GREEN'S FUNCTIONS AND SOLUTIONS OF COMPOSITE PRODUCTS OF LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor W. N. Everitt on the occasion of his 85th birthday

Abstract. Given a basis of solutions to k ordinary linear differential equations $\ell_j[y] = 0$ ($j = 1, 2, \ldots k$), we show how the classical Green's function can be used to construct a basis of solutions to the homogeneous differential equation $\ell[y] = 0$, where ℓ is the composite product $\ell = \ell_1\ell_2 \ldots \ell_k$. The construction of these solutions is elementary and classical. In particular, we consider the special case when $\ell = \ell_1^k$. Remarkably, in this case, if $\{y_1, y_2, \ldots, y_n\}$ is a basis of $\ell_1[y] = 0$, then our method produces a basis of $\ell_1^k[y] = 0$ for any $k \in \mathbb{N}$. We illustrate our results with several classical differential equations and their special function solutions.

1. Introduction

For a positive integer k, consider the set of k homogeneous linear differential equations

$$\ell_1[y](x) = 0, \quad \ell_2[y](x) = 0, \ldots, \quad \ell_k[y](x) = 0 \quad (x \in I), \tag{1.1}$$

where, for $j = 1, 2, \ldots k$, $\ell_j[\cdot]$ is the differential expression of order n_j defined by

$$\ell_j[y](x) := y^{(n_j)}(x) + a_{n_j-1,j}(x)y^{(n_j-1)}(x) + \ldots + a_{1,j}(x)y'(x) + a_{0,j}(x)y(x) \quad (x \in I). \tag{1.2}$$

Here, $I := (a, b)$ is an open interval (bounded or unbounded) of the real line $\mathbb{R}$ and, for $i = 0, 1, \ldots, n_j - 1$ and $j = 1, 2, \ldots k$, each of the coefficients $a_{i,j} \in C^{n_1+n_2+\ldots+n_{j-1}}(I)$. Let

$$\ell[y](x) := (\ell_1\ell_2 \cdots \ell_k)[y](x) \quad (x \in I); \tag{1.3}$$

this differential expression $\ell[\cdot]$ of order $n = n_1 + n_2 + \ldots + n_k$ is understood to be the composite product $\ell = \ell_1 \circ \ell_2 \circ \cdots \circ \ell_k$. In this paper, we show how to construct a basis of solutions to the homogeneous differential equation

$$\ell[y](x) = 0 \quad (x \in I), \tag{1.4}$$

when we know a basis $B_j = \{y_{1,j}, y_{2,j}, \ldots, y_{n_j,j}\}$ of each of the factor equations $\ell_j[y] = 0$. Of course, if B_j is a basis of solutions of $\ell_j[y](x) = 0$ for each $j = 1, 2, \ldots k$, it is generally not the case that $B = \cup_{j=1}^k B_j$ is a basis for $\ell[y] = 0$. Indeed, even though the functions in B_k are necessarily solutions of (1.4), in general, elements in $B \backslash B_k$ are not solutions of (1.4) (unless, say, each of the k equations in (1.2) has constant coefficients) since, in general, the composite product of two differential expressions need not be commutative (see the remark at the end of Example 3.1 below for an interesting example of two classical commutative, variable coefficient differential expressions).

The problem of factoring a differential equation into a product of the form (1.3) has a long history and it is well known to be, generally, a difficult problem in the subject of differential Galois

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groups. We refer the reader to the contributions [3], [4], [5], [9], [10], [11], [12], [13] and [14], and the references contained therein, for further information on factoring ordinary differential operators. In this paper, we assume we know a factorization of a homogeneous differential equation $\ell[y] = 0$ on I and that we also know a basis of solutions to each of the homogeneous factor equations $\ell_j[y] = 0$ on I. We then show by a judicial use of the classic Green's function associated with each expression $\ell_j[\cdot]$ that we are able to produce a basis of solutions to $\ell[y] = 0$ on I. In fact, the ubiquitous Green's function comes to our aid in spectacular fashion to render this basis of (1.4). This technique is powerful, yet straightforward; we are unaware of our results in the literature and we feel certain they should be better known and a part of the contents of standard elementary textbooks on differential equations and linear algebra. Typically, there are various 'reduction of order' methods employed to find a basis of solutions to a homogeneous differential equation given a proper subset of linearly independent solutions; see, for example, the discussion in [15, Section 85]. The methods developed in this paper allow us to obtain new (non-trivial) differential equations for special functions; in fact, the examples that we consider in this paper employ a considerable use of special functions in order for us to explicitly find new solutions. These new differential equations may be used to approximate special functions by using the Frobenius method, Olver's techniques, the Liouville-Neumann expansion or any other method based on differential equations.

