MATH 3326 FINAL EXAMINATION

Spring Semester 2009

Lance L. Littlejohn Name SOLUTIONS

Instructions: Show all work. Partial credit can only be given if sufficient work accompanies each answer. Calculators may be used but exact answers are required. This examination is out of 70 points. GOOD LUCK!

Problem No.	Points
1.	
2.	
3.	
4.	
5.	
6.	
Grade	/70

1. (10 POINTS) Using d'Alembert's formula, solve the Cauchy problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \quad (-\infty < x < \infty,\ t > 0) \\ u(x,0) &= \cos x,\ u_t(x,0) = e^{-x} \quad (-\infty < x < \infty) \end{aligned}$$

with $c = 4$. Simplify your answer as much as possible.

In this case, $\varphi(x) = \cos x$ & $\psi(x) = e^{-x}$

d'Alembert's solution in this case is:

$$u(x,t) = \frac{\varphi(x+4t)+\varphi(x-4t)}{2} + \frac{1}{8}\int_{x-4t}^{x+4t} e^{-u}\,du$$

$$= \frac{\cos(x+4t)+\cos(x-4t)}{2} - \frac{1}{8}e^{-u}\Big|_{x-4t}^{x+4t}$$

$$= \frac{\cos(x+4t)+\cos(x-4t)}{2} - \frac{1}{8}e^{-x-4t} + \frac{1}{8}e^{-x+4t}$$

or

$$= \boxed{\cos x \cos 4t - \frac{1}{8}e^{-x-4t} + \frac{1}{8}e^{-x+4t}}$$

2. Consider the first-order linear PDE

$$u_x + xu_y + 3u = 2. \tag{1}$$

(a) (3 POINTS) Find, and solve, the characteristic equation associated with (1).

$\frac{dy}{dx} = x$ so $y = \frac{1}{2}x^2 + C$ or $\boxed{2y - x^2 = K}$

(b) (4 POINTS) Using $\xi(x, y) = x$, find a transformation $\eta = \eta(x, y)$ with Jacobian $J \neq 0$ that transforms (1) into an equation of the form

$$w_\xi + h(\xi, \eta)w = f(\xi, \eta). \tag{2}$$

Write down this transformed equation.

With $\xi(x,y) = x$ & $\boxed{\eta(x,y) = 2y - x^2}$, the transformed equation is

$$\boxed{w_\xi + 3w = 2}$$

(c) (7 POINTS) Explicitly solve your transformed equation from (2) and, from this, solve (1).

Multiply $w_\xi + 3w = 2$ by $e^{3\xi}$:

$$(e^{3\xi} w)_\xi = 2e^{3\xi}$$

Integrate with respect to ξ:

$$e^{3\xi} w = \frac{2}{3} e^{3\xi} + f(\eta)$$

So that $\boxed{w(\xi, \eta) = \frac{2}{3} + e^{-3\xi} f(\eta)}$

Consequently, the general solution to (1) is given by

$$\boxed{u(x,y) = \frac{2}{3} + e^{-3x} f(2y - x^2)},$$

where f is an arbitrary differentiable real-valued function.

3. Let $f(x) = x^2$ for $x \in [-\pi, \pi]$.

(a) (10 POINTS) Find the Fourier series expansion of $f(x)$ on the interval $[-\pi, \pi]$. Simplify your coefficients as much as possible.

The Fourier series of $f(x) = x^2$ is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx],$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx \quad (n = 0, 1, \ldots)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$; since $x^2 \sin nx$ is odd, $b_n = 0$.

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2}{3}\pi^2$.

For $n \geq 1$, let $u = x^2$, $dv = \cos nx\, dx$, $du = 2x\,dx$, $v = \frac{\sin nx}{n}$ so $a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$

$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n}\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin nx\, dx\right] = -\frac{4}{\pi n}\int_0^{\pi} x \sin nx\, dx$.

Let $u = x$, $dv = \sin nx\, dx$, $du = dx$, $v = -\frac{\cos nx}{n}$ so $a_n = -\frac{4}{\pi n}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx\, dx\right]$

$= \frac{4}{\pi n^2}\pi(-1)^n = \frac{4(-1)^n}{n^2}$.

Hence $\boxed{FS\, f(x) = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}}$.

(b) (6 POINTS) Using Parseval's identity, and part (a), evaluate the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Parseval's identity says $\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 \quad (*)$

Now $\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2}{\pi}\int_0^{\pi} x^4 dx = \frac{2}{\pi}\frac{\pi^5}{5} = \frac{2\pi^4}{5}$.

