in } L^2_{1,1}(-1,1).$$

Hence

$$\begin{aligned} \|f - \pi_r\|_1^2 &= \left\|f' - \pi_r'\right\|_{1,1}^2 + k\,\|f - \pi_r\|^2 \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Hence the set $\mathcal{P}$ of polynomials is dense in H_1 and, consequently, the Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ form a complete orthogonal set in H_1. □

We now show that it is possible to simplify the original definition of the function space V_1.

Theorem 5.3. *The vector space V_1, defined in* (5.1)*, is given by*

$$V_1 = \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1,1); f' \in L^2_{1,1}(-1,1)\}.$$

Proof. Define

$$V_1' := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1,1); f' \in L^2_{1,1}(-1,1)\}.$$

Clearly, $V_1 \subset V_1'$. Conversely, let $f \in V_1'$ so $f' \in L^2_{1,1}(-1,1)$. From the completeness of the orthonormal set $\{P_m^{(1,1)}(\cdot)\}_{m=0}^{\infty}$ in $L^2_{1,1}(-1,1)$, we see that

$$\sum_{m=0}^{r} \alpha_m P_m^{(1,1)} \to f' \text{ as } r \to \infty \text{ in } L^2_{1,1}(-1,1),$$

where

$$\alpha_m = (f', P_m^{(1,1)})_{1,1} = \int_{-1}^{+1} f'(t) P_m^{(1,1)}(t)(1-t^2)dt \quad (m \in \mathbb{N}_0).$$

As in the proof of Theorem 5.2, define for $r \geq 1$,

$$p_r(t) = \sum_{m=1}^{r} \frac{d_{m-1}}{c(m,1)} P_m(t),$$

where $\{P_m(\cdot)\}_{m=0}^{\infty}$ are the Legendre polynomials and $c(m,1) = \sqrt{m(m+1)}$. As seen in Theorem 5.2, there exists $g \in L^2(-1,1)$ such that

$$\begin{aligned} p_r &\to g \text{ as } r \to \infty \text{ in } L^2(-1,1), \\ p_r' &\to f' \text{ as } r \to \infty \text{ in } L^2_{1,1}(-1,1) \end{aligned}$$

where f and g are connected through the identity

$$g(t) = f(t) + c \quad (\text{a.e. } t \in (-1,1))$$

for some constant $c \in \mathbb{C}$. In particular, $f \in L^2(-1,1)$. Consequently, $V_1' \subset V_1$ and this completes the proof. □

From Theorem 5.3 and Corollary 4.1, the following result is immediate.

Corollary 5.1. *The domain of the self-adjoint operator $A^{1/2}$, the positive square root of the Legendre operator A defined in (3.2) and (3.3), is given by*

$$\begin{aligned}\mathcal{D}(A^{1/2}) &= \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\mathrm{loc}}(-1,1), f^{(j)} \in L^2_{j,j}(-1,1) \text{ for } j = 0,1\} \\ &= \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\mathrm{loc}}(-1,1), f' \in L^2_{1,1}(-1,1)\}.\end{aligned}$$

6. The second and third left-definite spaces for $(L^2(-1,1), A)$

We begin by defining the following two inner product spaces.

Definition 6.1. *Let*

$$V_2 := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\mathrm{loc}}(-1,1); f^{(j)} \in L^2_{j,j}(-1,1) \ (j = 0,1,2)\}, \tag{6.1}$$

and

$$V_3 := \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{\mathrm{loc}}(-1,1); f^{(j)} \in L^2_{j,j}(-1,1) \ (j = 0,1,2,3)\}. \tag{6.2}$$

Define $(\cdot,\cdot)_j : V_j \times V_j \to \mathbb{C}$ $(j = 1,2)$, respectively, by

$$(f,g)_2 := \int_{-1}^{+1} \left[k^2 f(t)\overline{g(t)} + 2(k+1)f'(t)\overline{g'(t)}(1-t^2) + f''(t)\overline{g''(t)}(1-t^2)^2\right] dt, \tag{6.3}$$

and

$$\begin{aligned}(f,g)_3 := \int_{-1}^{+1} &[k^3 f(t)\overline{g(t)} + (3k^2+6k+4)f'(t)\overline{g'(t)}(1-t^2) \\ &+(3k+8)f''(t)\overline{g''(t)}(1-t^2)^2 + f'''(t)\overline{g'''(t)}(1-t^2)^3]dt\end{aligned}$$

