

# SOBOLEV ORTHOGONAL POLYNOMIALS IN TWO VARIABLES AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider polynomials in two variables which satisfy an admissible second order partial differential equation of the form

$$(*) \quad Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda u,$$

and are orthogonal relative to a symmetric bilinear form defined by

$$\varphi(p, q) = \langle \sigma, pq \rangle + \langle \tau, p_x q_x \rangle,$$

where  $A, \dots, E$  are polynomials in  $x$  and  $y$ ,  $\lambda$  is an eigenvalue parameter,  $\sigma$  and  $\tau$  are linear functionals on polynomials.

We find a condition for the partial differential equation (\*) to have polynomial solutions which are orthogonal relative to a symmetric bilinear form  $\varphi(\cdot, \cdot)$ . Also examples are provided.

## 1. INTRODUCTION

In 1967, Krall and Sheffer [6] investigated a second order partial differential equation of the form

$$(1.1) \quad A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} + D(x)u_x + E(y)u_y = \lambda u$$

and classified all weak orthogonal polynomials satisfying the partial differential equation (1.1), where  $A(x, y), \dots, E(y)$  are polynomials in  $x$  and  $y$ , and  $\lambda$  is an eigenvalue parameter.

As a generalization, we consider polynomial solutions to the partial differential equation (1.1) which are orthogonal relative to a symmetric bilinear form  $\phi(\cdot, \cdot)$  on polynomials defined by

$$(1.2) \quad \phi(p, q) = \langle \sigma, pq \rangle + \langle \tau, p_x q_x \rangle,$$

where  $\sigma$  and  $\tau$  are moment functionals, and  $p, q$  are polynomials in  $x$  and  $y$ .

The case  $\tau = 0$  was investigated by Krall and Sheffer. For the partial differential equation (1.1) considered by Krall and Sheffer, we know that

- (i)  $C_x = 0$  (up to a linear change of independent variables)  
and
- (ii) partial derivatives with respect to  $x$  satisfy the partial differential equation of the same type as the partial differential equation (1.1) (see [4]).

These facts remind us of the Hahn-Somine characterization theorem for classical orthogonal polynomials ([2, 5, 10]) which states that : The only polynomial sequences  $\{P_n(x)\}_{n=0}^{\infty}$  (up to a complex change of variable) which are simultaneously orthogonal with respect to bilinear forms of the form

$$\begin{aligned} (p, q)_0 &= \int_{\mathbb{R}} p(x)q(x) d\mu_0, \\ (p, q)_1 &= \int_{\mathbb{R}} p(x)q(x) d\mu_0 + \int_{\mathbb{R}} p'(x)q'(x) d\mu_1, \end{aligned}$$

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with  $\mu_i (i = 0, 1)$  real-valued signed Borel measures, are the classical orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel polynomials. Naturally they lead us to the problem of investigating polynomials orthogonal relative to  $\phi(\cdot, \cdot)$  in (1.2). But contrary to the classical orthogonal polynomials in one variable, orthogonal polynomials in two variables whose partial derivatives with respect to  $x$  or  $y$  are orthogonal do not satisfy the partial differential equation of the form (1.1) (See [3] for these materials). Instead, we consider polynomials in two variables which are orthogonal relative to a symmetric bilinear form  $\phi(\cdot, \cdot)$  in (1.2) and satisfy the partial differential equation (1.1).

In this paper, we give some basic facts on Sobolev orthogonal polynomials and the relationship between Sobolev orthogonal polynomials relative to a symmetric bilinear form  $\phi(\cdot, \cdot)$  in (1.2) and the partial differential equation (1.1). Also, we give some examples of the partial differential equation having Sobolev orthogonal polynomials as solutions.

## 2. PRELIMINARIES: BASIC THEORY OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

Let  $\mathcal{P}_n$  be the space of all polynomials in  $x$  and  $y$  of degree  $\leq n$ . The set of all polynomials in two variables is denoted by  $\mathcal{P}$ . By a polynomial system (in short, PS), we mean a sequence  $\{\phi_{mn}(x, y)\}_{m,n=0}^{\infty}$  of polynomials such that  $\deg \phi_{mn} = m + n$  for each  $m, n \geq 0$  and  $\{\phi_{n-j,j}\}_{j=0}^n$  is linearly independent modulo  $\mathcal{P}_{n-1}$ . We denote  $\{\phi_{n-j,j}(x, y)\}_{j=0}^n$  by an  $(n+1)$ -dimensional column vector  $\Phi_n$  and a PS  $\{\phi_{mn}(x, y)\}_{m,n=0}^{\infty}$  by  $\{\Phi_n\}_{n=0}^{\infty}$ .

We say that a PS  $\{\Phi_n\}_{n=0}^{\infty}$  is monic if  $\phi_{mn}(x, y) = x^m y^n$  modulo  $\mathcal{P}_{m+n-1}$  for each  $m, n \geq 0$ . To a given PS  $\{\Phi_n\}_{n=0}^{\infty}$ , there corresponds a unique monic PS  $\{\mathbb{P}_n\}_{n=0}^{\infty}$  which is defined by

$$\mathbb{P}_n = A_n^{-1} \Phi_n,$$

where  $A_n = (a_{jk}^n)_{j,k=0}^n$  and  $\phi_{n-j,j}(x, y) = \sum_{k=0}^n a_{j,k}^n x^{n-k} y^k$  modulo  $\mathcal{P}_{n-1}$ . It will be called the normalization of  $\{\Phi_n\}_{n=0}^{\infty}$ .

A linear functional on  $\mathcal{P}$  is called a moment functional. We denote the action of a moment functional  $\sigma$  on polynomial  $\pi$  by  $\langle \sigma, \pi \rangle$  instead of the customary  $\phi(\pi)$ . Similarly, for a matrix  $Q = (Q_{i,j})$  with  $Q_{i,j}$  being a polynomial,  $\langle \sigma, Q \rangle$  is defined to be the matrix  $(\langle \sigma, Q_{i,j} \rangle)$ . We see that  $\langle \sigma, AB^T \rangle = \langle \sigma, BA^T \rangle^T$  for any column vectors  $A$  and  $B$  of polynomials.

For a moment functional  $\sigma$  and any polynomial  $\phi$ , we define the partial derivatives of  $\sigma$  by the formulas

$$(2.1) \quad \langle \partial_x \sigma, \phi \rangle = -\langle \sigma, \partial_x \phi \rangle, \quad \langle \partial_y \sigma, \phi \rangle = -\langle \sigma, \partial_y \phi \rangle \quad \text{for } \phi \in \mathcal{P},$$

and define the multiplication on  $\sigma$  by a polynomial  $\psi$  through the formula

$$(2.2) \quad \langle \psi \sigma, \phi \rangle = \langle \sigma, \psi \phi \rangle \quad \text{for } \phi \in \mathcal{P}.$$

**Definition 2.1.** A PS  $\{\Phi_n\}_{n=0}^{\infty}$  is called an orthogonal basis (OB) relative to  $\sigma$  if there is a nonzero moment functional  $\sigma$  such that for all  $n \geq 0$

$$\langle \sigma, \phi_{n-k} \pi \rangle = 0, \quad \pi \in \mathcal{P}_{n-1}, 0 \leq k \leq n.$$

And  $\{\Phi_n\}_{n=0}^{\infty}$  is called a weak orthogonal polynomial set (WOPS) relative to  $\sigma$  if there is a nonzero moment functional  $\sigma$  such that

$$\langle \sigma, \phi_{m,n} \phi_{k,l} \rangle = K_{mn} \delta_{mk} \delta_{nl} \quad \text{if } m+n \neq k+l.$$

If  $K_{mn} \neq 0$  (respectively,  $K_{mn} > 0$ ) for each  $m, n \geq 0$ , we say that  $\{\Phi_n\}_{n=0}^{\infty}$  is an orthogonal polynomial set (in short, OPS) (respectively, a positive-definite OPS) relative to  $\sigma$ .

It is obvious that there is an OB relative to  $\sigma$  if and only if there is a WOPS relative to  $\sigma$ .

