GHOST MATRICES AND A CHARACTERIZATION OF SYMMETRIC SOBOLEV BILINEAR FORMS

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We dedicate this paper to Robert Piziak on the occasion of his 65th birthday

Abstract. In this paper, we characterize symmetric Sobolev bilinear forms defined on $\mathcal{P} \times \mathcal{P}$, where $\mathcal{P}$ is the space of polynomials. More specifically we show that symmetric Sobolev bilinear forms, like symmetric matrices, can be re-written with a diagonal representation. As an application, we introduce the notion of a ghost matrix, extending some classic work of T. J. Stieltjes.

1. Introduction

In this paper we discuss Sobolev bilinear forms of the type

$$\phi_N(p,q) = \sum_{i=0}^{N}\sum_{j=0}^{N} < \sigma_{i,j}, p^{(i)}q^{(j)} > \quad (p,q \in \mathcal{P}), \tag{1.1}$$

where $\mathcal{P}$ is the vector space of all polynomials $p : \mathbb{R} \to \mathbb{R}$, N is a fixed non-negative integer, $\sigma_{i,j}$ is a moment functional for $0 \le i,j \le N$, and $p^{(i)}$ denotes the i^{th} derivative of the polynomial $p(x)$. With A_{N+1} defined to be the $(N+1) \times (N+1)$ matrix of moment functionals

$$A_{N+1} := \begin{pmatrix} \sigma_{0,0} & \sigma_{0,1} & \cdots & \sigma_{0,N} \\ \sigma_{1,0} & \sigma_{1,1} & \cdots & \sigma_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N,0} & \sigma_{N,1} & \cdots & \sigma_{N,N} \end{pmatrix}, \tag{1.2}$$

we say that $\phi_N(\cdot,\cdot)$ is generated by A_{N+1} and, symbolically, we write (1.1) as

$$\phi_N(p,q) = (p, p', \ldots, p^{(N)})A_{N+1}\begin{pmatrix} q \\ q' \\ \vdots \\ q^{(N)} \end{pmatrix}; \tag{1.3}$$

in regards to the notation in (1.3), see Remark 2.1 in Section 2 below.

We ask, and answer, the following questions:

(1) Under what conditions on the moment functionals $\{\sigma_{i,j}\}$ will $\phi_N(\cdot,\cdot)$ be a symmetric bilinear form on polynomials? That is, when will

$$\phi_N(p,q) = \phi_N(q,p) \quad (p,q \in \mathcal{P})? \tag{1.4}$$

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Furthermore, are there necessary and sufficient conditions on these moment functionals $\{\sigma_{i,j}\}$ to guarantee that $\phi_N(\cdot,\cdot)$ is symmetric? In this paper, we will produce necessary and sufficient conditions for $\phi_N(\cdot,\cdot)$ to be symmetric.

(2) It is well known that every symmetric quadratic form can be diagonalized (for example, see [14, Chapter 7] and [19, Chapters 10 and 12]). In the case that the Sobolev bilinear form $\phi_N(\cdot,\cdot)$ is symmetric, do there exist moment functionals $\{\tau_k\}$ such that

$$\phi_N(p,q)=\sum_{k=0}^{N}<\tau_k, p^{(k)}q^{(k)}>? \tag{1.5}$$

If so, can we characterize these moment functionals $\{\tau_k\}$ in terms of the given moment functionals $\{\sigma_{i,j}\}$? For both questions, the answer is yes. In particular, under the condition of symmetry, $\phi_N(\cdot,\cdot)$ does have a representation of the form (1.5); furthermore, we explicitly determine each τ_k in terms of the given moment functionals $\{\sigma_{i,j}\}$.

(3) When $\phi_N(\cdot,\cdot)$ is the zero Sobolev bilinear form the associated matrix A_{N+1}, that generates $\phi_N(\cdot,\cdot)$, acts as a zero matrix; in this sense, we call A_{N+1} a ghost matrix. In the case $N=0$, the connection with ghost functions, which are non-trivial functions defined on the half or whole real line whose moments are all zero, is classical and can be traced back to work of Stieltjes [21]. Generalizing this idea, are there *non-trivial* ghost matrices that generate the zero Sobolev bilinear form? Can we characterize all ghost matrices? Again, the answers to these questions are yes.

In matrix theory, the connection between symmetrizability and diagonalizability is well known and classic; however, as the reader can see below in the details of a simple example (Example 3.2 below), the diagonalizability of a symmetric bilinear form $\phi_N(\cdot,\cdot)$ is somewhat surprising and unexpected. Indeed, we note that there are contributions in the literature that discuss non-diagonal symmetric Sobolev inner products; for example, see [1].

Every moment functional σ has two well known, and now classical, integral representations. The first one, due to R. P. Boas [5], shows that if σ is a moment functional, then there exists (a non-unique) signed measure μ_σ, generated from a function of bounded variation on the real line $\mathbb{R}$, such that

$$<\sigma,p>=\int_{\mathbb{R}} p\,d\mu_\sigma \quad (p\in\mathcal{P}).$$

The other representation, due to A. J. Duran [8], says that

$$<\sigma,p>=\int_{\mathbb{R}} p w_\sigma dx \quad (p\in\mathcal{P}),$$

where w_σ, also non-unique, belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$ of functions. Consequently, the form given in (1.1) is equivalent to the more standard looking bilinear forms

$$\phi_N(p,q)=\sum_{i=0}^{N}\sum_{j=0}^{N}\int_{\mathbb{R}} p^{(i)}q^{(j)}d\mu_{ij} \quad (p,q\in\mathcal{P},\mu_{ij}\in BV(\mathbb{R})),$$

or

$$\phi_N(p,q)=\sum_{i=0}^{N}\sum_{j=0}^{N}\int_{\mathbb{R}} p^{(i)}q^{(j)}w_{ij}dx \quad (p,q\in\mathcal{P},w_{ij}\in \mathcal{S}(\mathbb{R})).$$

We note that bilinear forms of the type given in (1.1) have been studied in detail for more than twenty years in conjunction with the development of the theory of Sobolev orthogonal polynomials. We refer the reader to [3], [13], [16], [17], and [18] for further information on this connection. Inner products of the form (1.1), when the underlying matrix (1.2) is symmetric, were earlier considered by Blankenagel [4] in his doctoral dissertation in 1971; this thesis was further emphasized in the 1977 survey paper by Danese [7]. The well-known classical theory of orthogonal polynomials - for example, the theory contained in the texts of Chihara [6] or Szegö [22] - is mainly concerned with the bilinear form $\phi_N(\cdot,\cdot)$ when $N = 0$. Although the theories for $N = 0$ and $N > 0$ share some commonalities, in general, the theory of Sobolev orthogonal polynomials is quite different from its classical counterpart; an excellent reference parlaying some distinct differences between classical and Sobolev orthogonal polynomials is the text [10] by Gautschi. For an earlier account discussing polynomials orthogonal with respect to the inner product (1.1), see the 1973 paper by Schäfke and Wolf [20]. On a more recent note, the authors in [2] discuss the diagonal Sobolev inner product

$$(p,q)_N = \sum_{j=0}^{N} \int_{-1}^{1} a_j(N)(1-x^2)^j p^{(j)}(x) q^{(j)}(x) dx \tag{1.6}$$

where the coefficients $\{a_j(N)\}$ are the so-called Legendre-Stirling numbers. The classical Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ are orthogonal with respect to this inner product (1.6) for each $N \in \mathbb{N}_0$; among other results, the paper [2] discusses a combinatorial interpretation of the Legendre-Stirling numbers and shows that these numbers behave remarkably similar to the classical Stirling numbers of the second kind.