Finally, let

$$H_2 := (V_2, (\cdot,\cdot)_2), \tag{6.4}$$

and

$$H_3 := (V_3, (\cdot,\cdot)_3). \tag{6.5}$$

We note that these spaces, and in particular their associated inner products, are essentially determined from the formal square $\ell^2[\cdot] = \ell[\ell[\cdot]]$ and cube $\ell^3[\cdot] = \ell[\ell^2[\cdot]]$ of the Legendre expression $\ell[\cdot]$ defined in (1.1); see Property 5 of Definition 4.1. Indeed,

$$\ell^2[y] = \left((1-t^2)^2 y''(t)\right)'' - 2(k+1)\left((1-t^2)y'(t)\right)' + k^2 y(t),$$

and

$$\begin{aligned}\ell^3[t] = &-\left((1-t^2)^3 y'''(t)\right)''' + (3k+8)\left((1-t^2)^2 y''(t)\right)'' \\ &-(3k^2+6k+4)\left((1-t^2)y'(t)\right)' + k^3 y(t).\end{aligned}$$

In this section, we outline only the proof that the space H_2 is the second left-definite space associated with $(L^2(-1,1), A)$. Many of the arguments carry over *mutatis mutandis* to those used in [4], [9], and in the last section. However, there are a few subtle differences that will warrant a careful analysis of H_2. We shall omit the proof that H_3 is the third left-definite space for $(L^2(-1,1), A)$.

We begin with the completeness of these spaces.

Theorem 6.1. *The inner product spaces H_2 and H_3 are Hilbert spaces.*

Proof. It is clear that $(\cdot,\cdot)_2$ is an inner product for on $V_2 \times V_2$; moreover (see (4) of Definition 4.1), it is easy to see that $H_2 \subset L^2(-1,1)$ and

$$(f,f)_2 \geq k^2(f,f) \quad (f \in V_2). \tag{6.6}$$

Suppose $\{f_m\}_{m=0}^{\infty}$ is a Cauchy sequence in H_2; since

$$\|f_m\|_2^2 = k^2 \|f_m\|^2 + 2(k+1) \|f_m'\|_{1,1}^2 + \|f_m''\|_{2,2}^2, \tag{6.7}$$

we see that $\{f_m''\}_{m=0}^{\infty}$ is Cauchy in $L^2_{2,2}(-1,1)$. Hence there exists $g_3 \in L^2_{2,2}(-1,1)$ such that

$$f_m'' \to g_3 \text{ in } L^2_{2,2}(-1,1). \tag{6.8}$$

Since each $f_m' \in AC_{\text{loc}}(-1,1)$, we have for fixed $t, t_0 \in (-1,1)$ with $t_0 \leq t$,

$$f_m'(t) - f_m'(t_0) = \int_{t_0}^{t} f_m''(u)du \to \int_{t_0}^{t} g_3(u)du \tag{6.9}$$

and $g_3 \in L^1_{\text{loc}}(-1,1)$. From (6.7), we see that $\{f_m'\}_{m=0}^{\infty}$ is also Cauchy in $L^2_{1,1}(-1,1)$ so there exists a function $g_2 \in L^2_{1,1}(-1,1)$ with

$$f_m' \to g_2 \text{ in } L^2_{1,1}(-1,1) \tag{6.10}$$

and, for all $t, t_1 \in (-1,1)$,

$$f_m(t) - f_m(t_1) = \int_{t_1}^{t} f_m'(u)du \to \int_{t_1}^{t} g_2(u)du. \tag{6.11}$$

There exists a subsequence $\{f_{m_{r,1}}'\}$ of $\{f_m'\}$ such that

$$f_{m_{r,1}}'(t) \to g_2(t) \quad \text{(a.e. } t \in (-1,1)).$$

Choose t_0 in (6.9) such that $f_{m_{r,1}}'(t_0) \to g_2(t_0)$ as $r \to \infty$. Passing through this subsequence, we see from (6.9) that

$$g_2(t) - g_2(t_0) = \int_{t_0}^{t} g_3(u)du \quad \text{(a.e. } t \in (-1,1)).$$