**Definition 2.2.** A moment functional  $\sigma$  is quasi-definite (respectively, weakly quasi-definite) if there is an OPS (respectively, a WOPS) relative to  $\sigma$ .

From Definition 2.1 and 2.2, we see that a PS  $\{\Phi_n\}_{n=0}^\infty$  is an OPS (respectively, a positive-definite OPS) relative to  $\sigma$  if and only if  $\langle \sigma, \Phi_m \Phi_n^T \rangle = H_n \delta_{mn}$  and  $H_n := \langle \sigma, \Phi_n \Phi_n^T \rangle$  is a nonsingular (respectively, a positive-definite) diagonal matrix.

For any PS  $\{\Phi_n\}_{n=0}^\infty$ , there is a unique moment functional  $\sigma$ , which is called the canonical moment functional of  $\{\Phi_n\}_{n=0}^\infty$ , defined by the conditions

$$\langle \sigma, 1 \rangle = 1, \quad \langle \sigma, \phi_{mn} \rangle = 0, \quad m + n \geq 1.$$

Note that if a PS  $\{\Phi_n\}_{n=0}^\infty$  is an OB relative to  $\sigma$ , then  $\sigma$  is a constant multiple of canonical moment functional of  $\{\Phi_n\}_{n=0}^\infty$ . Although  $\{\Phi_n\}_{n=0}^\infty$  is an OPS relative to  $\sigma$ , its normalization  $\{\mathbb{P}_n\}_{n=0}^\infty$  need not be an OPS relative to  $\sigma$  but  $\{\mathbb{P}_n\}_{n=0}^\infty$  is an OB relative to  $\sigma$ . It is not easy to produce an OPS relative to  $\sigma$ , if any, from an OB relative to  $\sigma$ .

**Theorem 2.1.** ([4, 6]) *For any moment functional  $\sigma$ , the following statements are equivalent.*

- (i)  $\sigma$  is quasi-definite.
- (ii) There is a unique monic OB  $\{\mathbb{P}_n\}_{n=0}^\infty$  relative to  $\sigma$ .
- (iii) There is a monic OB  $\{\mathbb{P}_n\}_{n=0}^\infty$  such that  $H_n := \langle \sigma, \mathbb{P}_n \mathbb{P}_n^T \rangle$  is nonsingular for all  $n \geq 0$ .

**Theorem 2.2** (Favard's Theorem). [11] *Let  $\{\Phi_n\}_{n=0}^\infty$  be a PS. Then the following statements are equivalent.*

- (i)  $\{\Phi_n\}_{n=0}^\infty$  is a WOPS relative to a quasi-definite moment functional  $\sigma$ .
- (ii) For  $n \geq 0$  and  $i = 1, 2$ , there are matrices  $A_{ni}$  of order  $(n+1) \times (n+2)$ ,  $B_{ni}$  of order  $(n+1) \times (n+1)$ , and  $C_{ni}$  of order  $(n+1) \times n$  such that
  - (a)  $x_i \Phi_n = A_{ni} \Phi_{n+1} + B_{ni} \Phi_n + C_{ni} \Phi_{n-1}$  (here  $x_1 = x, x_2 = y$ )
  - (b)  $\text{rank } C_n = n + 1$ , where  $C_n = (C_{n1}, C_{n2})$ .

**Lemma 2.3.** *Let  $\sigma$  be a moment functional and  $\psi$  be a polynomial. Then we have*

- (i)  $\sigma = 0$  if and only if  $\sigma_x = 0$  or  $\sigma_y = 0$
- (ii)  $(\psi\sigma)_x = \psi_x \sigma + \psi \sigma_x$  and  $(\psi\sigma)_y = \psi_y \sigma + \psi \sigma_y$ .

*Proof.* (i): The proof is obvious.

(ii) A computation shows that for any  $p \in \mathcal{P}$ , we have

$$\begin{aligned} \langle (\psi\sigma)_x, p \rangle &= \langle \psi\sigma, -p_x \rangle = \langle \sigma, -\psi p_x \rangle = \langle \sigma, -(\psi p)_x + \psi_x p \rangle \\ &= \langle \sigma_x, \psi p \rangle + \langle \sigma, \psi_x p \rangle = \langle \psi \sigma_x, p \rangle + \langle \psi_x \sigma, \psi p \rangle \\ &= \langle \psi_x \sigma + \psi \sigma_x, p \rangle, \end{aligned}$$

which means  $(\psi\sigma)_x = \psi_x \sigma + \psi \sigma_x$ . By a similar calculation, we have  $(\psi\sigma)_y = \psi_y \sigma + \psi \sigma_y$ .  $\square$

If the partial differential equation (1.1) has a PS  $\{\Phi_n\}_{n=0}^\infty$  as solutions, then it must be of the form

$$\begin{aligned} (2.3) \quad & Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y \\ &= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} + (ay^2 + d_3x + e_3y + f_3)u_{yy} \\ &\quad + (gx + h_1)u_x + (gy + h_2)u_y \\ &= \lambda_n u, \end{aligned}$$

where  $\lambda_n = an(n-1) + dn$ .

We say that the partial differential equation (2.3) is admissible if  $\lambda_m \neq \lambda_n$  for  $m \neq n$ . (2.3) has a unique monic PS as solutions if and only if it is admissible.

**Theorem 2.4.** [4] *Let  $\sigma$  be the canonical moment functional of a PS  $\{\Phi_n\}_{n=0}^\infty$ . If  $\{\Phi_n\}_{n=0}^\infty$  satisfies the partial differential equation (2.3), then  $\sigma$  satisfies the equation*

$$(2.4) \quad L^*[\sigma] = (A\sigma)_{xx} + 2(B\sigma)_{xy} + (C\sigma)_{yy} - (D\sigma)_x - (E\sigma)_y = 0,$$

where  $L^*[u] := (Au)_{xx} + 2(Bu)_{xy} + (Cu)_{yy} - (Du)_x - (Eu)_y$  is the formal Lagrange adjoint operator of  $L[\cdot]$ .

Furthermore, (2.4) has a unique solution up to a multiplication constant if the partial differential equation (2.3) is admissible.

**Theorem 2.5.** ([4, 6]) Let  $\{\Phi_n\}_{n=0}^\infty$  be an OB relative to  $\sigma$ . Then the following statements are all equivalent.

- (i)  $\{\Phi_n\}_{n=0}^\infty$  satisfies the partial differential equation (2.3).
- (ii)  $\sigma$  satisfies the moment equations

$$(2.5) \quad \begin{cases} M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0, \\ M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0. \end{cases}$$

**Remark 2.1.** For any moment functional  $\tau$ ,  $L^*[\tau]$  is written in the following form:

$$L^*[\tau] = (M_1[\tau])_x + (M_2[\tau])_y.$$

This formula will be used in section 4.

**Theorem 2.6.** [4] Let  $\{\Phi_n\}_{n=0}^\infty$  be a PS satisfying the admissible partial differential equation (2.3) and  $\sigma$  the canonical moment functional of  $\{\Phi_n\}_{n=0}^\infty$ . Then the following statements are equivalent.

- (i)  $\{\Phi_n\}_{n=0}^\infty$  is a WOPS relative to  $\sigma$ .
- (ii)  $M_1[\sigma] = 0$ .
- (iii)  $M_2[\sigma] = 0$ .

### 3. THEORY OF SOBOLEV ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

We know that any moment functional  $\sigma$  defines a symmetric bilinear form  $\varphi(\cdot, \cdot)$  on  $\mathcal{P} \times \mathcal{P}$  through the formula

$$\varphi(p, q) = \langle \sigma, pq \rangle.$$

Conversely, a symmetric bilinear form can be generated by a moment functional provided some conditions are fulfilled.

**Theorem 3.1.** Let  $\varphi(\cdot, \cdot)$  be a symmetric bilinear form on  $\mathcal{P} \times \mathcal{P}$ . Then the following statements are equivalent.

- (i) there is a moment functional  $\sigma$  such that  $\varphi(p, q) = \langle \sigma, pq \rangle$  for any  $p, q \in \mathcal{P}$ .
- (ii)  $\varphi(xp, q) = \varphi(p, xq)$  and  $\varphi(yp, q) = \varphi(p, yq)$  for any  $p, q \in \mathcal{P}$ .