The contents of this paper are as follows. In Section 2, we will review some classical properties of moment functionals that are necessary for the results and analysis that follow. Section 3 deals with some specific examples that precede our general results. In Section 4, we obtain a complete characterization of symmetric Sobolev bilinear forms. Lastly, in Section 5, we introduce the concept of a ghost matrix and offer a complete characterization, as well as several examples, of this type of matrix.

2. Definitions and Preliminaries

A polynomial system (PS) $\{p_n\}_{n=0}^{\infty}$ is a basis for $\mathcal{P}$ with $\deg(p_n) = n$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of positive integers. A moment functional σ is a real or complex-valued linear functional defined on $\mathcal{P}$; we use distributional notation $< \sigma, p >$ to indicate the action of σ on $p \in \mathcal{P}$ rather than the function notation $\sigma(p)$. Of course, σ is completely determined by its values on any PS; in particular, if the so-called *moments*

$$\sigma_n :=< \sigma, x^n > \quad (n \in \mathbb{N}_0)$$

of σ are known, then the value of $< \sigma, p >$ is known for any $p \in \mathcal{P}$.

If σ is a moment functional, then we define the derivative σ' to be the moment functional defined by

$$< \sigma', p >:= - < \sigma, p' > \quad (p \in \mathcal{P}). \tag{2.1}$$

If $q \in \mathcal{P}$, we define the moment functional $q\sigma$ by

$$< q\sigma, p >:=< \sigma, pq > \quad (p \in \mathcal{P}). \tag{2.2}$$

Remark 2.1. In view of (2.2), we note that

$$\phi_N(p,q) = \left\langle (p, p', \ldots, p^{(N)}) A_{N+1} \begin{pmatrix} q \\ q' \\ \vdots \\ q^{(N)} \end{pmatrix}, 1 \right\rangle \quad (p, q \in \mathcal{P}).$$

This is the proper interpretation of the notation used in (1.3).

The following lemma is well known and can be found, for example, in [13]; we make repeated use of this lemma in the results that follow.

Lemma 2.1. *Let σ be a moment functional.*

(i) *Then $\sigma = 0$ if and only if $\sigma' = 0$;*
(ii) *(Leibniz' rule) If $q \in \mathcal{P}$, then $(q\sigma)' = q'\sigma + q\sigma'$.*

The calculus of moment functionals has proven to be a very useful tool in understanding, and solving, some classical problems in the theory of orthogonal polynomial solutions to ordinary and partial differential equations during the past several years. One of the more spectacular applications of this calculus is due to Kwon et al in [11]. Indeed, they construct a real-valued weight function for the Bessel polynomials $\{y_n\}_{n=0}^{\infty}$ (a completely different, but also elegant, solution of this problem was given in 1993 by Duran in [9]). The Bessel PS was introduced into the mathematical literature by Krall and Frink [12] in 1949. For each $n \in \mathbb{N}_0$, the polynomial $y = y_n$ is a solution of the second-order differential equation

$$x^2y'' + 2(x+1)y' = n(n+1)y.$$

As discussed in [12], the orthogonality of these polynomials is considered in the complex plane $\mathbb{C}$; specifically,

$$\int_\gamma y_n(z)y_m(z)e^{-2/z}dz = 2\frac{(-1)^{n+1}}{(2n+1)}\delta_{n,m} \quad (n, m \in \mathbb{N}_0), \tag{2.3}$$

where γ is any closed, Jordan curve encircling the origin in $\mathbb{C}$. The moments $\{\mu_n\}_{n=0}^{\infty}$ associated with these polynomials are real and, as a consequence of the Residue Theorem, they are readily computed to be

$$\mu_n = \frac{(-1)^n 2^{n+1}}{(n+1)!} \quad (n \in \mathbb{N}_0).$$

Consequently, from Boas' Theorem [5], there must exist a real measure μ, originating from a function of bounded variation on $(-\infty, \infty)$, that generates the same orthogonality relation as in (2.3); furthermore, from the general theory of moments, μ cannot be a positive measure. To this end, Littlejohn [15] showed that a real orthogonalizing weight for the Bessel polynomials will satisfy the weight equation

$$x^2w' - 2w = 0 \tag{2.4}$$

in some distributional sense. The classical solution of (2.4) is

$$\widehat{w}(x) = \exp(-2/x); \tag{2.5}$$

however, this function cannot be an orthogonalizing weight function for the Bessel polynomials on any interval of the real line. It was at this point that Kwon et al considered (2.4) in the sense of moment functionals. In fact, they replaced (2.4) by

$$x^2w' - 2w = g, \tag{2.6}$$

where g is the classical Stieltjes *ghost* function defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-x^{1/4}}\sin(x^{1/4}) & \text{if } x > 0. \end{cases} \tag{2.7}$$

The remarkable feature of this function, as first noted by Stieltjes, is that all of its moments are zero:

$$\int_{-\infty}^{\infty} x^n g(x)dx = 0 \quad (n \in \mathbb{N}_0); \tag{2.8}$$

that is to say, g is a non-trivial representation of the zero moment functional. We call such a function a *ghost function* for the obvious reason. In (2.4), the singularity $x = 0$ is an essential singularity and this is reflected in the classical solution $\widehat{w}$ given in (2.5); however, in (2.6), the non-homogeneous term g tempers this singularity and actually results in a solution w that belongs to $C(-\infty,\infty) \cap L^1(-\infty,\infty)$. Indeed, this solution of (2.6) is given by

$$w(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -\exp(-2/x)\int_x^\infty \dfrac{\exp(2/t)\exp(-t^{1/4})\sin(t^{1/4})}{t^2}dt & \text{if } x > 0; \end{cases}$$

it is an orthogonalizing weight function for the Bessel polynomials $\{y_n\}_{n=0}^{\infty}$. It is possible, as shown in [11], to replace the Stieltjes ghost function given in (2.7) with other ghost functions, for example

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sin(2\pi \ln x)\exp(-\ln^2 x) & \text{if } x > 0 \end{cases} \tag{2.9}$$

has the same 'zero moment' property that is given in (2.8). In Section 5, we generalize ghost functions by introducing the concept of $m \times m$ ghost matrices.

3. Examples

To motivate our main results, in particular Theorem 4.2, we consider the bilinear form $\phi_N(\cdot,\cdot)$, defined in (1.1), in the cases $N = 1$ and $N = 2$.