Hence $\frac{2\pi^4}{5} = \frac{1}{2}\left(\frac{2}{3}\pi^2\right)^2 + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$ so that $\frac{2}{5}\pi^4 - \frac{2}{9}\pi^4 = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$

ie/ $\frac{8\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$ or $\boxed{\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}}$

4. (10 POINTS) Showing your steps using separation of variables, determine an infinite series solution of the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad (0 < x < \pi,\ 0 < y < \pi) \\ u(x,0) &= u(x,\pi) = 0 \quad (0 \le x \le \pi) \\ u(0,y) &= 0;\ u(\pi,y) = f(y) \quad (0 \le y \le \pi) \end{aligned}$$

on the rectangle $R = \{(x,y) \mid 0 \le x \le \pi; 0 \le y \le \pi\}$. Here, $f(y)$ is a continuous function on the y-interval $[0,y]$. Show all your steps in your solution.

We assume a non-trivial solution of the form $u(x,y) = X(x)Y(y)$

The boundary conditions $u(x,0) = u(x,\pi) = u(0,y) = 0$ become

$Y(0) = Y(\pi) = 0$ & $X(0) = 0$

Moreover, $X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$. so $Y'' + \lambda Y = 0$ & $X'' - \lambda X = 0$.

We simultaneously solve the 2 problems $\begin{matrix} Y'' + \lambda Y = 0 \\ Y(0) = Y(\pi) = 0 \end{matrix}$ (1) $\begin{matrix} X'' - \lambda X = 0 \\ X(0) = 0. \end{matrix}$ (2)

The solution to (1) is: $Y_n(y) = \sin ny$ & $\lambda_n = n^2$ $(n = 1, 2, \ldots)$

Then $X'' - n^2 X = 0 \Rightarrow X_n(x) = c_n e^{nx} + d_n e^{-nx}$

With $0 = X_n(0) = c_n + d_n$, we see that $d_n = -c_n$ so $X_n(x) = c_n[e^{nx} - e^{-nx}]$

Let $c_n = 1/2$ so $X_n(x) = \frac{e^{nx} - e^{-nx}}{2} = \sinh nx$.

Hence $u_n(x,y) = \sinh nx \sin ny$ $(n = 1, 2, \ldots)$

Now, let $u(x,y) = \sum_{n=1}^{\infty} k_n \sinh nx \sin ny$

Then, $f(y) = u(\pi, y) = \sum_{n=1}^{\infty} k_n \sinh n\pi \sin ny$ $(0 \le y \le \pi)$

This is the Fourier sine expansion of $f(y)$ on $[0,\pi]$ so

(*) $\boxed{k_n = \frac{2}{\pi \sinh n\pi} \int_0^{\pi} f(y) \sin ny \, dy}$ $(n = 1, 2, \ldots)$

In summary, the solution of this Dirichlet problem is

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} k_n \sinh nx \sin ny},$$

where each k_n is given in (*).

5. (10 POINTS) Using the method of separation of variables, determine an infinite series solution of the 'mixed' heat equation

$$\begin{aligned} u_t &= k u_{xx} \quad (0 < x < L,\ t > 0) \\ u_x(0,t) &= u(L,t) = 0 \quad (t \geq 0) \\ u(x,0) &= f(x) \quad (0 \leq x \leq L); \end{aligned}$$

here $f(x)$ is a continuous function on the x-interval $[0, L]$. Show all the steps in your solution.

We assume a non-trivial solution of the form $u(x,t) = X(x)T(t)$.
The boundary conditions $u_x(0,t) = u(L,t) = 0$ then become

$$X'(0) = X(L) = 0.$$

Substituting $u = XT$ into $u_t = ku_{xx}$ yields $XT' = kX''T$ so

$\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda \Rightarrow$ we need to simultaneously solve $X'' + \lambda X = 0$, $X'(0) = X(L) = 0$ & $T' + \lambda k T = 0$

(1) $\lambda = 0 \Rightarrow X(x) = c_1 + c_2 x$ so $X'(x) = c_2$
Then $X'(0) = 0 \Rightarrow c_2 = 0$ & $X(L) = 0 \Rightarrow c_1 = 0$; ie, $X(x) \equiv 0$ (we don't want that)
(2) $\lambda \neq 0 \Rightarrow X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ so $X'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$
Then $0 = X'(0) = c_2\sqrt{\lambda}$ so $c_2 = 0$; $0 = X(L) = c_1 \cos\sqrt{\lambda}L$ so $\sqrt{\lambda}L = \frac{(2n-1)\pi}{2}$
Hence $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$ & $X_n(x) = \cos\left(\frac{(2n-1)\pi x}{2L}\right)$ $(n = 1, 2, \ldots)$

The general solution of $T' + \lambda_n k T = 0$ is $T_n(t) = e^{-\frac{(2n-1)^2\pi^2 k t}{4L^2}}$.