*Proof.* ( $\Rightarrow$ ) It is obvious.

( $\Leftarrow$ ) Define a moment functional  $\sigma$  by

$$\langle \sigma, p \rangle = \varphi(p, 1), \quad p \in \mathcal{P}.$$

Then we have for any  $p, q \in \mathcal{P}$

$$\begin{aligned} \varphi(p, q) &= \varphi(p, \sum_{i+j=0}^{\deg q} a_{ij} x^i y^j) = \sum_{i+j=0}^{\deg q} a_{ij} \varphi(p, x^i y^j) = \sum_{i+j=0}^{\deg q} a_{ij} \varphi(x^i y^j p, 1) \\ &= \varphi(p \sum_{i+j=0}^{\deg q} a_{ij} x^i y^j, 1) = \varphi(pq, 1) = \langle \sigma, pq \rangle. \end{aligned}$$

□

For any symmetric bilinear form  $\varphi(\cdot, \cdot)$  on  $\mathcal{P} \times \mathcal{P}$ , we call

$$\varphi_{k,l}^{m,n} := \varphi(x^k y^l, x^m y^n)$$

the  $\binom{m,n}{k,l}$ -th moment of  $\varphi(\cdot, \cdot)$  and the matrix

$$D_n(\varphi) := \begin{pmatrix} \varphi_{0,0}^{0,0} & \varphi_{1,0}^{0,0} & \varphi_{0,1}^{0,0} & \cdots & \varphi_{n,0}^{0,0} & \cdots & \varphi_{0,n}^{0,0} \\ \varphi_{0,0}^{1,0} & \varphi_{1,0}^{1,0} & \varphi_{0,1}^{1,0} & \cdots & \varphi_{n,0}^{1,0} & \cdots & \varphi_{0,n}^{1,0} \\ \varphi_{0,0}^{0,1} & \varphi_{1,0}^{0,1} & \varphi_{0,1}^{0,1} & \cdots & \varphi_{n,0}^{0,1} & \cdots & \varphi_{0,n}^{0,1} \\ \varphi_{0,0}^{0,0} & \varphi_{1,0}^{0,0} & \varphi_{0,1}^{0,0} & \cdots & \varphi_{n,0}^{0,0} & \cdots & \varphi_{0,n}^{0,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \varphi_{0,0}^{n,0} & \varphi_{1,0}^{n,0} & \varphi_{0,1}^{n,0} & \cdots & \varphi_{n,0}^{n,0} & \cdots & \varphi_{0,n}^{n,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{0,0}^{0,n} & \varphi_{1,0}^{0,n} & \varphi_{0,1}^{0,n} & \cdots & \varphi_{n,0}^{0,n} & \cdots & \varphi_{0,n}^{0,n} \end{pmatrix}$$

the  $n$ -th Hankel matrix, and  $\Delta_n(\varphi) := \det D_n(\varphi)$  the  $n$ -th Hankel determinant of  $\varphi(\cdot, \cdot)$ .

We define the value of  $\varphi(\cdot, \cdot)$  on a pair  $(\mathbf{u}, \mathbf{v})$  of column vectors of polynomials. Let  $\mathbf{u} = (u_1, u_2, \dots, u_m)^T$  and  $\mathbf{v}^T = (v_1, v_2, \dots, v_n)$ . We define

$$\varphi(\mathbf{u}, \mathbf{v}^T) = (\varphi(u_i, v_j))_{i=1, j=1}^{m, n}.$$

Then we see that for any matrices  $A$  and  $B$  (when the matrix multiplication can be defined)

$$(3.1) \quad \varphi\left(\mathbf{A}\mathbf{u}, (\mathbf{B}\mathbf{v})^T\right) = \mathbf{A}\varphi(\mathbf{u}, \mathbf{v}^T)\mathbf{B}^T.$$

**Definition 3.1.** Let  $\{\Phi_n\}_{n=0}^\infty$  be a PS.

- (i)  $\{\Phi_n\}_{n=0}^\infty$  is a Sobolev orthogonal basis (SOB) if there is a nonzero symmetric bilinear form  $\varphi(\cdot, \cdot)$  such that for all  $n \geq 0$

$$\varphi(\phi_{n-k,k}, \pi) = 0, \quad 0 \leq k \leq n, \pi \in \mathcal{P}_{n-1}.$$

- (ii)  $\{\Phi_n\}_{n=0}^\infty$  is a weak Sobolev orthogonal polynomial set (WSOPS) if there is a nonzero symmetric bilinear form  $\varphi(\cdot, \cdot)$  such that

$$\varphi(\phi_{m,n}\phi_{k,l}) = K_{m,n}\delta_{m,k}\delta_{n,l} \quad K_{mn} \in \mathbb{R}.$$

If  $K_{m,n} \neq 0$ , then we say that  $\{\Phi_n\}_{n=0}^\infty$  is a Sobolev orthogonal polynomial set (SOPS). In this case, we say that  $\{\Phi_n\}_{n=0}^\infty$  is a WSOPS or SOPS relative to  $\varphi(\cdot, \cdot)$ .

It is obvious that if  $\{\Phi_n\}_{n=0}^\infty$  is a WSOPS relative to  $\varphi(\cdot, \cdot)$ , then  $\varphi(\Phi_n, \Phi_n^T)$  is a diagonal matrix for  $n \geq 0$ .

**Definition 3.2.** A symmetric bilinear form  $\varphi(\cdot, \cdot)$  is quasi-definite (respectively, weakly quasi-definite) if there is a SOPS (respectively, a WSOPS) relative to  $\varphi(\cdot, \cdot)$ .

**Theorem 3.2.** For any symmetric bilinear form  $\varphi(\cdot, \cdot)$ , the following statements are all equivalent.

- (i)  $\varphi(\cdot, \cdot)$  is weakly quasi-definite.
- (ii) there is a SOB relative to  $\varphi(\cdot, \cdot)$ .
- (iii) there is a monic SOB relative to  $\varphi(\cdot, \cdot)$ .

*Proof.* (i)  $\Rightarrow$  (ii) : Let  $\{\Phi_n\}_{n=0}^\infty$  be a WSOPS relative to  $\varphi(\cdot, \cdot)$ . Then  $\{\Phi_n\}_{n=0}^\infty$  itself is a SOB relative to  $\varphi(\cdot, \cdot)$ .

(ii)  $\Rightarrow$  (iii) : Let  $\{\Phi_n\}_{n=0}^\infty$  be a SOB relative to  $\varphi(\cdot, \cdot)$  and  $\{\mathbb{P}_n\}_{n=0}^\infty$  be the normalization of  $\{\Phi_n\}_{n=0}^\infty$ . Then  $\{\mathbb{P}_n\}_{n=0}^\infty$  is a monic SOB relative to  $\varphi(\cdot, \cdot)$ .

(iii)  $\Rightarrow$  (i) : Let  $\{\mathbb{P}_n\}_{n=0}^\infty$  be a monic SOB relative to  $\varphi(\cdot, \cdot)$  and  $\mathbf{H}_n := \varphi(\mathbb{P}_n, \mathbb{P}_n^T)$ . Then  $\mathbf{H}_n$  is a symmetric matrix so that there is a nonsingular matrix  $A_n$  such that  $A_n\mathbf{H}_nA_n^T = D_n$  is diagonal. Then  $\Phi_n := A_n\mathbb{P}_n$  is a WSOPS relative to  $\varphi(\cdot, \cdot)$  since

$$\varphi(\Phi_n, \Phi_n^T) = \varphi(A_n\mathbb{P}_n, (A_n\mathbb{P}_n)^T) = A_n\varphi(\mathbb{P}_n, \mathbb{P}_n^T)A_n^T = D_n.$$

□

**Lemma 3.3.** *For any homogeneous polynomial  $H(x, y) = \sum_{i=0}^n a_i x^{n-i} y^i$ , there exists a unique polynomial  $R_{n-1}(x, y) \in \mathcal{P}_{n-1}$  such that  $\varphi(H + R_{n-1}, \pi) = 0$  for all  $\pi \in \mathcal{P}_{n-1}$  if and only if  $\Delta_{n-1}(\varphi) \neq 0$ .*

*Proof.* Let  $R_{n-1}(x, y) = \sum_{i+j=0}^n r_{ij} x^i y^j$ . Then

$$\begin{aligned} \varphi(H + R_{n-1}, \pi) = 0, \pi \in \mathcal{P}_{n-1} &\iff \varphi(H + R_{n-1}, x^r y^s) = 0, 0 \leq r + s \leq n - 1 \\ &\iff \sum_{i+j=0}^{n-1} r_{ij} \varphi(x^i y^j, x^r y^s) = -\varphi(H, x^r y^s), 0 \leq r + s \leq n - 1. \end{aligned}$$

This is a linear system for the unknowns  $r_{ij}$  whose coefficient matrix is  $D_{n-1}(\varphi)$ .