Example 3.1. $N = 1$. In this case,

$$\phi_1(p,q) =< \sigma_{0,0}, pq > + < \sigma_{0,1}, pq' > + < \sigma_{1,0}, p'q > + < \sigma_{1,1}, p'q' > .$$

Since p_0 is a nonzero constant, we may assume that $p_0 = 1$ in which case we see that for $n \in \mathbb{N}_0$,

$$\phi_1(p_0,p_n) =< \sigma_{00}, p_n > + < \sigma_{01}, p_n' >$$

while

$$\phi_1(p_n,p_0) =< \sigma_{00}, p_n > + < \sigma_{10}, p_n' > .$$

Consequently, for symmetry, we see that we must have

$$< \sigma_{0,1}, p_n' > = < \sigma_{1,0}, p_n' > \quad (n \in \mathbb{N}_0).$$

However, since $\{p_n'\}_{n=1}^{\infty}$ is also a PS, it follows from Lemma 2.1 (i) that

$$\sigma_{0,1} = \sigma_{1,0}.$$

With this condition, we see that

$$\begin{aligned} \phi_1(p,q) &=< \sigma_{0,0}, pq > + < \sigma_{1,0}, pq' + p'q > + < \sigma_{1,1}, p'q' > \\ &=< \sigma_{0,0}, pq > + < \sigma_{1,0}, (pq)' > + < \sigma_{1,1}, p'q' > \\ &=< \sigma_{0,0}, pq > - < \sigma_{1,0}', pq > + < \sigma_{1,1}, p'q' > \\ &= \langle \sigma_{0,0} - \sigma_{1,0}', pq \rangle + \langle \sigma_{1,1}, p'q' \rangle. \end{aligned}$$

That is to say, $\phi_1(\cdot,\cdot)$ has the diagonal representation

$$\phi_1(p,q) = \langle \sigma_{0,0} - \sigma_{1,0}', pq \rangle + \langle \sigma_{1,1}, p'q' \rangle. \tag{3.1}$$

Example 3.2. $N = 2$. In this case,

$$\begin{aligned}\phi_2(p,q) = &< \sigma_{0,0}, pq > + < \sigma_{0,1}, pq' > + < \sigma_{0,2}, pq'' > + < \sigma_{1,0}, p'q > + < \sigma_{1,1}, p'q' > \\ &+ < \sigma_{1,2}, p'q'' > + < \sigma_{2,0}, p''q > + < \sigma_{2,1}, p''q' > + < \sigma_{2,2}, p''q'' > .\end{aligned} \tag{3.2}$$

A similar analysis to Example 3.1 shows that $\phi_2(\cdot,\cdot)$ is symmetric if and only if the following two "symmetry" equations are satisfied:

$$-\sigma_{1,0} + \sigma'_{2,0} + \sigma_{0,1} - \sigma'_{0,2} = 0 \tag{3.3}$$

$$\sigma_{2,1} - \sigma_{1,2} = 0. \tag{3.4}$$

Indeed, equation (3.3) is found by simplifying

$$\phi_2(p_0, p_n) - \phi_2(p_n, p_0) = 0 \quad (n \in \mathbb{N}_0),$$

while equation (3.4) is found by simplifying

$$\phi_2(p_1, p_n) - \phi_2(p_n, p_1) = 0 \quad (n \in \mathbb{N}_0).$$

From Leibniz' rule (Lemma 2.1(ii)) and (2.2), we see that

$$\begin{aligned}&< \sigma_{0,2}, pq'' >=< p\sigma_{0,2}, q'' >=< -(p\sigma_{0,2})', q' > \\ &= - < p'\sigma_{0,2} + p\sigma'_{0,2}, q' >= - < \sigma_{0,2}, p'q' > - < \sigma'_{0,2}, pq' >;\end{aligned}$$

similarly

$$< \sigma_{1,0}, p'q >= - < \sigma_{1,0}, pq' > - < \sigma'_{1,0}, pq >,$$

$$< \sigma_{1,2}, p'q'' >= - < \sigma_{1,2}, p''q' > - < \sigma'_{1,2}, p'q' >,$$

$$\begin{aligned}< \sigma_{2,0}, p''q > &=< q\sigma_{2,0}, p'' >= - < (q\sigma_{2,0})', p' >= - < \sigma_{2,0}, p'q' > - < \sigma'_{2,0}, p'q > \\ &= - < \sigma_{2,0}, p'q' > + < \sigma'_{2,0}, pq' > + < \sigma''_{2,0}, pq > .\end{aligned}$$

Substituting this into (3.2) yields

$$\begin{aligned}\phi_2(p,q) =&< \sigma_{0,0}, pq > + < \sigma_{0,1}, pq' > - < \sigma_{0,2}, p'q' > - < \sigma'_{0,2}, pq' > - < \sigma_{1,0}, pq' > - < \sigma'_{1,0}, pq > \\ &+ < \sigma_{1,1}, p'q' > - < \sigma_{1,2}, p''q' > - < \sigma'_{1,2}, p'q' > - < \sigma_{2,0}, p'q' > + < \sigma'_{2,0}, pq' > + < \sigma''_{2,0}, pq > \\ &+ < \sigma_{2,1}, p''q' > + < \sigma_{2,2}, p''q'' > \\ =&< \sigma_{0,0} - \sigma'_{1,0} + \sigma''_{2,0}, pq > + < \sigma_{0,1} - \sigma'_{0,2} - \sigma_{1,0} + \sigma'_{2,0}, pq' > \\ &+ < -\sigma_{0,2} + \sigma_{1,1} - \sigma'_{1,2} - \sigma_{2,0}, p'q' > + < -\sigma_{1,2} + \sigma_{2,1}, p''q' > + < \sigma_{2,2}, p''q'' > \\ =& \langle \sigma_{0,0} - \sigma'_{0,1} + \sigma''_{0,2}, pq\rangle + \langle \sigma_{1,1} - \sigma_{0,2} - \sigma_{2,0} - \sigma'_{2,1}, p'q'\rangle + \langle \sigma_{2,2}, p''q''\rangle,\end{aligned}$$

the last equality coming on account of (3.3) and (3.4). Hence, when the Sobolev bilinear form $\phi_2(\cdot,\cdot)$ is symmetric, it has the diagonal form

$$\phi_2(p,q) = \langle \sigma_{0,0} - \sigma'_{0,1} + \sigma''_{0,2}, pq\rangle + \langle \sigma_{1,1} - \sigma_{0,2} - \sigma_{2,0} - \sigma'_{2,1}, p'q'\rangle + \langle \sigma_{2,2}, p''q''\rangle.$$

From these two examples, it is natural to ask: given any $N \in \mathbb{N}$, does a symmetric Sobolev bilinear form always have a diagonal representation? We show in the next section that the answer is yes; furthermore, we will explicitly compute each moment functional in this diagonal representation.

4. Main Results

A well known classical result in matrix theory, the Principle Axes Theorem (see [14, Theorem 4, Section 7.2]) asserts that whenever A is a symmetric matrix, there is a change of variables $x = Py$ that transforms the quadratic form $x^T Ax$ into a quadratic form $y^T Dy$ with no cross-product term. In this section, we give necessary and sufficient conditions on extending this result to the case of symmetric bilinear forms.

