Math 3326 Quiz #10

SPRING SEMESTER 2009

Name SOLUTIONS

1. Solve the heat equation

$$\begin{aligned} &(1) \quad u_t = k u_{xx} \quad (\text{for } 0 < x < L, t > 0) \\ &(2) \quad u_x(0,t) = u(L,t) = 0 \text{ for } t \geq 0 \\ &(3) \quad u(x,0) = f(x) \quad (0 \leq x \leq L). \end{aligned}$$

Assume that a non-trivial solution of (1) & (2) is of the form $u(x,t) = X(x)T(t)$; then $X(x)T'(t) = kX''(x)T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{kT} = -\lambda$

ie, $X'' + \lambda X = 0$ & $T' + k\lambda T = 0$

From (2), we see that $0 = X'(0)T(t) \Rightarrow X'(0) = 0$ & $0 = X(L)T(t) \Rightarrow X(L) = 0$

Summarizing, (1) & (2) give us $\begin{matrix} X'' + \lambda X = 0 \\ X'(0) = X(L) = 0 \end{matrix}$ (4) & $T' + k\lambda T = 0$ (5)

<u>We first solve (4)</u>: Case 1 $\lambda = 0 \Rightarrow X(x) = c_1 + c_2 x$ so $X'(x) = c_2$

Now $0 = X'(0) = c_2$ and $0 = X(L) = c_1 \Rightarrow X(x) \equiv 0$ so $u(x,t) \equiv 0$

Hence $\lambda = 0$ is not an eigenvalue.

Case 2 $\lambda \neq 0 \Rightarrow X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ so $X'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

Now $0 = X'(0) = c_2\sqrt{\lambda} \Rightarrow c_2 = 0$ & $0 = X(L) = c_1 \cos\sqrt{\lambda}L$

Hence $\sqrt{\lambda}L = (2n-1)\frac{\pi}{2}$ so $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$ and $X_n(x) = \cos\frac{(2n-1)\pi x}{2L}$ $(n = 1, 2, \dots)$

<u>We now solve (5)</u>: With $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$, the solution of $T' + k\lambda_n T = 0$

is $T_n(t) = e^{-\frac{(2n-1)^2\pi^2 kt}{4L^2}}$.

Hence $u_n(x,t) = \cos\frac{(2n-1)\pi x}{2L} e^{-\frac{(2n-1)^2\pi^2 kt}{4L^2}}$ is a solution of (1) and (2)

for every $n \in \mathbb{N}$. Let

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L} e^{-\frac{(2n-1)^2\pi^2 kt}{4L^2}}$$

In order for $u(x,t)$ to be a solution of (3), we require that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L}$$

Notice that this is not a Fourier cosine series expansion of $f(x)$.

However, we find that $c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx \quad (n=1,2,\dots)$

Hence the solution to (1), (2), and (3) is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L}\, e^{\frac{-(2n-1)^2\pi^2 kt}{4L^2}},$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx \quad (n=1,2,\dots)$$