If  $\Delta_{n-1}(\varphi) \neq 0$ , then  $r_{ij}$  are uniquely determined. Conversely, if this linear system has a unique solution, we must have  $\Delta_{n-1}(\varphi) \neq 0$ .  $\square$

**Theorem 3.4.** *For any symmetric bilinear form  $\varphi(\cdot, \cdot)$ , the following statements are all equivalent.*

- (i)  $\Delta_n(\varphi) \neq 0$  for  $n \geq 0$ .
- (ii)  $\varphi(\cdot, \cdot)$  is quasi-definite.
- (iii) there is a unique monic SOB relative to  $\varphi(\cdot, \cdot)$ .
- (iv) there is a monic SOB  $\{\mathbb{P}_n\}_{n=0}^\infty$  relative to  $\varphi(\cdot, \cdot)$  such that

$$\mathbf{H}_n := \varphi(\mathbb{P}_n, \mathbb{P}_n^T), \quad n \geq 0,$$

is nonsingular.

*Proof.* (i)  $\iff$  (iii) : It is obvious by Lemma 3.3.

(iv)  $\Rightarrow$  (ii) : Since  $\varphi(\mathbb{P}_n, \mathbb{P}_n^T)$  is a symmetric nonsingular matrix, there is a nonsingular symmetric matrix  $A_n$  of order  $(n+1) \times (n+1)$  such that  $A_n \varphi(\mathbb{P}_n, \mathbb{P}_n^T) A_n^T := D_n$  is diagonal. Then

$$A_n \varphi(\mathbb{P}_n, \mathbb{P}_n^T) A_n^T = \varphi(A_n \mathbb{P}_n, \mathbb{P}_n^T A_n^T) = \varphi(A_n \mathbb{P}_n, (A_n \mathbb{P}_n)^T) = D_n.$$

Thus  $\{A_n \mathbb{P}_n\}_{n=0}^\infty$  is a SOPS relative to  $\varphi(\cdot, \cdot)$ . This proves that  $\varphi(\cdot, \cdot)$  is quasi-definite.

(ii)  $\Rightarrow$  (iii) : Let  $\{\mathbb{P}_n\}_{n=0}^\infty$  and  $\{\mathbb{Q}_n\}_{n=0}^\infty$  be monic SOB's relative to  $\varphi(\cdot, \cdot)$ . Let  $R(x, y) = P_{mn}(x, y) - Q_{mn}(x, y)$  for  $m+n \geq 1$  since  $P_{00} = Q_{00}$ . Then  $R(x, y) \in \mathcal{P}_{m+n-1}$  and is orthogonal to  $\mathcal{P}_{m+n-1}$ . Thus  $R(x, y) \equiv 0$  and so  $P_{mn}(x, y) = Q_{mn}(x, y)$  for all  $m+n \geq 0$ .

(iii)  $\Rightarrow$  (iv) : Assume that  $\det \varphi(\mathbb{P}_n, \mathbb{P}_n^T) = 0$  for  $n \geq 0$ . Then there is a nonzero  $(n+1)$ -dimensional row vector  $C$  such that

$$0 = C \varphi(\mathbb{P}_n, \mathbb{P}_n^T) = \varphi(C \mathbb{P}_n, \mathbb{P}_n^T).$$

This implies that

$$P_{n+1,0}, P_{n+1,0} + C \mathbb{P}_n \perp \mathcal{P}_n,$$

which is a contradiction to the assumption that there is a unique monic SOB relative to  $\varphi(\cdot, \cdot)$ .  $\square$

**Theorem 3.5.** *Let  $\{\Phi_n\}_{n=0}^\infty$  be a SOPS relative to  $\varphi(\cdot, \cdot)$ . If there is a polynomial  $\alpha(x, y)$  of degree  $t$  such that*

$$\varphi(\alpha p, q) = \varphi(p, \alpha q) \quad \text{for all } p, q \in \mathcal{P},$$

then  $\{\Phi_n\}_{n=0}^\infty$  satisfies the  $(2t+1)$  term recurrence relation

$$\alpha(x, y) \Phi_n = \sum_{i=n-t}^{n+t} C_{ni} \Phi_i, \quad C_{n, n-t} \neq 0,$$

where  $C_{ni}$  is a constant matrix of order  $(n+1) \times (i+1)$  for  $n-t \leq i \leq n+t$ .

*Proof.* Let  $\alpha(x, y) \Phi_n = \sum_{i=0}^{n+t} C_{ni} \Phi_i$ . Then for  $0 \leq k < n-t$

$$C_{nk} \varphi(\Phi_k, \Phi_k^T) = \varphi \left( \sum_{i=n-t}^{n+t} C_{ni} \Phi_i, \Phi_k^T \right) = \varphi \left( \alpha(x, y) \Phi_n, \Phi_k^T \right) = \varphi \left( \Phi_n, \alpha(x, y) \Phi_k^T \right) = 0$$

since  $\deg \alpha(x, y)\Phi_k^T < n$ . Thus we have  $C_{nk} = 0$  for  $0 \leq k < n - t$ . For  $k = n - t$ , we see that

$$C_{n, n-t} = \varphi(\Phi_n, \Phi_n^T) \tilde{C}_n^T \varphi(\Phi_{n-t}, \Phi_{n-t}^T)^{-1} \neq 0$$

since if we write  $\alpha(x, y)\Phi_{n-t} = \sum_{i=0}^n \tilde{C}_i \Phi_i$  ( $\tilde{C}_n \neq 0$ ), we have

$$\begin{aligned} C_{n, n-t} \varphi(\Phi_{n-t}, \Phi_{n-t}^T) &= \varphi(\alpha(x, y)\Phi_n, \Phi_{n-t}^T) = \varphi(\Phi_n, \alpha(x, y)\Phi_{n-t}^T) \\ &= \varphi\left(\Phi_n, \sum_{i=0}^n \Phi_i^T \tilde{C}_i^T\right) \\ &= \varphi(\Phi_n, \Phi_n^T) \tilde{C}_n^T. \end{aligned}$$

□

#### 4. SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS AND SOBOLEV ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

In this section, we are concerned with polynomials in two variables which satisfy an *admissible* second order partial differential equation

$$\begin{aligned} L[u] : &= Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y \\ (4.1) \quad &= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} + (ay^2 + d_3x + e_3y + f_3)u_{yy} \\ &\quad + (gx + h_1)u_x + (gy + h_2)u_y \\ &= \lambda_n u, \end{aligned}$$

and are orthogonal relative to a symmetric bilinear form  $\varphi(\cdot, \cdot)$  of the form

$$(4.2) \quad \varphi(p, q) = \langle \sigma, pq \rangle + \langle \tau, p_x q_x \rangle,$$

where  $\sigma$  and  $\tau$  are moment functionals.

If  $\{\Phi_n\}_{n=0}^\infty$  is a SOPS relative to  $\varphi(\cdot, \cdot)$  in (4.2), then  $\sigma$  is a constant multiple of the canonical moment functional of  $\{\Phi_n\}_{n=0}^\infty$  since  $\varphi(\phi_{ij}, 1) = \langle \sigma, \phi_{ij} \rangle + \langle \tau, \partial_x \phi_{ij} \partial_x 1 \rangle = \langle \sigma, \phi_{ij} \rangle$  for  $i + j \geq 1$ .

