MATH 3325 MIDTERM #1

FALL SEMESTER 2009

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Name SOLUTIONS

Instructions: Show all work. Partial credit can only be given if sufficient work accompanies each answer. Calculators may be used but exact answers are required. This examination is out of 50 points. GOOD LUCK!

Problem No.	Points
1.	
2.	
3.	
4.	
5.	
6.	
Grade	/50

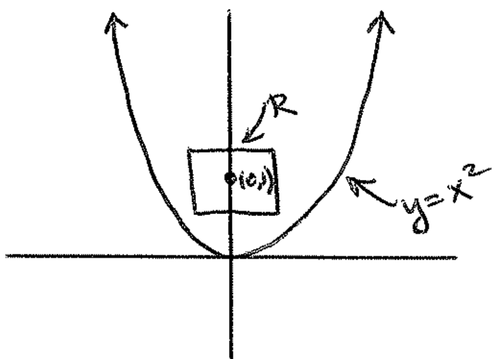
1. (6 Points) With precise, clear statements and a picture with an appropriate rectangle R, prove that the initial value problem

$$\begin{aligned} (x^2 - y)\frac{dy}{dx} &= 3x^2y \\ y(0) &= 1 \end{aligned}$$

has a unique solution. Be sure to correctly identify the function $f(x, y)$ in the E/U Theorem, simplify any partial derivative that is part of that theorem, and identify in your picture where the "problem(s)" of $f(x, y)$ occur and draw an appropriate R that avoids the problem(s).

Here $f(x,y) = \dfrac{3x^2y}{x^2-y}$

so $$\frac{\partial f}{\partial y} = \frac{3x^2(x^2-y)+3x^2y}{(x^2-y)^2} = \frac{3x^4}{(x^2-y)^2}$$



Both f & $\frac{\partial f}{\partial y}$ are continuous in any rectangle, containing $(0,1)$, that avoids the parabola $y=x^2$. For example, see the R in the above picture. Therefore, by the Existence/Uniqueness Theorem, this IVP has a unique solution.

2. Consider the IVP

$$\begin{aligned} x\frac{dy}{dx} &= y + y^2 \\ y(1) &= 1. \end{aligned}$$

(a) (6 Points) Find the explicit solution of this equation. Calculate $y(\frac{3}{2})$.

Rewrite as $\frac{dy}{y(y+1)} = \frac{dx}{x}$; since $\int \frac{dy}{y(y+1)} = \int \frac{dy}{y} - \int \frac{dy}{y+1}$,

we see that $\ln|y| - \ln|y+1| = \ln|x| + c$ so that

$$\frac{y}{y+1} = Ax \Rightarrow y = Axy + Ax \Rightarrow y = \frac{Ax}{1-Ax}.$$

Now $1 = y(1) = \frac{A}{1-A} \Rightarrow A = 1/2$ & $y(x) = \frac{\frac{1}{2}x}{1-\frac{1}{2}x} = \frac{x}{2-x}$

ie/ the explicit solution is $y(x) = \frac{x}{2-x}$

In particular, $y(\frac{3}{2}) = \frac{\frac{3}{2}}{2-\frac{3}{2}} = 3$

(b) (5 Points) Using the Runge-Kutta 4^{th}-order method developed in class with $h = 0.05$, approximate $y(\frac{3}{2})$ showing at least four places of decimal accuracy. Fill in the table below.

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.05$	$y_1 = 1.1053$
$x_2 = 1.1$	$y_2 = 1.2222$
$x_3 = 1.15$	$y_3 = 1.3529$
$x_4 = 1.2$	$y_4 = 1.5$
$x_5 = 1.25$	$y_5 = 1.6667$
$x_6 = 1.3$	$y_6 = 1.8571$
$x_7 = 1.35$	$y_7 = 2.0769$
$x_8 = 1.4$	$y_8 = 2.3333$
$x_9 = 1.45$	$y_9 = 2.6364$
$x_{10} = 1.5$	$y_{10} = 3$ (or 2.999982969)

Hence, from Runge-Kutta, $y(\frac{3}{2}) \approx$ 3.

3. (10 Points) "Apple Juice" Sampson, the famous football player known to his fans as "AJ", is accused of murdering his wife, Micki. Here are the undisputed facts concerning this case:

   (i) Micki's body was discovered at 10:22 pm by police, who responded to a neighbor's phone call. She died instantly from a knife wound. Her internal body temperature at that exact time was measured to be 97.5°F. Three minutes later, her internal body temperature was measured to be 97°F.

   (ii) Micki's normal internal body temperature is 99°F.

   (iii) Micki was murdered in her home, which is kept at a constant temperature of 75°F.

   (iv) AJ admits that he was the only person with Micki between 9 pm and 10 pm on the evening of the murder. However, reliable witnesses say that AJ was with them from 10:05 pm until 11:30 pm on the evening of the murder.

   Using Newton's Law of Cooling, which states that the temperature $T(t)$ of an object satisfies the first order equation

$$\frac{dT}{dt} = k(T - T_0)$$

   where T_0 is the temperature of the surrounding medium,

   (a) Determine the time that Micki was murdered in her home. (Hint: it may be wise to set up a time scale so that $10{:}22 \simeq t = 0$. Once you determine her internal body temperature $T(t)$, find when $T(t) = 99$).