Theorem 4.1. *Let $N \in \mathbb{N}$ and let $\phi_N(\cdot,\cdot)$ be a bilinear form defined in (1.1). Then there are moment functionals*

$$\{\sigma_k^N\}_{k=0}^N, \{\tau_k^N\}_{k=0}^{N-1}, \text{ and } \{\tilde{\tau}_k^N\}_{k=0}^{N-1}$$

such that

$$\phi_N(p,q) = \sum_{k=0}^{N} < \sigma_k^N, p^{(k)}q^{(k)} > + \sum_{k=0}^{N-1} < \tau_k^N, p^{(k+1)}q^{(k)} > + \sum_{k=0}^{N-1} < \tilde{\tau}_k^N, p^{(k)}q^{(k+1)} >; \tag{4.1}$$

more specifically,

$$\begin{cases} \sigma_0^N = \sigma_{0,0}, \\ \sigma_N^N = \sigma_{N,N}, \\ \sigma_k^N = \sigma_{k,k} + \sum_{i=0}^{k-1}\sum_{j=k+i+1}^{N}(-1)^{j-k}\binom{j-k-1}{i}(\sigma_{j,k-i-1}^{(j-i-k-1)} + \sigma_{k-i-1,j}^{(j-i-k-1)}) \end{cases} \tag{4.2}$$

$$\tau_k^N = \sum_{i=0}^{k}\sum_{j=k+i+1}^{N}(-1)^{j-k-1}\binom{j-k-1}{i}\sigma_{j,k-i}^{(j-i-k-1)} \tag{4.3}$$

$$\tilde{\tau}_k^N = \sum_{i=0}^{k}\sum_{j=k+i+1}^{N}(-1)^{j-k-1}\binom{j-k-1}{i}\sigma_{k-i,j}^{(j-i-k-1)}. \tag{4.4}$$

Proof. Using Leibniz' rule, we may rewrite $\phi(\cdot,\cdot)$ in (1.1) as

$$\phi_N(p,q) = \sum_{k=0}^{N} < \sigma_k, p^{(k)}q^{(k)} > + \sum_{k=0}^{N-1} < \tau_k, p^{(k+1)}q^{(k)} > + \sum_{k=0}^{N-1} < \tilde{\tau}_k, p^{(k)}q^{(k+1)} >, \tag{4.5}$$

and then we shall show that the moment functionals σ_k, τ_k, and $\tilde{\tau}_k$ are expressed as in (4.2), (4.3), (4.4), respectively.

First we decompose the summation as

$$\sum_{i=0}^{N}\sum_{j=0}^{N} < \sigma_{i,j}, p^{(i)}q^{(j)} >= L_N(p,q) + D_N(p,q) + U_N(p,q),$$

where

$$L_N(p,q) = \sum_{i=1}^{N}\sum_{j=0}^{i-1} < \sigma_{i,j}, p^{(i)}q^{(j)} >, \tag{4.6}$$

$$D_N(p,q) = \sum_{i=0}^{N} < \sigma_{i,i}, p^{(i)}q^{(i)} >, \tag{4.7}$$

$$U_N(p,q) = \sum_{j=1}^{N}\sum_{i=0}^{j-1} < \sigma_{i,j}, p^{(i)}q^{(j)} > . \tag{4.8}$$

First, we shall show that $L_N(p,q)$ can be simplified to

$$L_N(p,q) = \sum_{k=1}^{N-1} < \alpha_k^N, p^{(k)}q^{(k)} > + \sum_{k=0}^{N-1} < \tau_k^N, p^{(k+1)}q^{(k)} >, \tag{4.9}$$

where each τ_k^N is given as in (4.3) and each α_k^N is given as

$$\alpha_k^N = \sum_{i=0}^{k-1}\sum_{j=k+i+1}^{N} (-1)^{j-k}\binom{j-k-1}{i}\sigma_{j,k-i-1}^{(j-i-k-1)}.$$

Obviously, in case when $N = 1$, we have

$$L_1(p,q) =< \sigma_{1,0}, p'q >,$$

that is, $\tau_0^1 = \sigma_{1,0}$. Now we consider the case of $N = 2$. Using Leibniz' rule step by step, we have

$$\begin{aligned} < \sigma_{2,0}, p''q > &=< \sigma_{2,0}, (p'q)' - p'q' > \\ &= - < \sigma_{2,0}', p'q > - < \sigma_{2,0}, p'q' > . \end{aligned}$$

In this case, we have

$$\alpha_1^2 = -\sigma_{2,0}, \quad \tau_0^2 = \sigma_{1,0} - \sigma_{2,0}' \quad \text{and} \quad \tau_1^2 = \sigma_{2,1}.$$

We now prove, by induction on $N = 2, 3, 4\ldots$, that the expression $L_N(p,q)$, defined in (4.6) can be written in the form (4.9). Assume that there exists an integer $\ell \geq 2$ such that any bilinear form of order $N \leq \ell$ can be written in the form given in (4.9). Let $L(p,q)$ be a bilinear form of order $N = \ell + 1$;

$$L(p,q) = \sum_{i=1}^{\ell+1}\sum_{j=0}^{i-1} < \sigma_{i,j}, p^{(i)}q^{(j)} > .$$

We split $L(p,q)$ into two parts,

$$L(p,q) = \sum_{i=1}^{\ell}\sum_{j=0}^{i-1} < \sigma_{i,j}, p^{(i)}q^{(j)} > + \sum_{j=0}^{\ell} < \sigma_{\ell+1,j}, p^{(\ell+1)}q^{(j)} >,$$

and apply Leibniz' rule to the second summation on the right-hand side, then we get the expression

$$\begin{aligned} L(p,q) = &\sum_{i=1}^{\ell}\sum_{j=0}^{i-1} < \sigma_{i,j}, p^{(i)}q^{(j)} > + \sum_{j=0}^{\ell-2} < \sigma_{\ell+1,j}, (p^{(\ell)}q^{(j)})' - p^{(\ell)}q^{(j+1)} > - < \sigma_{\ell+1,\ell-1}', p^{(\ell)}q^{(\ell-1)} > \\ &- < \sigma_{\ell+1,\ell-1}, p^{(\ell)}q^{(\ell)} > + < \sigma_{\ell+1,\ell}, p^{(\ell+1)}q^{(\ell)} > . \end{aligned}$$

Now we obtain a bilinear form $\tilde{L}$ of order $\leq \ell$

$$\begin{aligned} \tilde{L}(p,q) = &\sum_{i=1}^{\ell}\sum_{j=0}^{i-1} < \sigma_{i,j}, p^{(i)}q^{(j)} > - \sum_{j=0}^{\ell-2}(< \sigma_{\ell+1,j}', p^{(\ell)}q^{(j)} > + < \sigma_{\ell+1,j}, p^{(\ell)}q^{(j+1)} >) \\ &- < \sigma_{\ell+1,\ell-1}', p^{(\ell)}q^{(\ell-1)} > \\ &= \sum_{i=1}^{\ell}\sum_{j=0}^{i-1} < \check{\sigma}_{i,j}, p^{(i)}q^{(j)} > \end{aligned}$$

where

$$\check{\sigma}_{i,j} = \sigma_{i,j} \quad \text{for} \quad 0 \le i \le \ell - 1, 0 \le j \le i-1,$$
$$\check{\sigma}_{\ell,j} = \sigma_{\ell,j} - \sigma'_{\ell+1,j} - \sigma_{\ell+1,j-1} \quad \text{for} \quad 0 \le j \le \ell - 1 \quad (\sigma_{\ell+1,-1} \equiv 0).$$