**Theorem 4.1.** *Let  $\varphi(\cdot, \cdot)$  be a symmetric bilinear form in (4.2). The following statements (i) and (ii) are equivalent.*

(i) *The partial differential operator  $L[\cdot]$  in (4.1) is symmetric on polynomials in the sense that*

$$(4.3) \quad \varphi(L[p], q) = \varphi(p, L[q]) \quad \text{for all } p, q \in \mathcal{P}.$$

(ii)  *$\sigma$  and  $\tau$  satisfy the relations*

$$(4.4) \quad \begin{cases} M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0, \\ M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0, \end{cases}$$

$$(4.5) \quad \begin{cases} M_1^{(x)}[\tau] := (A\tau)_x + (B\tau)_y - (D + A_x)\tau = 0, \\ M_2^{(x)}[\tau] := (B\tau)_x + (C\tau)_y - (E + 2B_x)\tau = 0, \end{cases}$$

$$(4.6) \quad C_x \tau = 0.$$

Furthermore, if  $\{\Phi_n\}_{n=0}^\infty$  is a SOPS relative to  $\varphi(\cdot, \cdot)$ , the statements (i) and (ii) are equivalent to

(iii)  $\{\Phi_n\}_{n=0}^\infty$  satisfies the partial differential equation (4.1).

*Proof.* (i)  $\iff$  (ii) : Since we have, by (2.1), (2.2) and Lemma 2.3, for all  $p, q \in \mathcal{P}$

$$\begin{aligned} \varphi(L[p], q) &= \langle L^*[q\sigma] - L^*[q_{xx}\tau] - L^*[q_x\tau_x], p \rangle, \\ \varphi(p, L[q]) &= \langle L[q]\sigma - (L[q])_{xx}\tau - (L[q])_x\tau_x, p \rangle, \end{aligned}$$

we can see that (4.3) is equivalent to

$$L^*[q\sigma] - L[q]\sigma + (L[q])_x\tau_x - L^*[q_x\tau_x] + (L[q])_{xx}\tau - L^*[q_{xx}\tau] = 0 \quad \text{for all } q \in \mathcal{P},$$

which can be written as

$$\begin{aligned}
& qL^*[\sigma] + q_x(2M_1[\sigma] + D_x\tau_x - L^*[\tau_x]) + q_yM_2[\sigma] \\
& + q_{xx}(A_x\tau_x - 2M_1[\tau_x] + (A_{xx} + 2D_x)\tau - L^*[\tau]) + q_{xy}(2B_x\tau_x - 2M_2[\tau_x]) + q_{yy}C_x\tau_x \\
& + q_{xxx}(2A_x\tau - 2M_1[\tau]) + q_{xxy}(4B_x\tau - 2M_2[\tau]) + 2q_{xyy}C_x\tau \\
& = 0.
\end{aligned}$$

Thus we have the following set of equations for  $\sigma$  and  $\tau$  :

$$\begin{aligned}
(4.7) \quad & L^*[\sigma] = 0; \\
(4.8) \quad & 2M_1[\sigma] + D_x\tau_x - L^*[\tau_x] = 0; \\
(4.9) \quad & M_2[\sigma] = 0; \\
(4.10) \quad & A_x\tau_x - 2M_1[\tau_x] + (A_{xx} + 2D_x)\tau - L^*[\tau] = 0; \\
(4.11) \quad & B_x\tau_x - M_2[\tau_x] = 0; \\
(4.12) \quad & C_x\tau_x = 0; \\
(4.13) \quad & A_x\tau - M_1[\tau] = 0; \\
(4.14) \quad & 2B_x\tau - M_2[\tau] = 0; \\
(4.15) \quad & C_x\tau = 0.
\end{aligned}$$

Here, we observe that

$$\begin{aligned}
(4.16) \quad & L^*[\sigma] = (M_1[\sigma])_x + (M_2[\sigma])_y, \\
& M_1[\tau_x] = \left(M_1^{(x)}[\tau]\right)_x + D_x\tau - (B_x\tau)_y, \\
& M_2[\tau_x] = \left(M_2^{(x)}[\tau]\right)_x + B_x\tau_x - C_x\tau_y, \\
& M_1[\tau] = M_1^{(x)}[\tau] + A_x\tau, \\
& M_2[\tau] = M_2^{(x)}[\tau] + 2B_x\tau.
\end{aligned}$$

By using our observations (4.16), Lemma 2.3 and  $C_x = d_3$ , after the tedious calculations, we can write (4.8), (4.10), (4.11) and (4.14) in a simpler form as the followings:

$$\begin{aligned}
& (2M_1[\sigma] + D_x\tau_x - L^*[\tau_x] = 2M_1[\sigma] - \left(M_1^{(x)}[\tau]\right)_{xx} - \left(M_2^{(x)}[\tau]\right)_{xy} + (C_x\tau)_{xy}, \\
& A_x\tau_x - 2M_1[\tau_x] + (A_{xx} + 2D_x)\tau - L^*[\tau] = -3\left(M_1^{(x)}[\tau]\right)_x - \left(M_1^{(x)}[\tau]\right)_y, \\
& B_x\tau_x - M_2[\tau_x] = -\left(M_2^{(x)}[\tau]\right)_x - (C_x\tau)_y, \\
& A_x\tau - M_1[\tau] = -M_1^{(x)}[\tau], \\
& 2B_x\tau - M_2[\tau] = -M_2^{(x)}[\tau].
\end{aligned}$$

Note that all the relations (4.7)-(4.15) are expressed in terms of  $M_i[\sigma]$ ,  $M_i^{(x)}[\tau]$  and  $C_x\tau$ .

If  $M_i[\sigma] = 0$ ,  $M_i^{(x)}[\tau] = 0$  for  $i = 1, 2$  and  $C_x\tau = 0$ , we have (4.7)-(4.15), which implies (4.3). Conversely, if (4.7)-(4.15) hold true, then we can easily see that (4.4), (4.5) and (4.6) hold.

Now assume that  $\{\Phi_n\}_{n=0}^\infty$  is a SOPS relative to  $\varphi(\cdot, \cdot)$  and  $\{\mathbb{P}_n\}_{n=0}^\infty$  be the normalization of  $\{\Phi_n\}_{n=0}^\infty$ .

(i)  $\Rightarrow$  (iii) : Since  $L[\mathbb{P}_n]$  is a vector of polynomials of degree  $\leq n$ , we may write

$$(4.17) \quad L[\mathbb{P}_n] = \sum_{k=0}^n C_{n,k} \mathbb{P}_k$$

for some constant matrices  $C_{n,k}$  of order  $(n+1) \times (k+1)$  for  $0 \leq k \leq n$ . Then for  $0 \leq j < n$ ,

$$C_{n,j}\varphi(\mathbb{P}_j, \mathbb{P}_j^T) = \varphi\left(\sum_{k=0}^n C_{n,k}\mathbb{P}_j, \mathbb{P}_j^T\right) = \varphi(L[\mathbb{P}_n], \mathbb{P}_j^T) = \varphi(\mathbb{P}_n, L[\mathbb{P}_j^T]) = 0.$$

Hence  $C_{n,j} = 0$  for  $0 \leq j < n$  and  $L[\mathbb{P}_n] = \lambda_n \mathbb{P}_n$  by comparing the coefficients in both sides of (4.17).

(iii)  $\Rightarrow$  (i) : Since  $L[\mathbb{P}_n] = \lambda_n \mathbb{P}_n$  for  $n \geq 0$ , we have for  $m \neq n$

$$\varphi(L[\mathbb{P}_n], \mathbb{P}_m^T) - \varphi(\mathbb{P}_n, L[\mathbb{P}_m^T]) = (\lambda_n - \lambda_m)\varphi(\mathbb{P}_n, \mathbb{P}_m^T) = 0.$$

Thus we have (i) by linearity.  $\square$

As we remarked in Introduction, all partial differential equations investigated by Krall and Sheffer satisfy  $C_x = 0$  up to a linear change of variables. And if  $C_x \neq 0$ , we have no new result because we have  $\tau = 0$ . Thus it is natural to assume that  $C_x = 0$ . Then we have the following result

**Theorem 4.2.** *Let  $\{\Phi_n\}_{n=0}^\infty$  a SOPS with respect to a symmetric bilinear form  $\varphi(\cdot, \cdot)$  in (4.2). If  $C_x = 0$ , then the followings are equivalent.*

- (i)  $\{\Phi_n\}_{n=0}^\infty$  satisfies the partial differential equation (4.1).
- (ii)  $\sigma$  and  $\tau$  satisfy moment equations (4.4) and (4.5), respectively.