As an application of our main results, we construct a basis of solutions to the power differential equation

$$m^k[y](x) = 0 \quad (x \in I)$$

of order kn, where

$$m[y](x) := y^{(n)} + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x),$$

and where $m^k[\cdot]$ is defined iteratively by

$$m^1[y] = m[y],\ m^2[y] = m(m[y]),\ m^3[y] = m[m^2[y]],\ \text{etc.}$$

In this special case, we show that knowing only a basis $\{y_1, y_2, \ldots, y_n\}$ of $m[y] = 0$ on I, we can produce a basis of solutions to $m^k[y] = 0$ on I for any positive integer k. In many applications, in particular the spectral analysis of the square of a classical second-order equation $m[y] = 0$ (like Legendre's or Bessel's equation, for example), it is important to know properties of each of the solutions in a basis of $m^2[y] = 0$ in order to prescribe the appropriate boundary conditions to construct a self-adjoint differential operator generated by $m^2[\cdot]$.

The contents of this paper are as follows. In Section 2, we review basic facts of Green's functions; we also state and prove our main results in this section. In Section 3, we illustrate our results by discussing several classical examples. As the reader will see, knowing explicit properties of special functions quickly become an absolute necessity in order to explicitly represent solutions to the product of differential equations.

2. Main Results

Suppose that $\{y_1, y_2, \ldots, y_n\}$ is a basis of solutions to an n^{th}-order linear, homogeneous differential equation of the kind defined in (1.2). Recall the definition of the Green's function $G(x,t)$

associated with $\ell[\cdot]$:

$$G(x,t) := \begin{cases} y_1(x)/y_1(t) & \text{if } n=1 \\ \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ y_1''(t) & y_2''(t) & \cdots & y_n''(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix} / W(y_1, y_2, \ldots, y_n)(t) & \text{if } n \geq 2, \end{cases} \tag{2.1}$$

where $W(y_1, y_2, \ldots, y_n)(t)$ is the Wronskian determinant defined by

$$W(y_1, y_2, \ldots, y_n)(t) := \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ y_1''(t) & y_2''(t) & \cdots & y_n''(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix} \quad (t \in I). \tag{2.2}$$

We recommend the texts in [7] and [15] for further information on the basic theory of differential equations. One of the principle uses of the Green's function is that $y_p(x)$, defined by

$$y_p(x) := \int_{x_0}^{x} G(x,t) f(t) dt, \tag{2.3}$$

where x_0 is an arbitrary point in I, provides a particular solution to the non-homogeneous differential equation

$$\ell[y](x) = f(x) \quad (x \in I);$$

that is to say

$$\ell[y_p](x) = f(x) \quad (x \in I) \tag{2.4}$$

or, using different notation,

$$y_p(x) = \ell^{-1} f(x) = \int_{x_0}^{x} G(x,t) f(t) dt. \tag{2.5}$$

The construction of y_p, in (2.3), follows from the well-known *method of variation of parameters.*

We prove the following lemma which will be a special case of our main result below.

