Math 3326 Quiz #7

FALL SEMESTER 2008

Name SOLUTIONS

1. Solve the Dirichlet mixed boundary value problem

$$(*)\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = 0, \quad u_y(x,M) = 0 \quad (0 \le x \le L) \\ u(0,y) = 0, \quad u(L,y) = f(y) \quad (0 \le y \le M). \end{cases}$$

Let $u(x,y) = X(x)Y(y)$; we assume $u(x,y)$ is a non-trivial solution of Laplace's equation & the three BC's that do not involve $f(y)$. Substitute $u(x,y)$ into Laplace's equation and separate the variables as we have done many times previously:

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} := \lambda \quad , \lambda \text{ a constant.}$$

Hence $X'' - \lambda X = 0$ & $Y'' + \lambda Y = 0$ and, from $(*)$, we see that:

$$X(0) = 0 \quad \& \quad Y(0) = Y'(M) = 0.$$

We first solve $Y'' + \lambda Y = 0$

$Y(0) = Y'(M) = 0$

Case 1 $\lambda = 0$

Then $Y(y) = c_1 + c_2 y$ so $Y'(y) = c_2$

Moreover $0 = Y(0) \Rightarrow c_1 = 0$ & $Y'(M) = 0 \Rightarrow c_2 = 0$

This implies that $Y(y) \equiv 0$ so $u(x,y)$ is not non-trivial.

Case 2 $\lambda \neq 0$

Then $Y(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y$ so $Y'(y) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}y + c_2\sqrt{\lambda}\cos\sqrt{\lambda}y$

Hence $0 = Y(0) \Rightarrow c_1 = 0$ & $Y'(M) = 0 \Rightarrow c_2\sqrt{\lambda}\cos\sqrt{\lambda}M = 0.$

Consequently $\sqrt{\lambda}M = \frac{(2n-1)\pi}{2}$ where we can take $n = 1, 2, \ldots$

Therefore, we take $\lambda_n = \frac{(2n-1)^2\pi^2}{4M^2}$ $(n = 1, 2, \ldots)$, and in this case, we see that we can take $Y_n(y) = \sin\left(\frac{(2n-1)\pi}{2M}y\right)$ $(n = 1, 2, \ldots)$

We now solve the BVP $X'' - \lambda X = 0$ where $\lambda_n = \frac{(2n-1)^2\pi^2}{4M^2}$

$X(0) = 0$

In this case, the general solution to $X'' - \lambda_n X = 0$ is

$$X_n(x) = k_1 e^{\frac{(2n-1)\pi x}{2M}} + k_2 e^{-\frac{(2n-1)\pi x}{2M}}$$

The condition $X(0)=0$ forces $k_1+k_2=0$ so $k_2=-k_1$ and

$$X_n(x) = k_1\left[e^{\frac{(2n-1)\pi x}{2M}} - e^{-\frac{(2n-1)\pi x}{2M}}\right]$$

It is convenient to choose $k_1 = 1/2$ so

$$X_n(x) = \sinh\left(\frac{(2n-1)\pi x}{2M}\right).$$

Consequently, a solution to Laplace's equation & the 3 BC's not involving $f(y)$ is:

$$u_n(x,y) := X_n(x)Y_n(y) = \sinh\left(\frac{(2n-1)\pi x}{2M}\right)\sin\left(\frac{(2n-1)\pi y}{2M}\right)$$

or, more generally, for any $N \in \mathbb{N}$

$$U_N(x,y) = \sum_{n=1}^{N} d_n u_n(x,y) = \sum_{n=1}^{\infty} d_n \sinh\left(\frac{(2n-1)\pi x}{2M}\right)\sin\left(\frac{(2n-1)\pi y}{2M}\right).$$

Define $u(x,y) := \sum_{n=1}^{\infty} d_n u_n(x,y) = \sum_{n=1}^{\infty} d_n \sinh\left(\frac{(2n-1)\pi x}{2M}\right)\sin\left(\frac{(2n-1)\pi y}{2M}\right)$

In order for $u(x,y)$ to satisfy the <u>entire</u> Dirichlet mixed BVP, we require that

$$f(y) = u(L,y) = \sum_{n=1}^{\infty} d_n \sinh\left(\frac{(2n-1)\pi L}{2M}\right)\sin\left(\frac{(2n-1)\pi y}{2M}\right)$$

This is a Fourier sine series expansion of $f(y)$ so it follows that

$$d_n \sinh\left(\frac{(2n-1)\pi L}{2M}\right) = \frac{2}{M}\int_0^M f(y)\sin\left(\frac{(2n-1)\pi y}{2M}\right)dy.$$

Therefore, the solution to the Dirichlet mixed BVP is

$$u(x,y) = \sum_{n=1}^{\infty} d_n \sinh\left(\frac{(2n-1)\pi x}{2M}\right)\sin\left(\frac{(2n-1)\pi y}{2M}\right), \text{ where}$$

$$d_n = \frac{2}{M\sin\left(\frac{(2n-1)\pi L}{2M}\right)}\int_0^M f(y)\sin\left(\frac{(2n-1)\pi y}{2M}\right)dy \quad (n=1,2,\dots)$$