If a SOPS  $\{\Phi_n\}_{n=0}^\infty$  satisfies the partial differential equation (4.1) with  $C_x = 0$ , we can see that by Theorem 2.5 and Theorem 4.1,  $\{\Phi_n\}_{n=0}^\infty$  is a WOPS  $\{\Phi_n\}_{n=0}^\infty$  relative to  $\sigma$ . Similarly, we know that any PS is a WOPS relative to  $\tau$  if it satisfies the partial differential equation

$$(4.18) \quad Av_{xx} + 2Bv_{xy} + Cv_{yy} + (D + A_x)v_x + (E + 2B_x)v_y = \mu_n u,$$

where  $\mu_n = an(n+1) + gn$ . In fact, the PS consisting of partial derivatives of  $\{\Phi_n\}_{n=0}^\infty$  is a WOPS relative to  $\tau$  since it satisfies the differential equation (4.18) (see Theorem 3.8 in [9]).

**Theorem 4.3.** *If  $L[p] = \lambda p$  and  $L[q] = \mu q$  for  $\lambda \neq \mu$ , then polynomials  $p$  and  $q$  are orthogonal with respect to a symmetric bilinear form (4.2), i. e. ,*

$$\varphi(p, q) = \langle \sigma, pq \rangle + \langle \tau, p_x q_x \rangle = 0$$

for any solutions  $\sigma$  and  $\tau$  of (4.4) and (4.5).

*Proof.* It suffices to observe that

$$(\lambda - \mu)\varphi(p, q) = \varphi(\lambda p, q) - \varphi(p, \mu q) = \varphi(L[p], q) - \varphi(p, L[q]) = 0$$

by (i) in Theorem 4.1.  $\square$

**Theorem 4.4.** *Let  $\{\Phi_n\}_{n=0}^\infty$  be a SOPS relative to  $\varphi(\cdot, \cdot)$  and satisfy the admissible partial differential equation (4.1) with  $C_x = 0$ . Suppose that there is a polynomial  $f(x, y)$  of degree  $\leq 2$  such that*

$$\begin{cases} Af_x + Bf_y - A_x f = 0, \\ Bf_x + Cf_y - 2B_x f = 0. \end{cases}$$

Then

- (i) *If  $\sigma$  is quasi-definite, then  $\{\Phi_n\}_{n=0}^\infty$  is an OB relative to  $\sigma$ . Moreover, there is a polynomial  $f(x, y)$  such that*

$$\tau = kf(x, y)\sigma$$

for some constant  $k$ . If  $\tau \neq 0$ , then  $\{\mathbb{P}_n^{(x)}\}_{n=0}^\infty$  is a monic OB relative to  $\tau$ , where  $\{\mathbb{P}_n^{(x)}\}_{n=0}^\infty$  is a monic PS obtained from the normalization  $\{\mathbb{P}_n\}_{n=0}^\infty$  of  $\{\Phi_n\}_{n=0}^\infty$  through the partial differentiation with respect to  $x$ , defined by

$$(4.19) \quad P_{n-k, k}^{(x)} = \frac{1}{n+1-k} \partial_x P_{n+1-k, k} \quad \text{for } 0 \leq k \leq n.$$

(ii) If  $\tau$  is quasi-definite, then  $\{\Phi_n\}_{n=0}^\infty$  is an OB relative to  $\sigma$ . Moreover, there is a polynomial  $f(x, y)$  such that

$$f(x, y)\sigma = k\tau$$

for some constant  $k$ . Hence  $f(x, y)\sigma = 0$  or  $f(x, y)\sigma$  is quasi-definite.

*Proof.* (i) : Let  $\{\mathbb{Q}_n\}_{n=0}^\infty$  be a monic OB relative to  $\sigma$ . Since  $\sigma$  satisfies (4.4),  $\{\mathbb{Q}_n\}_{n=0}^\infty$  satisfy the partial differential equation (4.1). Then  $\mathbb{Q}_n = \mathbb{P}_n$  for all  $n \geq 0$  by the uniqueness of monic PS solutions to the partial differential equation (4.1) where  $\{\mathbb{P}_n\}_{n=0}^\infty$  is the normalization of  $\{\Phi_n\}_{n=0}^\infty$ . Thus  $\{\Phi_n\}_{n=0}^\infty$  is an OB relative to  $\sigma$ .

On the other hand,  $\tau$  and  $f(x, y)\sigma$  satisfy the same equation (4.5), which are the moment equations corresponding to the partial differential equation (4.18). Since the partial differential equation (4.18) is admissible, the moment equations (4.5) have the unique solution by Theorem 2.4. Thus there is a constant  $k$  such that

$$\tau = kf(x, y)\sigma$$

since  $\{\mathbb{P}_n^{(x)}\}_{n=0}^\infty$  is a unique monic PS satisfying the partial differential equation (4.18) and it is a monic OB relative to  $f(x, y)\sigma$  satisfying the partial differential equation (4.18) if  $\tau \neq 0$ .

(ii) : Since  $\sigma$  is a constant multiple of the canonical moment functional of  $\{\Phi_n\}_{n=0}^\infty$  and satisfies the moment equations (4.4),  $\{\Phi_n\}_{n=0}^\infty$  is an OB relative to  $\sigma$  by Theorem 2.5.

Next, we see that there is a constant  $k$  such that

$$f(x, y)\sigma = k\tau$$

since  $\tau$  and  $f(x, y)\sigma$  satisfy the same moment equation (4.5) corresponding to the admissible partial differential equation (4.18). Hence  $f(x, y)\sigma = 0$  or  $f(x, y)\sigma$  is quasi-definite.  $\square$

**Remark 4.1.** *The assumption in Theorem 4.4 holds for almost OPS's investigated by Krall and Sheffer [6] and Kwon, Lee and Littlejohn [7]. Further, we refer [9] for interesting properties of polynomial solutions satisfying the differential equation (4.1) with  $A_y = 0$  and  $C_x = 0$ .*

## 5. EXAMPLES

In this section, we provide examples of SOPS's which satisfy the partial differential equation (4.1) with  $C_x = 0$  and are orthogonal with respect to a symmetric bilinear form (4.2). All differential equations were dealt by Krall and Sheffer [6].

**Example 5.1.** Consider the differential equation

$$(5.1) \quad xu_{xx} + u_{yy} + (1 + \alpha - x)u_x - yu_y + nu = 0.$$

We know that (5.1) has a PS  $\{\Phi_n\}_{n=0}^\infty$  as solutions, where

$$\phi_{n-k,k}(x, y) = L_{n-k}^{(\alpha)}(x)H_k\left(\frac{1}{\sqrt{2}}y\right),$$

$\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$  are Laguerre polynomials and  $\{H_n(y)\}_{n=0}^\infty$  are Hermite polynomials given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},$$

$$H_n(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!(n-2k)!} \frac{y^{n-2k}}{4^k}.$$

Note that  $\frac{d}{dx}L_n^{(\alpha)}(x) = (-1)L_{n-1}^{(\alpha+1)}(x)$ .