Lemma 2.1. *Suppose* $\{z_1, z_2, \ldots, z_{n_1}\}$ *is a basis of solutions to*

$$\ell_1[y](x) = 0 \quad (x \in I)$$

and $\{y_1, y_2, \ldots, y_{n_2}\}$ *is a basis of solutions of*

$$\ell_2[y](x) = 0 \quad (x \in I),$$

where $\ell_j[\cdot]$ $(j = 1, 2)$ *is the differential expression defined in (1.2) of order* n_j. *Let*

$$\ell[y] := \ell_1(\ell_2[y])$$

be the composite product differential expression of order $n_1 + n_2$ *and let* $G_2(x,t)$ *denote the Green's function associated with* $\ell_2[\cdot]$. *For any fixed* $x_0 \in I$, *define*

$$y_{n_2+j}(x) := \int_{x_0}^{x} G_2(x,t) z_j(t) dt \quad (j = 1, 2, \ldots, n_1).$$

Then $\{y_1, y_2, \ldots, y_{n_2}, y_{n_2+1}, y_{n_2+2}, \ldots, y_{n_1+n_2}\}$ *is a basis of solutions to the product differential equation*

$$\ell[y](x) = 0 \quad (x \in I).$$

Proof. Since $\ell_2[y_j] = 0$ for $j = 1, 2, \ldots, n_2$, it is clear that

$$\ell[y_j](x) = \ell_1(\ell_2[y_j])[x] = 0 \quad (j = 1, 2, \ldots, n_2;\ x \in I); \tag{2.6}$$

that is to say, $\{y_1, y_2, \ldots, y_{n_2}\}$ are solutions of $\ell[y] = 0$ on I. Furthermore, from (2.3) and (2.4), we see that

$$\ell_2[y_{n_2+j}](x) = z_j(x) \quad (j = 1, 2, \ldots, n_1;\ x \in I) \tag{2.7}$$

so that

$$\ell[y_{n_2+j}](x) = \ell_1(\ell_2[y_{n_2+j}](x)) = \ell_1[z_j](x) = 0 \quad (j = 1, 2, \ldots, n_1;\ x \in I) \tag{2.8}$$

That is to say, $\{y_1, y_2, \ldots, y_{n_2}, y_{n_2+1}, y_{n_2+2}, \ldots, y_{n_1+n_2}\}$ are solutions of $\ell[y] = \ell_1\ell_2[y] = 0$ on I. To see that these solutions are linearly independent, set

$$\begin{aligned}\alpha_1 y_1(x) + \alpha_2 y_2(x) + \ldots + \alpha_{n_2} y_{n_2}(x) \\ + \alpha_{n_2+1} y_{n_2+1}(x) + \alpha_{n_2+2} y_{n_2+2}(x) + \ldots + \alpha_{n_1+n_2} y_{n_1+n_2}(x) = 0 \quad (x \in I).\end{aligned} \tag{2.9}$$

Apply ℓ_2 to both sides of (2.9); from (2.7) and since $\ell_2[y_j] = 0$ for $j = 1, 2, \ldots, n_2$, we obtain

$$\alpha_{n_2+1} z_1(x) + \alpha_{n_2+1} z_2(x) \ldots + \alpha_{n_1+n_2} z_{n_1}(x) = 0 \quad (x \in I).$$

From the independence of $\{z_1, z_2, \ldots, z_{n_1}\}$, we see that

$$\alpha_{n_2+1} = \alpha_{n_2+1} = \ldots = \alpha_{n_1+n_2} = 0.$$

This simplifies (2.9) to

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) + \ldots + \alpha_{n_2} y_{n_2}(x) = 0 \quad (x \in I);$$

however, the linear independence of $\{y_1, y_2, \ldots, y_{n_2}\}$ now forces

$$\alpha_1 = \alpha_2 = \ldots = \alpha_{n_2} = 0$$

and this completes the proof. □

This result generalizes to our main result which we now state and prove.

Theorem 2.1. *Consider the* k *differential equations* $\ell_j[y] = 0$, *defined in (1.2), of orders* n_j *for* $j = 1, 2, \ldots k$. *Let*

$$B_j = \{z_1^{(j)}, z_2^{(j)}, \ldots, z_{n_j}^{(j)}\} \quad (j = 1, 2, \ldots k)$$

be a basis of solutions for $\ell_j[y] = 0$ *on* I. *Fix* $x_0 \in I$ *and let* $G_j(x, t)$ *be the Green's function for* $\ell_j[\cdot]$ *(as defined in (2.1)) so that*

$$\ell_j^{-1}[y](x) := \int_{x_0}^{x} G_j(x, t) y(t) dt \quad (j = 1, 2, \ldots k).$$

