Math 3326 Quiz #12
Spring Semester 2009

Name SOLUTIONS

1. Solve the Dirichlet problem

$$\nabla^2 u = 0 \text{ for } 0 \le r < \rho, \quad -\pi \le \theta \le \pi$$
$$u(\rho, \theta) = \sin^2\theta \quad (-\pi \le \theta \le \pi).$$

Assume a solution of the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta \quad (0 \le r < \rho, -\pi \le \theta \le \pi)$$

Then

$$\sin^2\theta = u(\rho,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \rho^n \cos n\theta + b_n \rho^n \sin n\theta$$

where $a_n = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi} \sin^2\theta \cos n\theta \, d\theta$ & $b_n = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi} \sin^2\theta \sin n\theta \, d\theta$

Since $\sin^2\theta \sin n\theta$ is an odd function, we see that $b_n = 0$, for $n = 1, 2, \ldots$

Moreover, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2\theta \, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{\pi}(2\pi)\left(\frac{1}{2}\right) = 1$

Also, $a_n = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)\cos n\theta \, d\theta = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi}\frac{\cos n\theta \, d\theta}{2} - \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi}\frac{\cos 2\theta \cos n\theta \, d\theta}{2}$

Since $\int_{-\pi}^{\pi} \cos 2\theta \cos n\theta \, d\theta = \begin{cases} 0 & n \ne 2 \\ \pi & n = 2 \end{cases}$, we see that

$a_n = \begin{cases} 0 & n \ne 2 \\ \frac{-1}{2\rho^2} & n = 2 \end{cases}$. Hence the solution in this case is given by

$$\boxed{u(r,\theta) = \frac{1}{2} - \frac{r^2}{2\rho^2}\cos 2\theta}$$