Since  $C_x = 0$ , by Theorem 4.1,  $\sigma$  and  $\tau$  satisfy the equations

$$(5.2) \quad \begin{cases} (x\sigma)_x = (1 + \alpha - x)\sigma, \\ \sigma_y = -y\sigma, \end{cases} \quad \begin{cases} (x\tau)_x = (2 + \alpha - x)\tau, \\ \tau_y = -y\tau. \end{cases}$$

**Case 1.**  $\alpha > -1$  :

By solving (5.2), we have the distributional representations for  $\sigma$  and  $\tau$

$$\begin{cases} \sigma = x^\alpha e^{-x} e^{-\frac{1}{2}y^2} dx dy, \\ \tau = x^{\alpha+1} e^{-x} e^{-\frac{1}{2}y^2} dx dy, \end{cases}$$

and  $\{\Phi_n\}_{n=0}^\infty$  is an OPS relative to  $\sigma$ . Furthermore,  $\{\Phi_n\}_{n=0}^\infty$  is a SOPS relative to the Sobolev inner product

$$\varphi(p, q) = \int_0^\infty \int_{-\infty}^\infty p(x, y) q(x, y) x^\alpha e^{-x} e^{-\frac{1}{2}y^2} dx dy + \int_0^\infty \int_{-\infty}^\infty p_x(x, y) q_x(x, y) x^{\alpha+1} e^{-x} e^{-\frac{1}{2}y^2} dx dy.$$

In fact, we have the orthogonality relation

$$\begin{aligned} \varphi(\phi_{n-k, k} \phi_{m-j, j}) &= \langle \sigma, \phi_{n-k, k} \phi_{m-j, j} \rangle + \langle \tau, \partial_x \phi_{n-k, k} \partial_x \phi_{m-j, j} \rangle \\ &= \left\langle \sigma, L_{n-k}^{(\alpha)}(x) L_{m-j}^{(\alpha)}(x) H_k \left( \frac{y}{\sqrt{2}} \right) H_j \left( \frac{y}{\sqrt{2}} \right) \right\rangle + \left\langle \tau, L_{n-k}^{(\alpha)'}(x) L_{m-j}^{(\alpha)'}(x) H_k \left( \frac{y}{\sqrt{2}} \right) H_j \left( \frac{y}{\sqrt{2}} \right) \right\rangle \\ &= \left\langle H(x) x^\alpha e^{-x}, L_{n-k}^{(\alpha)}(x) L_{m-j}^{(\alpha)}(x) \right\rangle \left\langle e^{-\frac{1}{2}y^2}, H_k \left( \frac{y}{\sqrt{2}} \right) H_j \left( \frac{y}{\sqrt{2}} \right) \right\rangle \\ &\quad + \left\langle H(x) x^{\alpha+1} e^{-x}, L_{n-k}^{(\alpha)'}(x) L_{m-j}^{(\alpha)'}(x) \right\rangle \left\langle e^{-\frac{1}{2}y^2}, H_k \left( \frac{y}{\sqrt{2}} \right) H_j \left( \frac{y}{\sqrt{2}} \right) \right\rangle \\ &= \delta_{mn} \delta_{kj} \left[ \left\| L_{n-k}^{(\alpha)} \right\|^2 + \left\| L_{n-k-1}^{(\alpha+1)} \right\|^2 \right] \left\langle e^{-\frac{1}{2}y^2}, H_k^2 \left( \frac{y}{\sqrt{2}} \right) \right\rangle. \end{aligned}$$

**Case 2.**  $\alpha = -1$  :

Then we have

$$\begin{cases} \sigma = \delta(x) dx \otimes e^{-\frac{1}{2}y^2} dy, \\ \tau = e^{-x} e^{-\frac{1}{2}y^2} dx dy, \end{cases}$$

( $\otimes$  means the tensor product) and we know that  $\{L_{n-k}^{(-1)}(x) H_k(y/\sqrt{2})\}_{n=0, k=0}^\infty$  is a WOPS relative to  $\sigma$ . Moreover,  $\{L_{n-k}^{(-1)}(x) H_k(y/\sqrt{2})\}_{n=0, k=0}^\infty$  is a SOPS relative to the Sobolev inner product

$$\varphi(p, q) = \int_{-\infty}^\infty p(0, y) q(0, y) e^{-\frac{1}{2}y^2} dy + \int_0^\infty \int_{-\infty}^\infty p_x(x, y) q_x(x, y) e^{-x} e^{-\frac{1}{2}y^2} dx dy.$$

We can see that that  $\varphi(\phi_{n-k,k}\phi_{m-j,j}) = K_{n,k}\delta_{mn}\delta_{kj}$  ( $K_{n,k} \neq 0$  for each  $n, k \geq 0$ ) from the following calculation

$$\begin{aligned}
\varphi(\phi_{n-k,k}\phi_{m-j,j}) &= \langle \sigma, \phi_{n-k,k}\phi_{m-j,j} \rangle + \langle \tau, \partial_x \phi_{n-k,k}\partial_x \phi_{m-j,j} \rangle \\
&= \left\langle \sigma, L_{n-k}^{(-1)}(x)L_{m-j}^{(-1)}(x)H_k\left(\frac{y}{\sqrt{2}}\right)H_j\left(\frac{y}{\sqrt{2}}\right) \right\rangle \\
&\quad + \left\langle \tau, L_{n-k}^{(-1)'}(x)L_{m-j}^{(-1)'}(x)H_k\left(\frac{y}{\sqrt{2}}\right)H_j\left(\frac{y}{\sqrt{2}}\right) \right\rangle \\
&= \left\langle \delta(x), L_{n-k}^{(-1)}(x)L_{m-j}^{(-1)}(x) \right\rangle \left\langle e^{-\frac{1}{2}y^2}, H_k\left(\frac{y}{\sqrt{2}}\right)H_j\left(\frac{y}{\sqrt{2}}\right) \right\rangle \\
&\quad + \left\langle H(x)e^{-x}, L_{n-k-1}^{(0)}(x)L_{m-j-1}^{(0)}(x) \right\rangle \left\langle e^{-\frac{1}{2}y^2}, H_k\left(\frac{y}{\sqrt{2}}\right)H_j\left(\frac{y}{\sqrt{2}}\right) \right\rangle \\
&= \delta_{kj} \left[ \delta_{n-k,0}\delta_{m-j,0} + \delta_{mn} \left\langle H(x)e^{-x}, \left(L_{n-k-1}^{(0)}\right)^2 \right\rangle \right] \left\langle e^{-\frac{1}{2}y^2}, H_k^2\left(\frac{y}{\sqrt{2}}\right) \right\rangle \\
&= \begin{cases} 0 & (k \neq j) \\ \left\langle e^{-\frac{1}{2}y^2}, H_0^2\left(\frac{y}{\sqrt{2}}\right) \right\rangle & m = n = 0 \\ \left[ \delta_{n-k,0} + \left\| L_{n-k-1}^{(0)}(x) \right\|^2 \right] \left\langle e^{-\frac{1}{2}y^2}, H_k^2\left(\frac{y}{\sqrt{2}}\right) \right\rangle & (m = n \geq 1, k = j). \end{cases}
\end{aligned}$$

See [8] for the Sobolev orthogonality of  $\{L_n^{(-k)}(x)\}$  with  $k$  a positive integer.

**Example 5.2.** Consider the differential equation

$$(5.3) \quad (x^2 - 1)u_{xx} + 2xyu_{xy} + (y^2 - 1)u_{yy} + gxu_x + gyu_y = n(n + g - 1)u.$$

In [7], we showed that the partial differential equation (5.3) has an OPS  $\{\Phi_n\}_{n=0}^\infty$  as solutions if  $g \neq -1, -2, -3, \dots$ , where

$$(5.4) \quad \phi_{n-k,k}(x, y) = P_{n-k}^{\left(\frac{g+2k-2}{2}, \frac{g+2k-2}{2}\right)}(1-x^2)^{\frac{k}{2}} P_k^{\left(\frac{g-3}{2}, \frac{g-3}{2}\right)}\left(\frac{y}{\sqrt{1-x^2}}\right) \quad (0 \leq k \leq n),$$

and  $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^\infty$  are Jacobi polynomials given by

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k} (x+1)^k.$$

It is known [6] that the partial differential equation (5.3) has a positive-definite OPS as solutions if  $g > 1$ .