Consider the set of $n_1 + n_2 + \ldots + n_k$ *functions*

$$B = \{y_1, y_2, \ldots, y_{n_k}, y_{n_k+1}, y_{n_k+2}, \ldots, y_{n_k+n_{k-1}}, \ldots, y_{n_k+n_{k-1}+\ldots+n_2+1}, \ldots, y_{n_k+n_{k-1}+\ldots+n_1}\}$$

defined by

$$y_j(x) := z_j^{(k)}(x) \quad (j = 1, 2, \ldots, n_k),$$

$$y_{n_k+j}(x) := \ell_k^{-1}[z_j^{(k-1)}](x) \quad (j = 1, 2, \ldots, n_{k-1}),$$

$$y_{n_k+n_{k-1}+j}(x) := \ell_k^{-1}\ell_{k-1}^{-1}[z_j^{(k-2)}](x) \quad (j = 1, 2, \ldots, n_{k-2}),$$

$$\vdots$$

$$y_{n_k+n_{k-1}+\ldots+n_2+j}(x) := \ell_k^{-1}\ell_{k-1}^{-1}\ldots\ell_2^{-1}[z_j^{(1)}](x) \quad (j = 1, 2, \ldots, n_1).$$

Then B is a basis of solutions to the composite product differential equation

$$\ell[y](x) = (\ell_1\ell_2\ldots\ell_k)[y](x) = 0 \quad (x \in I).$$

Proof. Since $\ell_k[y_j] = \ell_k[z_j^{(k)}] = 0$, it is clear that $\ell[y_j] = (\ell_1\ell_2\cdots\ell_k)[y_j](x) = 0$ for $j = 1, 2, \ldots n_k$. Similarly,

$$(\ell_{k-1}\ell_k)[y_{n_k+j}] = \ell_{k-1}\ell_k\ell_k^{-1}[z_j^{(k-1)}] = \ell_{k-1}[z_j^{(k-1)}] = 0$$

and hence $\ell[y_{n_k+j}] = 0$ for $j = 1, 2, \ldots, n_{k-1}$. Continuing in this fashion, we can see that each function in B is a solution of $\ell[y] = 0$. We prove that these functions in B are linearly independent by induction on k. Clearly, if $k = 1$, there is nothing to prove. The proof of $k = 2$ is given in Lemma 2.1. Let us assume linear independence in the case of any positive integer $m < k$. For $x \in I$, set

$$\begin{aligned}(2.10)\qquad 0 = \alpha_1 y_1(x) + \ldots + \alpha_{n_k} y_{n_k}(x) + \ldots + \alpha_{n_k+n_{k-1}+\ldots+n_2} y_{n_k+n_{k-1}+\ldots+n_2}(x)\\ + \alpha_{n_k+n_{k-1}+\ldots+n_2+1} y_{n_k+n_{k-1}+\ldots+n_2+1}(x) + \ldots + \alpha_{n_k+n_{k-1}+\ldots+n_1} y_{n_k+n_{k-1}+\ldots+n_1}(x).\end{aligned}$$

Apply $\ell_2\ell_3\cdots\ell_k$ to both sides of this equation to obtain

$$\begin{aligned}\alpha_{n_k+n_{k-1}+\ldots+n_2+1} z^{(1)}_{n_k+n_{k-1}+\ldots+n_2+1}(x) + \alpha_{n_k+n_{k-1}+\ldots+n_2+2} z^{(1)}_{n_k+n_{k-1}+\ldots+n_2+2}(x)\\ + \ldots + \alpha_{n_k+n_{k-1}+\ldots+n_2+n_1} z^{(1)}_{n_k+n_{k-1}+\ldots+n_2+n_1}(x) = 0 \quad (x \in I).\end{aligned}$$

Since $\{z^{(1)}_{n_k+n_{k-1}+\ldots+n_2+1}, z^{(1)}_{n_k+n_{k-1}+\ldots+n_2+2}, \ldots, z^{(1)}_{n_k+n_{k-1}+\ldots+n_2+n_1}\}$ is linearly independent, we see that

$$\alpha_{n_k+n_{k-1}+\ldots+n_2+1} = \alpha_{n_k+n_{k-1}+\ldots+n_2+2} = \ldots = \alpha_{n_k+n_{k-1}+\ldots+n_1} = 0.$$