Hence, from our assumption, $\tilde{L}(p,q)$ is expressed as

$$\tilde{L}(p,q) = \sum_{k=1}^{\ell-1} < \check{\alpha}_k^\ell, p^{(k)} q^{(k)} + \sum_{k=0}^{\ell-1} < \check{\tau}_k^\ell, p^{(k+1)} q^{(k)} >$$

where

$$\begin{aligned}
\check{\alpha}_k^\ell =& \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{\ell} (-1)^{j-k} \binom{j-k-1}{i} \check{\sigma}_{j,k-i-1}^{(j-i-k-1)} \\
=& \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{\ell} (-1)^{j-k} \binom{j-k-1}{i} \sigma_{j,k-i-1}^{(j-i-k-1)} \\
& - \sum_{i=0}^{k-1} (-1)^{\ell-k} \binom{\ell-k-1}{i} [\sigma_{\ell+1,k-i-1}^{(\ell-i-k)} + \sigma_{\ell+1,k-i-2}^{(\ell-i-k-1)}] \\
=& \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{\ell} (-1)^{j-k} \binom{j-k-1}{i} \sigma_{j,k-i-1}^{(j-i-k-1)} \\
& + \sum_{i=0}^{k-1} (-1)^{\ell+1-k} \binom{\ell-k}{i} \sigma_{\ell+1,k-i-1}^{(\ell-i-k)} \\
=& \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{\ell+1} (-1)^{j-k} \binom{j-k-1}{i} \sigma_{j,k-i-1}^{(j-i-k-1)} \\
=& \alpha_k^{\ell+1}, \quad k = 1, 2, \ldots, \ell - 1.
\end{aligned}$$

and for $k = 0, 1, 2, \ldots, \ell - 1$,

$$\begin{aligned}
\check{\tau}_k^\ell =& \sum_{i=0}^{k} \sum_{j=k+i+1}^{\ell} (-1)^{j-k-1} \binom{j-k-1}{i} \check{\sigma}_{j,k-i}^{(j-i-k-1)} \\
=& \sum_{i=0}^{k} \sum_{j=k+i+1}^{\ell} (-1)^{j-k-1} \binom{j-k-1}{i} \sigma_{j,k-i}^{(j-i-k-1)} \\
& - \sum_{i=0}^{k} (-1)^{\ell-k-1} \binom{\ell-k-1}{i} [\sigma_{\ell+1,k-i}^{(\ell-i-k)} + \sigma_{\ell+1,k-i-1}^{(\ell-i-k-1)}] \\
=& \sum_{i=0}^{k} \sum_{j=k+i+1}^{\ell+1} (-1)^{j-k-1} \binom{j-k-1}{i} \sigma_{j,k-i}^{(j-i-k-1)} \\
=& \ \tau_k^{\ell+1}, \quad k = 0, 1, 2, \ldots, \ell - 1.
\end{aligned}$$

Hence, the bilinear form $L(p,q)$ is written as

$$\begin{aligned} L(p,q) &= \sum_{k=1}^{\ell-1} < \alpha_k^{\ell+1}, p^{(k)}q^{(k)} > + \sum_{k=0}^{\ell-1} < \tau_k^{\ell+1}, p^{(k+1)}q^{(k)} > \\ &\quad - < \sigma_{\ell+1,\ell-1}, p^{(\ell)}q^{(\ell)} > + < \sigma_{\ell+1,\ell}, p^{(\ell+1)}q^{(\ell)} > \\ &= \sum_{k=1}^{\ell} < \alpha_k^{\ell+1}, p^{(k)}q^{(k)} > + \sum_{k=0}^{\ell} < \tau_k^{\ell+1}, p^{(k+1)}q^{(k)} > \end{aligned}$$

where for $k = 0, 1, \dots \ell$,

$$\alpha_k^{\ell+1} = \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{\ell+1} (-1)^{j-k} \binom{j-k-1}{i} \sigma_{j,k-i-1}^{(j-i-k-1)} \quad (\alpha_0^{\ell+1} \equiv 0)$$

and

$$\tau_k^{\ell+1} = \sum_{i=0}^{k} \sum_{j=k+i+1}^{\ell+1} (-1)^{j-k-1} \binom{j-k-1}{i} \sigma_{j,k-i}^{(j-i-k-1)}.$$

Therefore, we have proved the expression (4.9) for $L_N(p,q)$.

Since the subscripts are symmetric, we obtain the expression for $U_N(p,q)$ as in (4.8) :

$$\begin{aligned} U_N(p,q) &= \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} < \sigma_{i,j}, p^{(i)}q^{(j)} > \\ &= \sum_{k=1}^{\ell} < \tilde{\alpha}_k^N, p^{(k)}q^{(k)} > + \sum_{k=0}^{\ell} < \tilde{\tau}_k{}^N, p^{(k)}q^{(k+1)} > \end{aligned}$$

where $\tilde{\tau}_k^N$ are given as (4.4) and for $k = 1, 2, \dots, N-1$,

$$\tilde{\alpha}_k^N = \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{\ell+1} (-1)^{j-k} \binom{j-k-1}{i} \sigma_{k-i-1,j}^{(j-i-k-1)}.$$

Consequently, we have established the identity in (4.1) for the Sobolev bilinear form $\phi_N(p,q)$, which completes the proof. □

Lemma 4.1. *Let $\phi(\cdot,\cdot)$ be the bilinear form given by*

$$\phi(p,q) = \sum_{k=0}^{N} < \sigma_k, p^{(k)}q^{(k+1)} > . \tag{4.10}$$

Then $\phi(\cdot,\cdot)$ is symmetric if and only if $\sigma_k = 0, \quad k = 0, 1, \dots, N$.

Proof. Assume that $\phi(\cdot,\cdot)$ is symmetric. Suppose that there is an integer $\ell \le N$ such that $\sigma_\ell \ne 0$ and $\sigma_k = 0$ for any $k < \ell$. Substituting $p(x) = x^\ell$ into (4.10) yields that for every polynomial q,

$$< \sigma_\ell, q^{(\ell+1)} >= 0.$$

Thus, we have shown that $\sigma_\ell = 0$, which leads to a contradiction. The converse is obvious and this completes the proof. □

We are now in position to prove one of our main results of this paper.

Theorem 4.2. *Let $\phi_N(\cdot,\cdot)$ be the bilinear form given in (1.1). Then $\phi_N(\cdot,\cdot)$ is symmetric if and only if the moment functionals $\sigma_{i,j}$ satisfy the following N "symmetry equations"*

$$\sum_{i=0}^{k}\sum_{j=k+i+1}^{N}(-1)^{j-k-1}\binom{j-k-1}{i}\left(\sigma_{k-i,j}^{(j-i-k-1)}-\sigma_{j,k-i}^{(j-i-k-1)}\right)=0, \tag{4.11}$$

for $k=0,1,2,\ldots,N-1$. Moreover, in this case, $\phi_N(\cdot,\cdot)$ is diagonalizable and can be rewritten as

$$\phi_N(p,q)=\sum_{k=0}^{N}<\mu_k,p^{(k)}q^{(k)}>, \tag{4.12}$$

where $\mu_k:=\sigma_k-\tau_k'$, $k=0,1,\cdots,N-1$ and $\mu_N:=\sigma_N$, and where each σ_k is given in (4.2) and each τ_k is given in (4.3).

Remark 4.1. The proof of Theorem 4.1 involves mathematical induction on N; consequently, it was necessary for the superscripts in each of the moment functionals defined in (4.2) and (4.3). For fixed $N\in\mathbb{N}$, however, this notation is unnecessary and, for this reason, we now drop the superscripts in the moment functionals given in (4.12).

