Math 3326 Quiz #8

FALL SEMESTER 2008

Name SOLUTIONS

1. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \text{ for } x^2 + y^2 < 16$$
$$u(x, y) = x^2y^2 \text{ for } x^2 + y^2 = 16.$$

(Hint: Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$; on the circle $x^2 + y^2 = 16$, we have $x^2y^2 = 256\sin^2\theta\cos^2\theta$)

$f(\theta) = x^2y^2 = 256\sin^2\theta\cos^2\theta$; here $\rho = 4$

Since $2\sin\theta\cos\theta = \sin 2\theta$, we see that $4\sin^2\theta\cos^2\theta = \sin^2(2\theta)$

& $256\sin^2\theta\cos^2\theta = 64\sin^2(2\theta) = 64\left(\frac{1-\cos 4\theta}{2}\right) = 32(1-\cos 4\theta) = f(\theta)$

the solution $u(r,\theta)$ to this BVP is

$$(*)\quad u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta, \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 32(1-\cos 4\theta)\,d\theta = \frac{1}{\pi}(2\pi)32 = 64$$

For $n \geq 1$, $a_n = \frac{1}{4^n\pi}\int_{-\pi}^{\pi} 32(1-\cos 4\theta)\cos n\theta\,d\theta = \frac{64}{4^n\pi}\int_0^{\pi}(\cos n\theta - \cos n\theta\cos 4\theta)\,d\theta$

$= \frac{-1}{4^{n-3}\pi}\int_0^{\pi}\cos n\theta\cos 4\theta\,d\theta$ since $\int_0^{\pi}\cos n\theta\,d\theta = 0$

$$= \begin{cases} 0 & n \neq 4 \\ -\frac{1}{4\pi}\cdot\frac{\pi}{2} & n = 4 \end{cases} = \begin{cases} 0 & n \neq 4 \\ -1/8 & n = 4. \end{cases}$$

Also, since $f(\theta)\sin n\theta$ is odd, for $n = 1, 2, \ldots$, each $b_n = 0$

$\therefore$ from $(*)$, the solution is $u(r,\theta) = 32 - \frac{1}{8}r^4\cos 4\theta$.

Now $r^4\cos 4\theta + i r^4\sin 4\theta = (r\cos\theta + i r\sin\theta)^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 = x^4 + i4x^3y - 6x^2y^2 - 4ixy^3 + y^4$

$\therefore r^4\cos 4\theta = x^4 - 6x^2y^2 + y^4$

Hence $\boxed{u(x,y) = 32 - \frac{1}{8}(x^4 - 6x^2y^2 + y^4)}$