Consequently, (2.10) reduces to

$$\begin{aligned}0 = \alpha_1 y_1(x) + \ldots + \alpha_{n_k} y_{n_k}(x) + \ldots\\ + \alpha_{n_k+n_{k-1}+\ldots+n_3+1} y_{n_k+n_{k-1}+\ldots+n_3+1} + \ldots + \alpha_{n_k+n_{k-1}+\ldots+n_2} y_{n_k+n_{k-1}+\ldots+n_2}(x)\end{aligned}$$

and from our induction hypothesis, with $m = k - 1$, we see that $\alpha_1 = \ldots = \alpha_{n_k} = \ldots = \alpha_{n_k+n_{k-1}+\ldots+n_2} = 0$, completing the proof of the theorem. □

The following result, in the special case that $\ell_1 = \ell_2 = \ldots = \ell_k := \ell$, is immediate. In this special case, our 'Green's function method' generates a basis of solutions to the power equation $\ell^k[y] = 0$ on I given, quite remarkably, only a basis of $\ell[y] = 0$.

Corollary 2.1. *Let $\{y_1, y_2, \ldots, y_n\}$ be a basis of solutions to the homogeneous differential equation*

$$\ell[y](x) = 0 \quad (x \in I),$$

where $\ell[\cdot]$ is a differential expression of order n of the form given in (1.2). Let $k \in \mathbb{N}$ and fix $x_0 \in I$. Define the kn functions $y_1, y_2, \ldots, y_{kn}$ by

$$(2.11)\qquad y_{n+j}(x) := \int_{x_0}^{x} G(x,t) y_j(t)dt \quad (j = 1, 2, \ldots, n),$$

$$(2.12)\qquad y_{2n+j}(x) := \int_{x_0}^{x} G(x,t) y_{n+j}(t)dt \quad (j = 1, 2, \ldots, n),$$

$$\vdots$$

$$y_{(k-2)n+j}(x) := \int_{x_0}^{x} G(x,t) y_{(k-3)n+j}(t)dt \quad (j = 1, 2, \ldots, n), \tag{2.13}$$

$$y_{(k-1)n+j}(x) := \int_{x_0}^{x} G(x,t) y_{(k-2)n+j}(t)dt \quad (j = 1, 2, \ldots, n). \tag{2.14}$$

Then $\{y_1, y_2, \ldots, y_{kn}\}$ *is a basis of solutions to*

$$\ell^k[y](x) = 0 \quad (x \in I).$$

Remark 2.1. *In order for the results in Lemma 2.1, Theorem 2.1, and Corollary 2.1 to be valid, it is important that the coefficient of the highest derivative in each differential equation is one. If this is not the case, then the denominator of the corresponding* $G(x,t)$ *must be modified slightly to include this leading coefficient. More specifically, if* $G(x,t)$ *is the Green's function associated with*

$$m[y](x) := a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_0(x)y(x),$$

then a particular solution to $m[y](x) = f(x)$ *is*

$$y_p(x) = \int_{x_0}^{x} \frac{G(x,t)}{a_n(t)} f(t)dt.$$

3. Examples

Example 3.1. *In the recent paper* [6], *the authors discuss the fourth-order differential equation*

$$\ell[y](x) = y^{(4)}(x) + \frac{2}{x}y'''(x) - \frac{9}{x^2}y''(x) + \frac{9}{x^2}y'(x) - \lambda^4 y(x) = 0 \quad (x \in (0,\infty));$$

this equation is a limiting form of the fourth-order Bessel type differential equation studied by the authors in [6] *and other papers cited therein. They prove that a basis of solutions of this equation is given by* $\{J_2(\lambda x), Y_2(\lambda x), I_2(\lambda x), K_2(\lambda x)\}$, *where* $J_2(\lambda x)$ *and* $Y_2(\lambda x)$ *are, respectively, the classical Bessel functions of the first and second kind and where* $I_2(\lambda x)$ *and* $K_2(\lambda x)$ *are, respectively, the modified Bessel functions of the first and second kind; see* [1, Chapter 9] *for properties of these Bessel and modified Bessel functions. We can establish this result by different means, specifically by using Lemma 2.1. Indeed, we first note that*

$$\ell[y](x) = (\ell_1 \circ \ell_2)[y](x),$$

where

$$\ell_1[y](x) = y''(x) + \frac{1}{x}y'(x) - \left(\frac{4}{x^2} + \lambda^2\right) y(x) \quad (x \in (0,\infty)),$$

and

$$\ell_2[y](x) = y'' + \frac{1}{x}y'(x) + \left(\lambda^2 - \frac{4}{x^2}\right) y(x) \quad (x \in (0,\infty)).$$