Hence $u_n(x,t) = \cos\left(\frac{(2n-1)\pi x}{2L}\right) e^{-\frac{(2n-1)^2\pi^2 k t}{4L^2}}$

Let $\boxed{u(x,t) = \sum_{n=1}^{\infty} k_n \cos\left(\frac{(2n-1)\pi x}{2L}\right) e^{-\frac{(2n-1)^2\pi^2 k t}{4L^2}}}$

Then $u(x,t)$ solves the above boundary value problem if

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} k_n \cos\left(\frac{(2n-1)\pi x}{2L}\right)$$

where $\boxed{k_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n-1)\pi x}{2L}\right)dx}$ $(n = 1, 2, \ldots)$

6. Consider the second-order PDE

$$u_{xx} + 6u_{xy} + 5u_{yy} = 0 \tag{3}$$

(a) (5 POINTS) Write down, and then solve, both of the characteristic equations associated with (3) for their implicit solutions

$$\xi(x, y) = k \text{ and } \eta(x, y) = K.$$

Verify that the Jacobian of this transformation is nonzero.

Here, $a = 1$, $b = 3$, $c = 5$ so $\frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{3 \pm \sqrt{9-5}}{1} = 5, 1$

Thus, the 2 characteristic eqns are

$$\boxed{\frac{dy}{dx} = 5 \quad \& \quad \frac{dy}{dx} = 1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$y = 5x + K \qquad y = x + C$$

Let $\boxed{\xi(x,y) = y - 5x \quad \& \quad \eta(x,y) = y - x}$

Then $J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -5 & 1 \\ -1 & 1 \end{vmatrix} = -4 \neq 0.$

(b) (5 POINTS) The canonical form of (3) is $w_{\xi\eta} = 0$, where $\xi(x, y)$ and $\eta(x, y)$ are as in (a). Solve this canonical equation and use this solution to find the general solution of (3).

From $W_{\xi\eta} = 0$, we get $W_\xi = f(\xi) \Rightarrow W = F(\xi) + G(\eta)$

Hence $\boxed{u(x,y) = F(y-5x) + G(y-x)}$ is the general solution to (3).

(c) (5 POINTS - BONUS PROBLEM - FOR EXTRA CREDIT) Solve the Cauchy data problem

$$\begin{aligned} u_{xx} + 6u_{xy} + 5u_{yy} &= 0 \\ u(x,y) &= 8x^2 \text{ on the line } y = 2x \\ u_x(x,y) &= 32x \text{ on the line } y = 2x. \end{aligned}$$

Simplify your answer as much as possible.

From (b), $u(x,y) = F(y-5x) + G(y-x)$

Now, $8x^2 = u(x,2x) = F(-3x) + G(x)$ (1)

Moreover, $u_x(x,y) = -5F'(y-5x) - G'(y-x)$ so that

$$32x = u_x(x,2x) = -5F'(-3x) - G'(x) \quad (2)$$

Differentiate (1):

$$16x = -3F'(-3x) + G'(x) \quad (3)$$

Add (2) and (3): $48x = -8F'(-3x)$ so $F'(-3x) = -6x$

& hence $F'(x) = -6\left(\frac{-x}{3}\right) = 2x \Rightarrow F(x) = x^2 + K$

From (1), $G(x) = 8x^2 - F(-3x) = 8x^2 - [9x^2 + K] = -x^2 - K$

ie/ $F(x) = x^2 + K$ & $G(x) = -x^2 - K$ so that

$$\begin{aligned} u(x,y) &= F(y-5x) + G(y-x) \\ &= (y-5x)^2 + K - (y-x)^2 - K \\ &= y^2 - 10xy + 25x^2 - y^2 + 2xy - x^2 \end{aligned}$$

ie/ $\boxed{u(x,y) = 24x^2 - 8xy}$