Proof. By Theorem 4.1, we know that $\phi_N(p,q)$ can be written in the form

$$\phi_N(p,q)=\sum_{k=0}^{N}<\sigma_k,p^{(k)}q^{(k)}>+\sum_{k=0}^{N-1}<\tau_k,p^{(k+1)}q^{(k)}>+\sum_{k=0}^{N-1}<\tilde{\tau}_k,p^{(k)}q^{(k+1)}>.$$

For each $k=0,1,2,\ldots,N-1$, we can rewrite

$$<\tau_k,p^{(k+1)}q^{(k)}>=-<\tau_k',p^{(k)}q^{(k)}>-<\tau_k,p^{(k)}q^{(k+1)}>.$$

Thus $\phi_N(p,q)$ is expressed as

$$\phi_N(p,q)=\sum_{k=0}^{N}<\sigma_k,p^{(k)}q^{(k)}>-\sum_{k=0}^{N-1}<\tau_k',p^{(k)}q^{(k)}>+\sum_{k=0}^{N-1}<\tilde{\tau}_k-\tau_k,p^{(k)}q^{(k+1)}>.$$

Lemma 4.1 implies that $\phi_N(p,q)$ is symmetric if and only if

$$\tau_k-\tilde{\tau}_k=0,\quad k=0,1,2,\ldots,N-1,$$

and this completes the proof of this theorem. □

5. Ghost matrices

In this section, we discuss a generalization of one-dimensional ghost functions, a topic that we discussed in Section 2, to $n\times n$ matrices with moment functional entries.

For $N\in\mathbb{N}_0$, let $\phi_N(\cdot,\cdot)$ be as defined in (1.1) and let A_{N+1} be the $(N+1)\times(N+1)$ matrix of moment functionals defined in (1.2). If $\phi_N(\cdot,\cdot)$ is symmetric, we show that $A_{N+1}-A_{N+1}^T$ is, in a sense, the zero matrix; see Theorem 5.1 below. We begin with a general definition of a *ghost matrix.*

Definition 5.1. *Let $m\in\mathbb{N}$. An $m\times m$ matrix $G_m=(g_{i,j})_{i,j=0}^{m-1}$ of moment functionals is a ghost matrix if*

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}<g_{i,j},p^{(i)}q^{(j)}>=0\quad(p,q\in\mathcal{P}).$$

Remark 5.1. With the Sobolev bilinear form $\psi_{m-1}(\cdot,\cdot)$ defined by

$$\psi_{m-1}(p,q)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} < g_{i,j}, p^{(i)}q^{(j)} > \quad (p,q\in\mathcal{P}),$$

it is clear that $\psi_{m-1}(\cdot,\cdot)$ is the zero bilinear form if and only if $G_m=(g_{i,j})_{i,j=0}^{m-1}$, the matrix that generates $\psi_{m-1}(\cdot,\cdot)$, is the ghost matrix.

When $m=1$, G_1 defines the zero moment functional which, as we saw in Section 2, can be represented by non-trivial ghost functions, as given in (2.7) and (2.9).

Example 5.1. Let

$$\begin{aligned}\phi_2(p,q) &=< \sigma_{0,0}, pq > + < \sigma_{0,1}, pq' > + < \sigma_{0,2}, pq'' > + < \sigma_{1,0}, p'q > + < \sigma_{1,1}, p'q' > \\ &\quad + < \sigma_{1,2}, p'q'' > + < \sigma_{2,0}, p''q > + < \sigma_{2,1}, p''q' > + < \sigma_{2,2}, p''q'' > \quad (p,q\in\mathcal{P}),\end{aligned}$$

so $\phi_2(\cdot,\cdot)$ is generated by

$$A_3=\begin{pmatrix}\sigma_{0,0} & \sigma_{0,1} & \sigma_{0,2}\\ \sigma_{1,0} & \sigma_{1,1} & \sigma_{1,2}\\ \sigma_{2,0} & \sigma_{2,1} & \sigma_{2,2}\end{pmatrix}.$$

If $\phi_2(\cdot,\cdot)$ is symmetric, then (see Example 3.2 in Section 3) the moment functionals in A_3 satisfy the two "symmetry" equations

$$\begin{aligned}-\sigma_{1,0}+\sigma_{2,0}'+\sigma_{0,1}-\sigma_{0,2}' &= 0\\ \sigma_{2,1}-\sigma_{1,2} &= 0.\end{aligned}$$

In this case,

$$\begin{aligned}A_3-A_3^T &= \begin{pmatrix}0 & \sigma_{0,1}-\sigma_{1,0} & \sigma_{0,2}-\sigma_{2,0}\\ -\sigma_{0,1}+\sigma_{1,0} & 0 & \sigma_{1,2}-\sigma_{2,1}\\ -\sigma_{0,2}+\sigma_{2,0} & -\sigma_{1,2}+\sigma_{2,1} & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & \sigma_{0,2}'-\sigma_{2,0}' & \sigma_{0,2}-\sigma_{2,0}\\ -\sigma_{0,2}'+\sigma_{2,0}' & 0 & 0\\ -\sigma_{0,2}+\sigma_{2,0} & 0 & 0\end{pmatrix}.\end{aligned}$$

Since σ_{02} and $\sigma_{2,0}$ are arbitrary moment functionals, we write this last matrix as,

$$G_3=\begin{pmatrix}0 & \omega' & \omega\\ -\omega' & 0 & 0\\ -\omega & 0 & 0\end{pmatrix}. \tag{5.1}$$

where ω is an arbitrary moment functional. Then G_3 is a 3×3 ghost matrix. This is a straightforward exercise to verify; it will also follow from the following general result.

Theorem 5.1. *Let $N\in\mathbb{N}_0$. If the Sobolev bilinear form $\phi_N(\cdot,\cdot)$, given in (1.1) and generated by A_{N+1}, is symmetric then*

$$G_{N+1}:=A_{N+1}-A_{N+1}^T \tag{5.2}$$

is an $(N+1)\times(N+1)$ ghost matrix. Moreover, if we write this (skew-symmetric) matrix as $G_{N+1}=(\omega_{i,j})_{i,j=0}^{N,N}$, then the entries $\{\omega_{i,j}\}$ satisfy the following conditions:

(i) $\omega_{i,j}=-\omega_{j,i}$ *for every* $i,j=0,1,\ldots,N$;
(ii) $\omega_{i,i}=0$ *for* $i=0,1,\ldots,N$;

(iii) *with* $N(N-1)/2$ *moment functionals* $\{\omega_{i,j}\}_{i\geq j+2}$ *(which we may regard as arbitrary moment functionals), the moment functionals* $\{\omega_{k+1,k}\}_{k=0}^{N-1}$ *are written as*

$$\omega_{1,0} = \sum_{j=2}^{N} (-1)^j \omega_{j,0}^{(j-1)},$$

$$\omega_{N,N-1} = 0,$$

and, for $k = 1, \ldots, N-2$,

$$\omega_{k+1,k} = \sum_{i=1}^{k} \sum_{j=k+i+1}^{N} (-1)^{j-k} \binom{j-k-1}{i} \omega_{j,k-i}^{(j-i-k-1)} + \sum_{j=k+2}^{N} (-1)^{j-k} \omega_{j,k}^{(j-k-1)}.$$

In particular, G_2 *is the trivial zero matrix.*

Proof. Let $\widetilde{\phi}_N(\cdot,\cdot)$ be the Sobolev bilinear form generated by A_{N+1}^T, the transpose of A_{N+1}. Since $\phi_N(\cdot,\cdot)$ is symmetric, we see that

$$\begin{aligned}\phi_N(p,q) = \phi_N(q,p) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} < \sigma_{i,j}, p^{(j)} q^{(i)} > \\ &= \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} < \sigma_{j,i}, p^{(i)} q^{(j)} > = \widetilde{\phi}_N(p,q) \quad (p, q \in \mathcal{P}).\end{aligned}$$