A basis of solutions of $\ell_1[y](x) = 0$ *on* $(0,\infty)$ *is* $\{I_2(\lambda x), K_2(\lambda x)\}$ *while a basis of solutions to* $\ell_2[y](x) = 0$ *on* $(0,\infty)$ *is* $\{J_2(\lambda x), Y_2(\lambda x)\}$. *Furthermore, since the Wronskian associated with* $\{J_2(\lambda x), Y_2(\lambda x)\}$ *is* $W(J_2(\lambda x), Y_2(\lambda x)) = 2/(\lambda\pi x)$, *we see that we can take the Green's function associated with* $\ell_2[\cdot]$ *to be*

$$G_2(x,t) = t[J_2(\lambda x)Y_2(\lambda t) - J_2(\lambda t)Y_2(\lambda x)].$$

Consequently, by Lemma 2.1, we see that two additional linearly independent solutions to $\ell[y](x) = 0$ *on* $(0,\infty)$ *are*

$$y_3(x) = \int_{x_0}^{x} t[J_2(\lambda x)Y_2(\lambda t) - J_2(\lambda t)Y_2(\lambda x)]I_2(\lambda t)dt,$$

and

$$y_4(x) = \int_{x_0}^{x} t[J_2(\lambda x)Y_2(\lambda t) - J_2(\lambda t)Y_2(\lambda x)]K_2(\lambda t)dt;$$

here $x_0 > 0$ *is fixed and* $x > 0$. *For* $y_3(x)$, *we can take* $x_0 = 0$; *in this case, we find* (*for example, from Mathematica*) *that*

$$\int_0^x tY_2(\lambda t)I_2(\lambda t)dt = \frac{1}{\pi\lambda^2} + \frac{x}{2\lambda}[Y_2(\lambda x)I_3(\lambda x) + Y_3(\lambda x)I_2(\lambda x)],$$

and

$$\int_0^x tJ_2(\lambda t)I_2(\lambda t)dt = \frac{x}{2\lambda}[J_2(\lambda x)I_3(\lambda x) + J_3(\lambda x)I_2(\lambda x)].$$

From these two integrals, we find that

$$\begin{aligned} y_3(x) &= \frac{1}{\pi\lambda^2}J_2(\lambda x) + \frac{x}{2\lambda}I_2(\lambda x)[J_2(\lambda x)Y_3(\lambda x) - J_3(\lambda x)Y_2(\lambda x)] \\ &= \frac{1}{\pi\lambda^2}J_2(\lambda x) - \frac{1}{\pi\lambda^2}I_2(\lambda x), \end{aligned} \tag{3.1}$$

where we have used the fact (*see* [1, Chapter 9, 9.1.16]) *that*

$$J_2(\lambda x)Y_3(\lambda x) - J_3(\lambda x)Y_2(\lambda x) = -W(J_2(\lambda x), Y_2(\lambda x)) = -\frac{2}{\lambda\pi x}. \tag{3.2}$$

Moreover, since $J_2(\lambda x)$ *is already a solution of* $\ell[y] = 0$ *on* $(0, \infty)$, *we can take a third linearly independent solution of this equation to be*

$$\widetilde{y}_3(x) = I_2(\lambda x).$$

A similar analysis shows that, for $x_0 > 0$,

$$\int_{x_0}^x tY_2(\lambda t)K_2(\lambda t)dt = \frac{x}{2\lambda}\left[K_2(\lambda x)Y_3(\lambda x) - K_3(\lambda x)Y_2(\lambda x)\right] + C_1$$