In this discussion, we consider the specific case  $g = 1$ . By Theorem 4.1,  $\sigma$  and  $\tau$  satisfy

$$(5.5) \quad \begin{cases} ((x^2 - 1)\sigma)_x + (xy\sigma)_y - x\sigma = 0, \\ (xy\sigma)_x + ((y^2 - 1)\sigma)_y - y\sigma = 0, \end{cases}$$

$$(5.6) \quad \begin{cases} ((x^2 - 1)\tau)_x + (xy\tau)_y - 3x\tau = 0, \\ (xy\tau)_x + ((y^2 - 1)\tau)_y - 3y\tau = 0. \end{cases}$$

Rewriting (5.5) in terms of moments  $\sigma_{m,n} = \langle \sigma, x^m y^n \rangle$ , we have

$$(5.7) \quad \begin{aligned} (m+n+1)\sigma_{m+1,n} + m\sigma_{m-1,n} &= 0, \\ (m+n+1)\sigma_{m,n+1} + n\sigma_{m,n-1} &= 0. \end{aligned}$$

On the other hand, we have, by solving moment equations (5.5) and (5.6),

$$\begin{cases} \sigma = \delta(1 - x^2 - y^2), \\ \tau = H(1 - x^2 - y^2) dx dy, \end{cases}$$

where  $\delta(1 - x^2 - y^2)$  is a distribution on the unit circle  $S^1$  which acts by the formula (see [1])

$$(5.8) \quad \langle \delta(1 - x^2 - y^2), \phi(x, y) \rangle = \frac{1}{2} \int_{-\pi}^{\pi} \phi(\cos \theta, \sin \theta) d\theta.$$

**Lemma 5.1.** *Let  $\sigma = \delta(1 - x^2 - y^2)$  and*

$$\sigma_{mn} = \langle \delta(1 - x^2 - y^2), x^m y^n \rangle = \frac{1}{2} \int_{-\pi}^{\pi} \cos^m \theta \sin^n \theta d\theta.$$

Then

- (i)  $\sigma_{mn}$  satisfies (5.7) for  $m, n \geq 0$ ;
- (ii)  $\sigma$  is not quasi-definite;
- (iii) The circle polynomials satisfying the partial differential equation (5.3) with  $g = 1$  is a WOPS relative to  $\sigma$ .
- (iv)  $(1 - x^2 - y^2)\sigma = 0$ .

*Proof.* (i) : It is not hard to see that

$$\begin{aligned} \sigma_{m+1, n} &= \frac{1}{2} \int_{-\pi}^{\pi} \cos^{m+1} \theta \sin^n \theta d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \cos^m \theta \left( \frac{1}{n+1} \sin^{n+1} \theta \right)' d\theta \\ &= \frac{1}{2} \left[ \frac{1}{n+1} \sin^{n+1} \theta \cos^m \theta \right]_{-\pi}^{\pi} + \frac{1}{2} \frac{m}{n+1} \int_{-\pi}^{\pi} \cos^{m-1} \theta \sin^{n+2} \theta d\theta \\ &= \frac{1}{2} \frac{m}{n+1} \int_{-\pi}^{\pi} \cos^{m-1} \theta \sin^n \theta (1 - \cos^2 \theta) d\theta = \frac{m}{n+1} \sigma_{m-1, n} - \frac{m}{n+1} \sigma_{m+1, n} \end{aligned}$$

and

$$\begin{aligned} \sigma_{m, n+1} &= \frac{1}{2} \int_{-\pi}^{\pi} \cos^m \theta \sin^{n+1} \theta d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \left( -\frac{1}{m+1} \cos^{m+1} \theta \right)' \sin^n \theta d\theta \\ &= \frac{1}{2} \left[ -\frac{1}{m+1} \cos^{m+1} \theta \sin^n \theta \right]_{-\pi}^{\pi} + \frac{1}{2} \frac{n}{m+1} \int_{-\pi}^{\pi} \cos^{m+2} \theta \sin^{n-1} \theta d\theta \\ &= \frac{1}{2} \frac{n}{m+1} \int_{-\pi}^{\pi} \cos^m \theta (1 - \sin^2 \theta) \sin^{n-1} \theta d\theta = \frac{n}{m+1} \sigma_{m, n-1} - \frac{n}{m+1} \sigma_{m, n+1}, \end{aligned}$$

which are (5.7).

(ii) : We compute a few moments of  $\delta(1 - x^2 - y^2)$  :

$$\begin{aligned} \sigma_{00} &= \pi, \\ \sigma_{10} &= 0, & \sigma_{01} &= 0, \\ \sigma_{20} &= \pi/2, & \sigma_{11} &= 0, & \sigma_{02} &= \pi/2, \\ \sigma_{30} &= 0, & \sigma_{21} &= 0, & \sigma_{12} &= 0, & \sigma_{03} &= 0, \\ \sigma_{40} &= 3\pi/8, & \sigma_{31} &= 0, & \sigma_{22} &= \pi/8, & \sigma_{13} &= 0, & \sigma_{04} &= 3\pi/8, \end{aligned}$$

and monic polynomial solutions of degree 2 :

$$\begin{aligned} P_{20} &= x^2 - \frac{1}{2}, \\ P_{11} &= xy, \\ P_{02} &= y^2 - \frac{1}{2}. \end{aligned}$$

Then we see that

$$\langle \sigma, \mathbb{P}_2 \mathbb{P}_2^T \rangle = \begin{pmatrix} \langle \sigma, P_{20} P_{20} \rangle & \langle \sigma, P_{20} P_{11} \rangle & \langle \sigma, P_{20} P_{02} \rangle \\ \langle \sigma, P_{11} P_{20} \rangle & \langle \sigma, P_{11} P_{11} \rangle & \langle \sigma, P_{11} P_{02} \rangle \\ \langle \sigma, P_{02} P_{20} \rangle & \langle \sigma, P_{02} P_{11} \rangle & \langle \sigma, P_{02} P_{02} \rangle \end{pmatrix} = \frac{\pi}{8} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

is not nonsingular. Thus  $\sigma = \delta(1 - x^2 - y^2)$  is not quasi-definite by Theorem 2.1.

(iii) : It follows from Theorem 2.6.

(iv) : See [1] for the proof. □

Next, we show that the partial differential equation (5.3) has a SOPS as solutions. To do this, we show that a symmetric bilinear form  $\varphi(\cdot, \cdot)$  on  $\mathcal{P} \times \mathcal{P}$  define by

$$(5.9) \quad \varphi(p, q) = \frac{1}{2} \int_{-\pi}^{\pi} p(\cos \theta, \sin \theta) q(\cos \theta, \sin \theta) d\theta + \iint_D p_x(x, y) q_x(x, y) dx dy$$

is positive-definite, where  $D = \{(x, y) | x^2 + y^2 \leq 1\}$ .

**Lemma 5.2.** *A symmetric bilinear form  $\varphi(\cdot, \cdot)$  defined by (5.9) is positive-definite on  $\mathcal{P} \times \mathcal{P}$ .*

*Proof.* We will prove that  $\varphi(p, p) = 0$  implies  $p = 0$ . If  $\varphi(p, p) = 0$ , then we have

$$(5.10) \quad p(\cos \theta, \sin \theta) = 0 \quad \text{for } -\pi \leq \theta \leq \pi,$$

$$(5.11) \quad p_x(x, y) = 0 \quad \text{for } x^2 + y^2 \leq 1.$$

By (5.11),  $p$  does not depend on  $x$  and we can write

$$p(x, y) = \sum_{i=0}^{\deg P} a_i y^i.$$

Then we have  $a_i = 0$  for all  $0 \leq i \leq n$  by (5.10). This completes the proof. □

Thus by Theorem 3.4, there exists a SOPS  $\{\Phi_n\}_{n=0}^{\infty}$  relative to a positive-definite symmetric bilinear form  $\varphi(\cdot, \cdot)$  in (5.9). But at this time, we can not find an explicit form of a SOPS  $\{\Phi_n\}_{n=0}^{\infty}$  relative to  $\varphi(\cdot, \cdot)$  in (5.9).

**Remark 5.1.** *Let  $\{\mathbb{P}_n\}_{n=0}^{\infty}$  be the normalization of  $\{\Phi_n\}_{n=0}^{\infty}$  defined in (4.19) with  $g = 1$ . Then the monic PS  $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$  are a monic OB relative to a positive-definite moment functiona (or distribution)  $H(1 - x^2 - y^2) dx dy$  since  $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$  satisfies the differential equation (5.3) with  $g = 3$ .*

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