Hence

$$g_2 \in AC_{\text{loc}}(-1,1), \tag{6.12}$$

and

$$g_2'(t) = g_3(t) \text{ a.e. } t \in (-1,1). \tag{6.13}$$

Returning to (6.7), we have that $\{f_m\}_{m=0}^{\infty}$ is Cauchy in $L^2(-1,1)$ so there exists $g_1 \in L^2(-1,1)$ with

$$f_m \to g_1 \text{ in } L^2(-1,1), \tag{6.14}$$

and there exists a subsequence $\{f_{m_{r,2}}\}$ of $\{f_m\}$ such that

$$f_{m_{r,2}}(t) \to g_1(t) \text{ a.e. } t \in (-1,1).$$

In (6.11), choose $t_1 \in (-1,1)$ such that $f_{m_{r,2}}(t_1) \to g_1(t_1)$; passing through this subsequence, we see from (6.11) that

$$g_1(t) - g_1(t_1) = \int_{t_1}^{t} g_2(u)du \quad \text{(a.e. } t \in (-1,1)).$$

Hence

$$g_1 \in AC_{\text{loc}}(-1,1), \tag{6.15}$$

and

$$g_1'(t) = g_2(t) \text{ a.e. } t \in (-1,1); \tag{6.16}$$

consequently, from (6.13),

$$g_1''(t) = g_2'(t) = g_3(t) \quad (\text{a.e. } t \in (-1,1)). \tag{6.17}$$

Equalities (6.16) and (6.17), together with (6.8), (6.10), (6.12), (6.14), and (6.15) show that $g_1 \in V_2$ and

$$\begin{aligned} \|f_m - g_1\|_2^2 &= k^2 \|f_m - g_1\|^2 + 2(k+1) \|f_m' - g_1'\|_{1,1}^2 + \|f_m'' - g_1''\|_{2,2}^2 \\ &= k^2 \|f_m - g_1\|^2 + 2(k+1) \|f_m' - g_2\|_{1,1}^2 + \|f_m'' - g_3\|_{2,2}^2 \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

This shows that H_2 is complete. □

We next outline the analog of Theorem 5.2 for the space H_2; the proof for H_3 is similar.

Theorem 6.2. *The Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ form a complete orthogonal set in H_2 and H_3.*

Proof. A calculation shows that

$$(P_m, P_n)_2 = (k + m(m+1))^2 \delta_{n,m} \quad (n, m \in \mathbb{N}_0), \tag{6.18}$$

so $\{P_m(\cdot)\}_{m=0}^{\infty}$ is an orthogonal set in H_2. As in Theorem 5.2, we show that $\mathcal{P}$ is dense in H_2. Let $f \in H_2$; in particular, $f'' \in L^2_{2,2}(-1,1)$. From the completeness of $\{P_m^{(2,2)}(\cdot)\}_{m=0}^{\infty}$ in $L^2_{2,2}(-1,1)$, we see that

$$\sum_{m=0}^{r} d_m P_m^{(2,2)} \to f'' \text{ as } r \to \infty \text{ in } L^2_{2,2}(-1,1), \tag{6.19}$$

where

$$d_m = \int_{-1}^{+1} f''(t) P_m^{(2,2)}(t)(1-t^2)^2 dt \quad (m \in \mathbb{N}_0).$$

Similar to the proof of Theorem 5.2, define the polynomials

$$p_r(t) = \sum_{m=2}^{r} \frac{d_{m-2}}{c(m,2)} P_m(t) \quad (r \geq 2),$$

where $c(m,j)$ is defined in (2.6). Then

$$p_r^{(j)}(t) = \sum_{m=j}^{r} \frac{d_{m-2}}{c(m,2)} c(m,j) P_m^{(j,j)}(t) \quad (j \in \mathbb{N}_0), \tag{6.20}$$

and, from 6.19, we see that

$$p_r'' = \sum_{m=2}^{r} d_{m-2} P_m^{(2,2)} \to f'' \text{ in } L^2_{2,2}(-1,1).$$