and

$$\int_{x_0}^x tJ_2(\lambda t)K_2(\lambda t)dt = \frac{x}{2\lambda}\left[K_2(\lambda x)J_3(\lambda x) - K_3(\lambda x)J_2(\lambda x)\right] - C_2,$$

where C_1 *and* C_2 *are arbitrary constants. Therefore, from* (*3.2*), *we obtain*

$$\begin{aligned} y_4(x) &= C_1J_2(\lambda x) + C_2Y_2(\lambda x) + \frac{xK_2(\lambda x)}{2\lambda}\left[J_2(\lambda x)Y_3(\lambda x) - J_3(\lambda x)Y_2(\lambda x)\right] \\ &= C_1J_2(\lambda x) + C_2Y_2(\lambda x) - \frac{K_2(\lambda x)}{\pi\lambda^2}. \end{aligned}$$

Consequently, it follows that we can take

$$\widetilde{y}_4(x) = K_2(\lambda x)$$

as a fourth linearly independent solution to $\ell[y] = 0$ *on* $(0, \infty)$.

Remark *We note that the easiest way to see that* $\{J_2(\lambda x), Y_2(\lambda x), I_2(\lambda x), K_2(\lambda x)\}$ *forms a basis of* $\ell[y] = 0$ *on* $(0, \infty)$ *is to simply observe that the two factors* $\ell_1[\cdot]$ *and* $\ell_2[\cdot]$ *commute; that is to say,* $\ell[y](x) = (\ell_1 \circ \ell_2)[y](x) = (\ell_2 \circ \ell_1)[y](x)$ *for sufficiently smooth functions* $y(x)$ *on* $(0, \infty)$.

Example 3.2. *Take $\ell_1[\cdot]$ to be the classical Bessel equation*

$$\ell_1[y](x) = y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{\nu^2}{x^2}\right)y(x) = 0 \quad (x \in (0,\infty)),$$

where $\nu \geq 0$, and let

$$\ell_2[y](x) = y'(x) + \frac{\nu+1}{x}y(x) = 0 \quad (x \in (0,\infty)).$$

Then

$$\begin{aligned}\ell[y](x) &= (\ell_1\ell_2)[y](x)\\ &= y'''(x) + \frac{\nu+2}{x}y''(x) + \left(1 - \frac{\nu^2+\nu+1}{x^2}\right)y'(x) + \frac{\nu+1}{x}\left(1 + \frac{1-\nu^2}{x^2}\right)y(x) \quad (x \in (0,\infty)).\end{aligned}$$

Using the notation from Lemma 2.1, we see that

$$z_1^{(1)}(x) = J_\nu(x), z_2^{(1)}(x) = Y_\nu(x), \text{ and } z_1^{(2)}(x) = \frac{1}{x^{\nu+1}}.$$

Since the Green's function associated with $\ell_2[\cdot]$ is given by $G_2(x,t) = t^{\nu+1}/x^{v+1}$, we see that two other solutions of $\ell[y] = 0$ on $(0,\infty)$ are

$$y_2(x) = \frac{1}{x^{\nu+1}}\int_0^x t^{\nu+1}J_\nu(t)dt = J_{\nu+1}(x),$$

$$y_3(x) = \frac{1}{x^{\nu+1}}\int_0^x t^{\nu+1}Y_\nu(t)dt = Y_{\nu+1}(x) + \frac{c}{x^{\nu+1}},$$

where c is a constant. Consequently, a basis of solutions of $\ell[y] = 0$ on $(0,\infty)$ is

$$\left\{\frac{1}{x^{\nu+1}}, J_{\nu+1}(x), Y_{\nu+1}(x)\right\}.$$

This third order differential equation $\ell[y] = 0$ is a new non-trivial equation for the Bessel functions of the first and the second kind. We now reverse the order of the factors $\ell_1[\cdot]$ and $\ell_2[\cdot]$ and consider

$$\begin{aligned}m[y](x) &= (\ell_2\ell_1)[y](x)\\ &= y'''(x) + \frac{\nu+2}{x}y''(x) + \left(1 + \frac{\nu-\nu^2}{x}\right)y'(x) + \left(\frac{\nu+1}{x} + \frac{\nu^2-\nu^3}{x^3}\right)y(x) \quad (x \in (0,\infty)).\end{aligned}$$

