Math 3326 Quiz #5

FALL SEMESTER 2008

Name SOLUTIONS

1. Find the Fourier series for $f(x) = x\cos x$ on $[-\pi, \pi]$ and then, using this result, obtain the Fourier series of $g(x) = x\sin x$ on $[-\pi, \pi]$.

Since $f(x)$ is odd, we see that each $a_n = 0$ $(n = 0, 1, \ldots)$

To calculate b_n, we first recall that

$$\cos x \sin nx = \frac{\sin(n+1)x + \sin(n-1)x}{2}$$

& , in particular, $\cos x \sin x = \frac{\sin 2x}{2}$

Hence $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin x = \frac{2}{\pi}\int_0^{\pi} x \frac{\sin 2x}{2}dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$

By parts, we see that $b_1 = \frac{1}{\pi}\left[-\frac{x}{2}\cos 2x\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos 2x\,dx\right]$ $\left\{\begin{array}{ll} u = x & dv = \sin 2x\,dx \\ du = dx & v = -\frac{1}{2}\cos 2x \end{array}\right.$

$$= -\frac{1}{2}$$

For $n = 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\left[\frac{\sin(n+1)x + \sin(n-1)x}{2}\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]dx$$

$u = x$ $du = dx$

$dv = [\sin(n+1)x + \sin(n-1)x]dx$

$v = \frac{-1}{n+1}\cos(n+1)x - \frac{1}{n-1}\cos(n-1)x$

$$= \frac{1}{\pi}\left[\frac{-x}{n+1}\cos(n+1)x - \frac{x}{n-1}\cos(n-1)x\Big|_0^{\pi} + \int_0^{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n+1}\cos(n+1)\pi - \frac{\pi}{n-1}\cos(n-1)\pi\right] = \frac{-1}{(n+1)}(-1)^{n+1} - \frac{1}{(n-1)}(-1)^{n-1} = (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^n\left(\frac{n-1+n+1}{n^2-1}\right)$$

$$= \frac{(-1)^n 2n}{n^2-1}$$

Hence

$$\boxed{\text{FS } x\cos x = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n 2n \sin nx}{n^2-1}}$$

For $g(x) = x\sin x$ on $[-\pi, \pi]$, we note that since $g(x)$ is an even function that $b_n = 0$ $(n=1,2,\dots)$

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$\boxed{\begin{array}{l} u = x \quad du = dx \\ dv = \sin x\,dx \\ v = -\cos x \end{array}}$$

$$= \frac{2}{\pi}\left[-x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx\right]$$

$$= 2$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} x\,\frac{\sin 2x}{2}\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x}{2}\cos 2x\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos 2x\,dx\right]$$

$$\boxed{\begin{array}{l} u = x \quad du = dx \\ dv = \sin 2x\,dx \\ v = -\frac{1}{2}\cos 2x \end{array}}$$

$$= -\frac{1}{2}.$$

To compute a_n $(n=2,3,\dots)$, we first note that

$$\sin x\cos nx = \frac{1}{2}[\sin(n+1)x - \sin(n-1)x]$$

Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\left[\frac{\sin(n+1)x - \sin(n-1)x}{2}\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$\boxed{\begin{array}{l} u = x \quad du = dx \\ dv = [\sin(n+1)x - \sin(n-1)x]\,dx \\ v = \frac{-1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x \end{array}}$$

$$= \frac{1}{\pi}\left[\frac{-x}{n+1}\cos(n+1)x + \frac{x}{n-1}\cos(n-1)x\Big|_0^{\pi} + \int_0^{\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n+1}\cos(n+1)\pi + \frac{\pi}{n-1}\cos(n-1)\pi\right] = \frac{(-1)^n}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{-2}{n^2-1}\right]$$

$$= \frac{2(-1)^{n+1}}{n^2-1}$$

Hence

$$\boxed{FS\ x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2-1}}$$