Furthermore, there exists a subsequence $\{p''_{r_{j,1}}\}$ of $\{p''_r\}$ such that

$$p''_{r_{j,1}}(t) \to f''(t) \quad (\text{a.e. } t \in (-1,1)).$$

With regards to (6.20), for $j = 0$ and 1, we observe that $\frac{c(m,j)}{c(m,2)} \to 0$ so that $\{\frac{d_{m-2}c(m,j)}{c(m,2)}\}_{m=2}^{\infty} \in \ell^2$ for $j = 0, 1$. Consequently, from the completeness of $\{P_m^{(j,j)}(\cdot)\}_{m=0}^{\infty}$, there exists $g_0 \in L^2(-1,1)$, $g_1 \in L^2_{1,1}(-1,1)$ such that

$$p_r \to g_0 \text{ in } L^2(-1,1), \tag{6.21}$$

and

$$p_r' \to g_1 \text{ in } L^2_{1,1}(-1,1). \tag{6.22}$$

Furthermore, there exist subsequences $\{p'_{r_{j,2}}\}$ of $\{p'_{r_{j,1}}\}$ and $\{p_{r_{j,3}}\}$ of $\{p_{r_{j,2}}\}$ such that

$$p'_{r_{j,2}}(t) \to g_1(t) \quad (\text{a.e. } t \in (-1,1)), \tag{6.23}$$

and

$$p_{r_{j,3}}(t) \to g_0(t) \quad (\text{a.e. } t \in (-1,1)). \tag{6.24}$$

Moreover, since $f' \in AC_{\text{loc}}(-1,1)$, we see that

$$p'_{r_{j,2}}(t) - p'_{r_{j,2}}(t_1) = \int_{t_1}^{t} p''_{r_{j,2}}(u)du \to \int_{t_1}^{t} f''(u)du = f'(t) - f'(t_1), \tag{6.25}$$

where t_1 is chosen such that $p'_{r_{j,2}}(t_1) \to g_1(t_1)$. Hence, from (6.23), we see that

$$g_1(t) = f'(t) + c \quad (\text{a.e. } t \in (-1,1)), \tag{6.26}$$

where $c = g_1(t_1) - f'(t_1)$. In (6.25), pass through the subsequence $\{p_{r_{j,3}}\}$ and integrate (6.25) from t_2 to t, where t_2 is chosen such that $p_{r_{j,3}}(t_2) \to g_0(t_2)$. As $j \to \infty$, we see

$$g_0(t) = f(t) + ct + c_1 \quad (\text{a.e. } t \in (-1,1)), \tag{6.27}$$

for some constant c_1. Define, for each integer $r \geq 2$, $\pi_r(t) = p_r(t) - ct - c_1$. Then

$$\pi_r \to g_0 - ct - c_1 = f \text{ in } L^2(-1,1),$$
$$\pi'_r = p'_r - c \to g_1 - c = f' \text{ in } L^2_{1,1}(-1,1),$$

and

$$\pi''_r = p''_r \to f'' \text{ in } L^2_{2,2}(-1,1).$$

Hence

$$\begin{aligned} \|f - \pi_r\|_2^2 &= k^2 \|f - \pi_r\|^2 + 2(k+1) \left\|f' - \pi'_r\right\|_{1,1}^2 + \left\|f'' - \pi''_r\right\|_{2,2}^2 \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Therefore $\mathcal{P}$ is dense in H_2 and, consequently, the Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ are a complete orthogonal set in H_2. □

We are now in position to prove the following theorem; again, we outline this proof for H_2 only.

Theorem 6.3. *Let A denote the self-adjoint operator in $L^2(-1,1)$ that is defined in* (3.2) *and* (3.3). *Then*

(i) the second left-definite space associated with $(L^2(-1,1), A)$ is H_2, defined in (6.4).
(ii) the third left-definite space associated with $(L^2(-1,1), A)$ is H_3, defined in (6.5).

Proof. We need to prove that H_2 satisfies the five conditions in Definition 4.1 when $r = 2$.
(i) H_2 is complete :
See Theorem 6.1.
(ii) $\mathcal{D}(A^2)$ is a linear manifold of V_2 :
First, a calculation shows that

$$(A^2 p, q) = (p,q)_2 \quad (p, q \in \mathcal{P}). \tag{6.28}$$

Let $f \in \mathcal{D}(A^2)$; since the Legendre polynomials $\{P_m(\cdot)\}_{m=0}^{\infty}$ are a complete orthonormal set in $L^2(-1,1)$, we see that

$$p_j \to f \text{ in } L^2(-1,1), \tag{6.29}$$

where

$$p_j(t) := \sum_{m=0}^{j} \alpha_m P_m(t) \text{ and } \alpha_m = (f, P_m) \text{ for each } m \in \mathbb{N}_0.$$