   (b) Based on the above evidence, is AJ guilty of personally murdering Micki?

(a) We first solve $\frac{dT}{dt} = k(T-75) \Rightarrow \int \frac{dT}{T-75} = \int k\,dt$

$\Rightarrow \ln|T-75| = kt + c \Rightarrow T(t) = 75 + Ae^{kt}$.

With 10:22pm $\sim t=0$, we see that $97.5 = T(0) = 75 + A$

So $A = 22.5$ & $T(t) = 75 + 22.5e^{kt}$.

Also, $97 = T(3) = 75 + 22.5e^{3k} \Rightarrow k = \frac{1}{3}\ln\left(\frac{22}{22.5}\right)$

Hence $T(t) = 75 + 22.5e^{\frac{t}{3}\ln\left(\frac{22}{22.5}\right)}$.

Now set $T(t) = 99$ & solve for t:

$$75 + 22.5e^{\frac{t}{3}\ln\left(\frac{22}{22.5}\right)} = 99$$

$$\Rightarrow \frac{t}{3}\ln\left(\frac{22}{22.5}\right) = \ln\left(\frac{24}{22.5}\right) \text{ so } t = \frac{3\ln\left(\frac{24}{22.5}\right)}{\ln\left(\frac{22}{22.5}\right)} \approx -8.62$$

ie, the murder occurred sometime between 10:13 pm and 10:14 pm.

(b) From the evidence, AJ is innocent!

4. (5 Points) Solve the following initial value problem for the explicit solution:

$$\begin{aligned} 2\frac{dy}{dx} + xy &= x \\ y(0) &= -1. \end{aligned}$$

Rewrite this equation as $\frac{dy}{dx} + \frac{x}{2}y = \frac{x}{2}$

Multiply both sides by $e^{\int \frac{x}{2}dx} = e^{x^2/4}$:

$$(e^{x^2/4}y(x))' = \frac{x}{2}e^{x^2/4}$$

Integrate to get $e^{x^2/4}y(x) = e^{x^2/4} + c$

$$\Rightarrow y(x) = 1 + ce^{-x^2/4}$$

Now $-1 = y(0) = 1 + c$ so $c = -2$

$$\therefore y(x) = 1 - 2e^{-x^2/4}$$

5. (5 Points) Solve the initial value problem

$$\begin{aligned} y'' + 4y' + 3y &= 0 \\ y(0) = -1, y'(0) &= 3. \end{aligned}$$

$\alpha^2 + 4\alpha + 3 = (\alpha+3)(\alpha+1) \Rightarrow r = -3$ & $r = -1$ are roots of the characteristic equation. Since e^{-3x} & e^{-x} are linearly independent, the general solution of $y'' + 4y' + 3y = 0$ is

$$y(x) = c_1 e^{-3x} + c_2 e^{-x}$$

$$\Rightarrow y'(x) = -3c_1 e^{-3x} - c_2 e^{-x}$$

Now

$$\begin{aligned} -1 &= y(0) = c_1 + c_2 \\ 3 &= y'(0) = -3c_1 - c_2 \end{aligned}$$

Add: $2 = -2c_1$ so $c_1 = -1 \Rightarrow c_2 = 0$

$\therefore$ the solution is $y(x) = -e^{-3x}$.

6. Find the general solution to each of the following differential equations. Make a clear statement on why your answer is the general solution.

(a) (3 Points) $y''' + 5y'' + 8y' + 4y = 0$.

$\alpha^3+5\alpha^2+8\alpha+4=0 \Rightarrow \alpha=-1$ is a root so $\alpha^3+5\alpha^2+8\alpha+4=(\alpha+1)(\alpha^2+4\alpha+4)$
$= (\alpha+1)(\alpha+2)^2$.

$\therefore$ the general solution is $y(x)=c_1e^{-x}+c_2e^{-2x}+c_3xe^{-2x}$.

(b) (3 Points) $y'' + 3y' + 3y = 0$.

$\alpha^2+3\alpha+3=0 \Rightarrow \alpha=\dfrac{-3\pm\sqrt{9-4(3)}}{2} = \dfrac{-3\pm i\sqrt{3}}{2}$

$\therefore$ general solution is $y(x)= c_1e^{-\frac{3}{2}x}\cos\frac{\sqrt{3}}{2}x+c_2e^{-\frac{3}{2}x}\sin\frac{\sqrt{3}}{2}x$.

(c) (3 Points) $y''' + 5y'' + 4y' = 0$.

$\alpha^3+5\alpha^2+4\alpha = \alpha(\alpha^2+5\alpha+4)=\alpha(\alpha+4)(\alpha+1)$

$\therefore$ the general solution is

$y(x)= c_1+c_2e^{-4x}+c_3e^{-x}$.

(d) (4 Points) $y^{(4)} - 81y = 0$.

$\alpha^4-81 = (\alpha^2-9)(\alpha^2+9)= (\alpha-3)(\alpha+3)(\alpha^2+9)$

$\therefore$ the general solution is

$y(x)= c_1e^{3x}+c_2e^{-3x}+c_3\cos 3x+c_4\sin 3x$.