Hence,

$$\phi_N(p,q) - \widetilde{\phi}_N(p,q) = 0 \quad (p, q \in \mathcal{P})$$

so $\phi_N - \widetilde{\phi}_N$, which is generated by $G_{N+1} = A_{N+1} - A_{N=1}^T$, is the zero bilinear form. By Remark 5.1, G_{N+1} is a ghost matrix. In general, conditions (i) and (ii) follow directly from the relation (5.2). Since the bilinear form $\psi_N(\cdot,\cdot)$ associated with G_{N+1} is symmetric, Theorem 4.2 implies that $\psi_N(\cdot,\cdot)$ can be written as

$$\psi_N(p,q) = \sum_{k=0}^{N} < \sigma_k, p^{(k)} q^{(k)} > - \sum_{k=0}^{N-1} < \tau_k', p^{(k)} q^{(k)} >,$$

where, for $k = 0, 1, \ldots, N$, σ_k and τ_k are given as

$$\begin{aligned}\sigma_k &= \omega_{k,k} + \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{N} (-1)^{j-k} \binom{j-k-1}{i} (\omega_{j,k-i-1}^{(j-i-k-1)} + \omega_{k-i-1,j}^{(j-i-k-1)}), \\ \tau_k &= \sum_{i=0}^{k} \sum_{j=k+i+1}^{N} (-1)^{j-k-1} \binom{j-k-1}{i} \omega_{j,k-i}^{(j-i-k-1)}.\end{aligned}$$

In this case, the conditions (i) and (ii) show that $\sigma_k = 0$ for $k = 0, \ldots, N$. Also, from the fact that a ghost matrix implies a zero bilinear form, we obtain that $\tau_k = 0$ for $k = 0, \ldots, N$, which proves the remaining claims in (iii). In the case $N = 1$, it is easy to verify, from the condition $\sigma_{0,1} = \sigma_{1,0}$ in Example 3.1, that

$$G_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

Example 5.2. If $\phi_3(\cdot,\cdot)$, as defined in (1.1), is symmetric, then the three symmetry equations in this case are

$$\begin{aligned} \sigma_{1,0} - \sigma'_{2,0} + \sigma''_{3,0} - \sigma_{0,1} + \sigma'_{0,2} - \sigma''_{0,3} &= 0 \\ \sigma_{0,3} - \sigma_{3,0} + \sigma_{2,1} - \sigma_{1,2} - \sigma'_{3,1} + \sigma'_{1,3} &= 0 \\ \sigma_{3,2} - \sigma_{2,3} &= 0. \end{aligned}$$

Let

$$\begin{aligned} \omega_{2,0} &= \sigma_{0,2} - \sigma_{2,0} \\ \omega_{3,0} &= \sigma_{0,3} - \sigma_{3,0} \\ \omega_{3,1} &= \sigma_{1,3} - \sigma_{3,1}. \end{aligned}$$

so that

$$\begin{aligned} \sigma_{0,1} - \sigma_{1,0} &= \omega'_{2,0} - \omega''_{3,0} \\ \sigma_{1,2} - \sigma_{2,1} &= \omega_{3,0} + \omega'_{3,1}. \end{aligned}$$

Following a similar analysis as in the preceding examples, we see that a 4×4 ghost matrix is given by

$$G_4 = \begin{pmatrix} 0 & \omega'_{2,0} - \omega''_{3,0} & \omega_{2,0} & \omega_{3,0} \\ -\omega'_{2,0} + \omega''_{3,0} & 0 & \omega_{3,0} + \omega'_{3,1} & \omega_{3,1} \\ -\omega_{2,0} & -\omega_{3,0} - \omega'_{3,1} & 0 & 0 \\ -\omega_{3,0} & -\omega_{3,1} & 0 & 0 \end{pmatrix};$$

observe that, when $\omega_{3,0} = \omega_{3,1} = 0$, the first principal submatrix is G_3, given in (5.1). Also, note that there are three arbitrary moment functionals (namely, $\omega_{2,0}$, $\omega_{3,0}$, and $\omega_{3,1}$) in G_4, in accordance with part (iii) of Theorem 5.1.

Example 5.3. Theorem 5.1 indicates that there are no 2×2 ghost matrices arising from the construction outlined above. We note, however, that there are non-trivial 2×2 ghost matrices. For example, for any moment functional σ, the matrix

$$\begin{pmatrix} \sigma' & \sigma \\ \sigma & 0 \end{pmatrix} \tag{5.3}$$

is a ghost matrix. Indeed, for any $p, q \in \mathcal{P}$,

$$\begin{aligned} (p, p') \begin{pmatrix} \sigma' & \sigma \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} &= < \sigma', pq > + < \sigma, pq' > + < \sigma, p'q > \\ &= < \sigma', pq > + < \sigma, (pq)' > \\ &= < \sigma', pq > - < \sigma', pq > = 0. \end{aligned}$$

The point of this example is that Theorem 5.1 does not characterize ghost matrices.

However, our final result in this paper does characterize ghost matrices.

Theorem 5.2. *For* $N \in \mathbb{N}_0$, *let* $\widetilde{G}_{N+1} = \{\omega_{i,j}\}_{i,j=0}^{N,N}$ *be an* $(N+1) \times (N+1)$ *matrix with moment functional entries. Then* $\widetilde{G}_{N+1}$ *is a ghost matrix if and only if*

$$\omega_{N,N} = 0 \tag{5.4}$$

and, for $k = 0, 1, \ldots, N-1$,

$$\sigma_k = \tau'_k \text{ and } \tau_k = \tilde{\tau}_k, \tag{5.5}$$

where σ_k, τ_k , and $\tilde{\tau}_k$ are moment functionals defined by

$$\sigma_k := \omega_{k,k} + \sum_{i=0}^{k-1} \sum_{j=k+i+1}^{N} (-1)^{j-k} \binom{j-k-1}{i} (\omega_{j,k-i-1}^{(j-i-k-1)} + \omega_{k-i-1,j}^{(j-i-k-1)}) \quad (\sigma_0 = \omega_{0,0}) \tag{5.6}$$

$$\tau_k := \sum_{i=0}^{k} \sum_{j=k+i+1}^{N} (-1)^{j-k-1} \binom{j-k-1}{i} \omega_{j,k-i}^{(j-i-k-1)} \tag{5.7}$$

$$\tilde{\tau}_k := \sum_{i=0}^{k} \sum_{j=k+i+1}^{N} (-1)^{j-k-1} \binom{j-k-1}{i} \omega_{k-i,j}^{(j-i-k-1)}. \tag{5.8}$$

Proof. Let $\widetilde{G}_{N+1} = \{\omega_{i,j}\}_{i,j=0}^{N,N}$ be an $(N+1) \times (N+1)$ matrix and let $\phi_N(\cdot, \cdot)$ be the associated Sobolev bilinear form as defined in (1.1). Then Theorem 4.1 implies that $\phi_N(\cdot, \cdot)$ can be written as

$$\phi_N(p, q) = \sum_{k=0}^{N} < \sigma_k, p^{(k)} q^{(k)} > + \sum_{k=0}^{N-1} < \tau_k, p^{(k+1)} q^{(k)} > + \sum_{k=0}^{N-1} < \tilde{\tau}_k, p^{(k)} q^{(k+1)} > . \tag{5.9}$$