Since $A^2 f \in L^2(-1,1)$,

$$\sum_{m=0}^{j} \beta_m P_m \to A^2 f \text{ in } L^2(-1,1),$$

where

$$\beta_m = (A^2 f, P_m) \quad (m \in \mathbb{N}_0).$$

From the self-adjointness of A^2, we see that

$$(A^2 f, P_m) = (f, A^2 P_m) = (m(m+1)+k)^2 (f, P_m) = (m(m+1)+k)^2 \alpha_m;$$

that is to say,

$$\beta_m = (m(m+1)+k)^2 \alpha_m \quad (m \in \mathbb{N}_0).$$

Hence,

$$\begin{aligned}\sum_{m=0}^{j} \beta_m P_m &= \sum_{m=0}^{j} (m(m+1)+k)^2 \alpha_m P_m \\ &= \sum_{m=0}^{j} \alpha_m A^2 P_m = A^2 \left(\sum_{m=0}^{j} \alpha_m P_m \right) \\ &= A^2 p_j \to A^2 f \text{ in } L^2(-1,1).\end{aligned}$$

Moreover, from (6.28), we see that

$$\begin{aligned}\|p_j - p_r\|_2^2 &= (A^2(p_j - p_r), p_j - p_r) \\ &\to 0 \text{ as } j, r \to \infty;\end{aligned}$$

that is to say, $\{p_j\}_{j=0}^{\infty}$ is Cauchy in H_2. From the completeness of H_2, there exists $g \in V_2 \subset L^2(-1,1)$ such that

$$p_j \to g \text{ in } H_2.$$

Furthermore, by definition of $(\cdot,\cdot)_2$, we see that

$$(p_j - g, p_j - g)_2 \geq k^2 (p_j - g, p_j - g);$$

hence

$$p_j \to g \text{ in } L^2(-1,1). \tag{6.30}$$

Comparing (6.29) and (6.30), we have $f \in V_2$.

<u>(iii) $\mathcal{D}(A^2)$ is dense in H_2</u> :

This follows since $\mathcal{P}$ is dense in H_2 (Theorem 6.2) and $\mathcal{P} \subset \mathcal{D}(A^2)$.

<u>(iv) $(f,f)_2 \geq k^2 (f,f) \quad (f \in V_2)$</u> :

This was seen in (6.6).

<u>(v) $(f,g)_2 = (A^2 f, g) \quad (f \in \mathcal{D}(A^2),\ g \in V_2)$</u> :

Let $f \in \mathcal{D}(A^2) \subset V_2$ and $g \in V_2$. Since polynomials are dense in both H_2 and $L^2(-1,1)$ and convergence in H_2 implies convergence in $L^2(-1,1)$ (from property (iv) above), there exist sequences of polynomials $\{p_j\}_{j=0}^{\infty}$, $\{q_j\}_{j=0}^{\infty}$ such that

$$p_j \to f \text{ in } H_2,\ A^2 p_j \to A^2 f \text{ in } L^2(-1,1)$$

and

$$q_j \to g \text{ in } H_2 \text{ and } L^2(-1,1).$$

Consequently, from (6.28), we see that

$$(A^2 f, g) = \lim_{j\to\infty} (A^2 p_j, q_j) = \lim_{j\to\infty} (p_j, q_j)_2 = (f,g)_2.$$

This proves (v) and completes the proof of the theorem. □

7. A new characterization of $\mathcal{D}(A)$ and a new proof of the Everitt-Marić result

We begin by simplifying the definition of V_2, given in Definition 6.1. We omit the proof since it is similar to the proof of Theorem 5.3 (using the arguments of Theorem 6.2).

Theorem 7.1. *The vector space* V_2, *defined in* (6.1), *is given by*

$$V_2 = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); (1-t^2)f'' \in L^2(-1,1)\}. \tag{7.1}$$

Our first application combines Corollary 4.1, Theorem 6.3, and Theorem 7.1.

Corollary 7.1. *The domain* $\mathcal{D}(A)$ *of the Legendre self-adjoint operator* A, *defined in* (3.2) *and* (3.3), *is given by*

$$\begin{aligned}\mathcal{D}(A) &= \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f^{(j)} \in L^2_{j,j}(-1,1)\ (j = 0,1,2)\} \\ &= \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); (1-t^2)f'' \in L^2(-1,1)\}.\end{aligned}$$

Our next result gives an alternative proof of the Everitt-Marić result concerning smoothness of functions in the domain of A. The first proof of the Everitt-Marić result was obtained by the authors in [5] using the Chisholm-Everitt inequality (see [2] and [3]).

