Math 3326 Quiz #3

FALL SEMESTER 2008

Name SOLUTIONS

1. Consider the PDE

$$u_{xx} + 4u_{xy} + u_{yy} = 0. \qquad (1)$$

(a) Verify that this PDE is hyperbolic in the region $\mathcal{D} = \mathbb{R}^2$.

Here $A=1, B=2, C=1$ so $B^2-AC = 4-1 = 3 > 0$

(b) Solve the two characteristic equations associated with this PDE for the change of variables $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$.

First of all, $\dfrac{B \pm \sqrt{B^2-AC}}{A} = 2 \pm \sqrt{3}$

$\dfrac{dy}{dx} = 2+\sqrt{3}$ & $\dfrac{dy}{dx} = 2-\sqrt{3}$

$\Rightarrow y = (2+\sqrt{3})x + k$ $\quad\Rightarrow y = (2-\sqrt{3})x + K$

so $\boxed{\xi(x,y) = y-(2+\sqrt{3})x}$ $\quad$ so $\boxed{\eta(x,y) = y-(2-\sqrt{3})x}$

$\xi_x = -(2+\sqrt{3})$ $\quad \eta_x = -(2-\sqrt{3})$

$\xi_y = 1$ $\quad \eta_y = 1$

(c) Using the change of variables, solve equation (1) in terms of two arbitrary, twice differentiable functions. Show all of your work (use the back page of this page as well).

$u_x = w_\xi \xi_x + w_\eta \eta_x = -(2+\sqrt{3})w_\xi - (2-\sqrt{3})w_\eta$

$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\xi + w_\eta$

$u_{xy} = (w_\xi)_y \xi_x + (w_\eta)_y \eta_x = (w_{\xi\xi}\xi_y + w_{\eta\xi}\eta_y)\xi_x + (w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y)\eta_x$

$= -(2+\sqrt{3})w_{\xi\xi} - (2+\sqrt{3})w_{\eta\xi} - (2-\sqrt{3})w_{\eta\xi} - (2-\sqrt{3})w_{\eta\eta}$

$= -(2+\sqrt{3})w_{\xi\xi} - 4w_{\eta\xi} - (2-\sqrt{3})w_{\eta\eta}$

$u_{xx} = (w_\xi)_x \xi_x + (w_\eta)_x \eta_x = (w_{\xi\xi}\xi_x + w_{\eta\xi}\eta_x)\xi_x + (w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x)\eta_x$

$= (2+\sqrt{3})^2 w_{\xi\xi} + 2w_{\eta\xi} + (2-\sqrt{3})^2 w_{\eta\eta}$

$u_{yy} = (w_\xi)_y \xi_y + (w_\eta)_y \eta_y = (w_{\xi\xi}\xi_y + w_{\eta\xi}\eta_y)\xi_y + (w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y)\eta_y$

$= w_{\xi\xi} + 2w_{\eta\xi} + w_{\eta\eta}$

$\therefore 0 = u_{xx} + 4u_{xy} + u_{yy} = (2+\sqrt{3})^2 w_{\xi\xi} + 2w_{\eta\xi} + (2-\sqrt{3})^2 w_{\eta\eta} + (-8-4\sqrt{3})w_{\xi\xi}$

$- 16w_{\eta\xi} + (-8-4\sqrt{3})w_{\eta\eta} + w_{\xi\xi} + 2w_{\eta\xi} + w_{\eta\eta}$

(OVER)

$$= (7+4\sqrt{3})w_{\xi\xi} + 2w_{\eta\xi} + (7-4\sqrt{3})w_{\eta\eta} + (-8-4\sqrt{3})w_{\xi\xi}$$
$$-16w_{\eta\xi} + (-8-4\sqrt{3})w_{\eta\eta} + w_{\xi\xi} + 2w_{\eta\xi} + w_{\eta\eta}$$

$$= -12w_{\eta\xi}$$

ie/ $\quad w_{\eta\xi} = 0 \Rightarrow w_\eta = f(\eta)$

$$\Rightarrow w = \int f(\eta)\,d\eta + G(\xi)$$
$$= F(\eta) + G(\xi)$$

Hence a solution of $u_{xx} + 4u_{xy} + u_{yy} = 0$ is

$$u(x,y) = F\left(y-(2-\sqrt{3})x\right) + G\left(y-(2+\sqrt{3})x\right),$$

where F & G are twice differentiable functions.