Applying Leibniz' rule to τ_k, we see that $\phi_N(\cdot, \cdot)$ becomes

$$\phi_N(p, q) = \sum_{k=0}^{N-1} < \sigma_k - \tau'_k, p^{(k)} q^{(k)} > + \sum_{k=0}^{N-1} < \tilde{\tau}_k - \tau_k, p^{(k)} q^{(k+1)} > + < \omega_{N,N}, p^{(N)} q^{(N)} > . \tag{5.10}$$

From (5.10) and Lemma 4.1, we see that $\widetilde{G}_{N+1}$ is a ghost matrix, equivalently, $\phi_N(\cdot, \cdot)$ is the zero bilinear form, if and only if $\omega_{N,N} = 0$ and, for $k = 0, 1, \ldots, N-1$, $\sigma_k - \tau'_k = 0$ and $\tau_k = \tilde{\tau}_k$. This completes the proof of the theorem. □

Example 5.4. In the case $N = 1$, we see that conditions (5.4)-(5.8) imply

$$\omega_{0,0} = \omega'_{1,0}, \ \omega_{1,0} = \omega_{0,1}, \ \text{and } \omega_{1,1} = 0,$$

in which case the most general 2×2 ghost matrix is given by

$$\widetilde{G}_2 = \begin{pmatrix} \omega'_{1,0} & \omega_{1,0} \\ \omega_{1,0} & 0 \end{pmatrix};$$

this is the same form of the matrix given in (5.3).

Example 5.5. When $N = 2$, the conditions in Theorem 5.2 yield

$$\begin{array}{ll} \omega_{0,0} = \omega'_{1,0} - \omega''_{2,0} & \omega_{0,1} = \omega_{1,0} - \omega'_{2,0} + \omega'_{0,2} \\ \omega_{1,1} = \omega'_{1,2} + \omega_{2,0} + \omega_{0,2} & \omega_{2,1} = \omega_{1,2} \\ \omega_{2,2} = 0. & \end{array}$$

Consequently, the most general 3×3 ghost matrix is given by

$$\widetilde{G}_3 = \begin{pmatrix} \omega'_{1,0} - \omega''_{2,0} & \omega_{1,0} - \omega'_{2,0} + \omega'_{0,2} & \omega_{0,2} \\ \omega_{1,0} & \omega'_{1,2} + \omega_{2,0} + \omega_{0,2} & \omega_{1,2} \\ \omega_{2,0} & \omega_{1,2} & 0 \end{pmatrix}.$$

Moreover, if $\widetilde{G}_3 = A_3 - A_3^T$ for some matrix A_3 then $\widetilde{G}_3 = -\widetilde{G}_3^T$; this condition implies, from Lemma 2.1(i), that

$$\omega_{1,0} = \omega'_{2,0}, \ \omega_{2,0} = -\omega_{0,2}, \ \omega_{1,2} = 0,$$

in which case $\widetilde{G}_3$ further simplifies to

$$\begin{pmatrix} 0 & \omega'_{0,2} & \omega_{0,2} \\ -\omega'_{0,2} & 0 & 0 \\ -\omega_{0,2} & 0 & 0 \end{pmatrix};$$

this is in agreement with the matrix G_3 given in (5.1) of Example 5.1.

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References

[1] M. Álvarez de Morales, T. E. Pérez, M. A. Piñar, A. Ronveaux, *Orthogonal polynomials associated with a nondiagonal Sobolev inner product with polynomial coefficients,* Orthogonal functions, moment theory, and continued fractions. Theory and Applications (Campinas, 1996), 343-358, Lecture Notes in Pure and Appl. Math., 199, Marcel Dekker, New York, 1998.

[2] G. E. Andrews and L. L. Littlejohn, *A combinatorial interpretation of the Legendre-Stirling numbers,* Proc. Amer. Math. Soc., to appear.

[3] D. Barrios Rolania, G. López-Lagomasino, H. Pijeira Cabrera, *The moment problem for a Sobolev inner product,* J. Approx. Theory, 100(2), 1999, 364-380.

[4] J. Blankenagel, *Anwendungenadjungierter Polynomoperatoren,* Doctoral Dissertation, Universität Köln, 1971.

[5] R. P. Boas, *The Stieltjes moment problem for functions of bounded variation,* Bull. Amer. Math. Soc. 45 (1939), 399-404.

[6] T. S. Chihara, *An introduction to orthogonal polynomials,* Gordon and Breach Publishers, New York, 1978.

[7] A. E. Danese, *Present status and current trends in the theory of orthogonal polynomials and special functions,* Rend. Sem. Math. Univers. Politecn. Torino 35 (1976-1977), 1-20.

[8] A. J. Duran, *The Stieltjes moment problem for rapidly decreasing functions,* Proc. Amer. Math. Soc. 107 (1989), 731-741.

[9] A. J. Duran, *Functions with given moments and weight functions for orthogonal polynomials,* Rocky Mountain J. Math., 23 (1993), 87-104.

[10] W. Gautschi, *Orthogonal polynomials: computation and approximation*, Numerical Mathematics and Scientific Computation Series, Oxford University Press, London, 2004.

[11] S. S. Han, S. S. Kim, and K. H. Kwon, *Orthogonalizing weights for Tchebycheff sets of polynomials,* Bull. London Math. Soc. 24 (1992), 361-367.

[12] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc., 65 (1949), 100-115.

[13] K. H. Kwon and L. L. Littlejohn, *Sobolev orthogonal polynomials and second-order differential equations*, Rocky Mountain J. Math., 28(2) (1998), 547-594.

[14] David C. Lay, *Linear algebra and its applications,* third edition, Addison-Wesley Publishing Co., New York, 2003.

[15] L. L. Littlejohn, *On the classification of differential equations having orthogonal polynomial solutions*, Ann. Mat. Pura Appl., 4 (1984), 35-53.

[16] F. Marcellán, M. Alfaro, and M. L. Rezola, *Orthogonal polynomials on Sobolev spaces: old and new directions,* J. Comput. Appl. Math., 49 (1993), 225-232.

[17] F. Marcellán and F. H. Szafraniec, *The Sobolev-type moment problem,* Proc. Amer. Math. Soc. 128 (2000), 2309-2317.

[18] F. Marcellán and F. H. Szafraniec, *A matrix algorithm towards solving the moment problem of Sobolev type,* Linear Algebra Appl., 331(1-2) (2001), 155-164.

[19] R. Piziak and P. L. Odell, *Matrix theory: from generalized inverses to Jordan form,* Pure and Applied Mathematics Series, Chapman and Hall/CRC, Boca Raton, 2007.

[20] F. W. Schäfke and G. Wolf, *Einfache verallgemeinerte Klassische Orthogonalpolynome,* J. Reine Angew. Math. 262-263 (1973), 339-355.

[21] T. J. Stieltjes, *Recherches sur les fractions continues,* Annales de la Faculte des Sciences de Toulousse (1) 8 (1894) T 1-122, 1 9 A 5-47.

[22] G. Szegö, *Orthogonal polynomials* (fourth edition), American Mathematical Society Colloquium Publications, Vol. 23, Providence, Rhode Island, 1975.

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