Corollary 7.2. (*Everitt-Marić* [5]) *If* $f \in \mathcal{D}(A)$, *then* $f' \in L^2(-1,1)$.

Proof. Let $f \in \mathcal{D}(A)$. From (3.2), (3.3), (6.1), and (7.1), we see that the functions

$$f,\ \ell[f] = -(1-t^2)f'' + 2tf' + kf, \text{ and } (1-t^2)f''$$

all belong to $L^2(-1,1)$. Taking linear combinations, we find $2tf' \in L^2(-1,1)$; that is

$$\int_{-1}^{+1} 4t^2 \left|f'(t)\right|^2 dt < \infty. \tag{7.2}$$

Since $f' \in AC_{\text{loc}}(-1,1)$, we see that

$$f' \in L^2[-1/2, 1/2]. \tag{7.3}$$

Moreover, since $1/(4t^2) \le 1$ on $[1/2, 1]$, we see from (7.2) that

$$\int_{1/2}^{1} \left|f'(t)\right|^2 dt = \int_{1/2}^{1} \frac{1}{4t^2} 4t^2 \left|f'(t)\right|^2 dt \le \int_{-1}^{+1} 4t^2 \left|f'(t)\right|^2 dt < \infty. \tag{7.4}$$

Similarly,

$$\int_{-1}^{-1/2} \left|f'(t)\right|^2 dt < \infty. \tag{7.5}$$

Combining (7.3), (7.4), and (7.5), we see that $f' \in L^2(-1,1)$. □

8. The domain $\mathcal{D}(A_1)$ of A_1

We state the following theorem; the proof is similar to the proofs of Theorems 5.3 and 7.1.

Theorem 8.1. *The vector space* V_3, *defined in* (6.2), *is also given by*

$$V_3 = \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{\text{loc}}(-1,1); (1-t^2)^{3/2} f^{(3)} \in L^2(-1,1)\}. \tag{8.1}$$

Observe, from (4.2), that V_3 is a subspace of V_1.

From Theorem 4.2, we have the following characterization of the domain of the first left-definite operator A_1 associated with $(L^2(-1,1), A)$. Everitt and Marić first obtained this characterization in [5] using the Chisholm-Everitt inequality (see [2] and [3]).

Theorem 8.2. *Let A denote the self-adjoint Legendre operator defined in* (3.2) *and* (3.3)*. The first left-definite* (*self-adjoint*) *operator* $A_1 : \mathcal{D}(A_1) \subset H_1 \to H_1$ *associated with the pair* $(L^2(-1,1), A)$ *is given by*

$$A_1 f = Af$$
$$f \in \mathcal{D}(A_1) = V_3,$$

where V_3 *is defined in* (6.2) *or* (8.1)*. Moreover, the Legendre polynomials* $\{P_m(\cdot)\}_{m=0}^{\infty}$ *form a complete orthogonal set of eigenfunctions of* A_1 *and the spectrum of* A_1 *is given by* $\sigma(A_1) = \sigma(A) = \{m(m+1) + k \mid m \in \mathbb{N}_0\}$.

Remark In [4], the operator A_1 was *first* defined through other means. Indeed, since A is bounded below in $L^2(-1,1)$ by kI where $k > 0$, it follows that $0 \in \rho(A)$, the resolvent set of A. Consequently, $R_0(A) = A^{-1}$ exists as a bounded operator from $L^2(-1,1)$ onto $\mathcal{D}(A)$. From (5.4) and the embeddings $\mathcal{D}(A) \subset H_1 \subset L^2(-1,1)$, it follows that the operator $B : H_1 \to H_1$ defined by

$$Bf = R_0 f$$
$$f \in \mathcal{D}(B) = H_1$$

is bounded, self-adjoint, and invertible with self-adjoint inverse B^{-1}. Using the uniqueness property of Theorem 4.2, it is the case that $B^{-1} = A_1$.

In [5] and [7], the authors describe A_1 by yet other means. Specifically, they define the operator $\mathcal{S} : H_1 \to H_1$ by

$$\mathcal{S}f = \ell[f]$$
$$f \in \mathcal{D}(\mathcal{S}) = \{f \in H_1 \mid f \in \mathcal{D}(A); \ell[f] \in H_1\}, \tag{8.2}$$

and show that $\mathcal{S}$ is self-adjoint. Since $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}(A)$, the uniqueness part of Theorem 4.2 yields $\mathcal{S} = A_1$.

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