Math 3326 Quiz #4

FALL SEMESTER 2008

Name SOLUTIONS

1. Verify that the following PDE is elliptical in $\mathcal{D} = \mathbb{R}^2$ and then transform it in canonical form:

$$(*) \qquad u_{xx} + 3u_{yy} - u = 0.$$

Here, $A=1$ $C=3$ so $B^2-AC=-3<0 \Rightarrow$ eqn. is elliptic on $\mathbb{R}^2$.

Since the coefficient of u_{xy} is already 0, we need not consider the first transformation.

We seek a transformation $\xi = \xi(x,y)$
$\eta = \eta(x,y)$

so that $(*)$ transforms to

$$(**) \quad w_{\xi\xi} + w_{\eta\eta} + \psi(\xi, \eta, w_\xi, w_\eta, w) = 0$$

We know we can do this by choosing η & ξ such that

$$(\sqrt{3}\,\eta_y)_y = \left(\frac{-1}{\sqrt{3}}\eta_x\right)_x \qquad (1)$$

&

$$\xi(x,y) = k + \int_{(x_0,y_0)}^{(x,y)} \sqrt{3}\,\eta_y\,dx - \frac{1}{\sqrt{3}}\eta_x\,dy \qquad (2)$$

Chose $\eta(x,y) = x+y$ so $\eta_x = \eta_y = 1$; clearly with this choice, (1) is satisfied. Then

$$\xi(x,y) = k + \int_{(x_0,y_0)}^{(x,y)} \sqrt{3}\,dx - \frac{1}{\sqrt{3}}dy$$

Set $\phi_x = \sqrt{3}$ so $\phi(x,y) = \sqrt{3}x + f(y)$

then $-\frac{1}{\sqrt{3}} = \phi_y = f'(y) \Rightarrow f(y) = -\frac{1}{\sqrt{3}}y \Rightarrow \phi(x,y) = \sqrt{3}x - \frac{1}{\sqrt{3}}y$

$$\therefore \quad \xi(x,y) = k + \phi(x,y) - \phi(x_0,y_0)$$
$$= \phi(x,y) = \sqrt{3}x - \frac{1}{\sqrt{3}}y \text{ if we chose } k = \phi(x_0,y_0).$$

Hence, the transformation we use is

$$\xi(x,y) = \sqrt{3}x - \frac{1}{\sqrt{3}}y \quad \text{so } \xi_x = \sqrt{3},\ \xi_y = \frac{-1}{\sqrt{3}}$$

$$\eta(x,y) = x+y \quad \text{so } \eta_x = \eta_y = 1.$$

Then

$$u_x = w_\xi \xi_x + w_\eta \eta_x = \sqrt{3}\, w_\xi + w_\eta$$

$$u_y = w_\xi \xi_y + w_\eta \eta_y = \frac{-1}{\sqrt{3}} w_\xi + w_\eta$$

$$u_{xx} = (w_\xi)_x \xi_x + (w_\eta)_x \eta_x = (w_{\xi\xi}\xi_x + w_{\eta\xi}\eta_x)\xi_x + (w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x)\eta_x$$

$$= 3w_{\xi\xi} + 2\sqrt{3}\, w_{\eta\xi} + w_{\eta\eta}$$

$$u_{yy} = (w_\xi)_y \xi_y + (w_\eta)_y \eta_y = (w_{\xi\xi}\xi_y + w_{\eta\xi}\eta_y)\xi_y + (w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y)\eta_y$$

$$= \frac{1}{3} w_{\xi\xi} - \frac{2}{\sqrt{3}} w_{\eta\xi} + w_{\eta\eta}$$

$$\therefore\ 0 = u_{xx} + 3u_{yy} - u$$

$$= 3w_{\xi\xi} + 2\sqrt{3}\, w_{\eta\xi} + w_{\eta\eta} + w_{\xi\xi} - \frac{6}{\sqrt{3}} w_{\eta\xi} + 3w_{\eta\eta} - w$$

$$= 4w_{\xi\xi} + 4w_{\eta\eta} - w$$

ie/ the canonical form is

$$\boxed{w_{\xi\xi} + w_{\eta\eta} - \frac{1}{4}w = 0}$$