Using the fact that Green's function for the Bessel expression $\ell_1[\cdot]$ is a nonzero multiple of

$$G_1(x,t) = t[J_\nu(t)Y_\nu(x) - J_\nu(x)Y_\nu(t)], \tag{3.3}$$

we see that a basis of solutions of $m[y] = 0$ on $(0,\infty)$ is given by $\{J_\nu(x), Y_\nu(x), z_3(x)\}$, where

$$z_3(x) = Y_\nu(x)\int_{x_0}^x t^{-\nu}J_\nu(t)dt - J_\nu(x)\int_{x_0}^x t^{-\nu}Y_\nu(t)dt,$$

here, for convergence reasons, we must have $x_0 > 0$.

Example 3.3. *Consider the classical second-order Airy differential equation, given by*

$$\ell_1[y](x) = y''(x) - xy(x) = 0 \quad (x \in (0,\infty)),$$

and the first order differential equation

$$\ell_2[y](x) = y'(x) + \frac{1}{x}y(x) = 0 \quad (x \in (0,\infty))$$

so that

$$\ell[y](x) = (\ell_1\ell_2)[y](x) = y'''(x) + \frac{1}{x}y''(x) - \left(\frac{2}{x^2} + x\right)y'(x) + \left(\frac{2}{x^3} - 1\right)y(x) \quad (x \in (0,\infty)).$$

A non-trivial solution of $\ell_2[y] = 0$ *is* $z_1^{(2)}(x) = 1/x$ *while a basis of solutions to* $\ell_1[y] = 0$ *is* $\{Ai(x), Bi(x)\}$, *where* $Ai(x)$ *and* $Bi(x)$ *are, respectively, the Airy functions of the first and second kind; see* [1, Chapter 9] *for properties of these Airy functions. In this case, the Green's function associated with* $\ell_2[\cdot]$ *is a non-constant multiple of*

$$G_2(x,t) = \frac{t}{x}.$$

From Lemma 2.1 or Theorem 2.1, two solutions of $\ell[y](x) = 0$ *are given by*

$$y_2(x) = \frac{1}{x}\int_0^x tAi(t)dt = \frac{Ai'(x) + c_1}{x},$$

and

$$y_3(x) = \frac{1}{x}\int_0^x tBi(t)dt = \frac{Bi'(x) + c_2}{x},$$

and, consequently, we can take a basis of $\ell[y] = 0$ *on* $(0,\infty)$ *to be*

$$\left\{\frac{1}{x}, \frac{Ai'(x)}{x}, \frac{Bi'(x)}{x}\right\}.$$

The equation $\ell[y] = 0$ *on* $(0,\infty)$ *is a new third-order differential equation for the special functions* $Ai'(x)/x$ *and* $Bi'(x)/x$.

Example 3.4. *Let* $\ell_1[y] = y^{(n)}$ *and let* $\ell_2[y] = 0$ *be an* m^{th} *order differential equation on the interval* I *with basis of solutions* $\{y_1, y_2, \ldots, y_m\}$ *and associated Green's function* $G_2(x,t)$. *Define* $\ell[\cdot]$ *to be the differential expression of order* $n+m$ *by* $\ell[y] := (\ell_1\ell_2)[y]$ *on* I. *Then a basis of solutions to* $\ell[y] = 0$ *on* I *is* $\{y_1, y_2, \ldots y_m, y_{m+1}, \ldots y_{n+m}\}$, *where*

$$y_{m+j}(x) := \int_{x_0}^x G_2(x,t)t^{j-1}dt \quad (j = 1, 2, \ldots, n).$$

On the other hand, let $m[y] = (\ell_2\ell_1)[y]$; *since the Green's function for* $\ell_1[\cdot]$ *is a non-zero constant multiple of*

$$G_1(x,t) = \frac{1}{(n-1)!}(x-t)^{n-1},$$

we see that a basis of solutions of $m[y] = 0$ *on* I *is* $\{1, x, \ldots, x^{n-1}, Y_1(x), Y_2(x), \ldots, Y_m(x)\}$, *where*

$$Y_j(x) := \frac{1}{(n-1)!}\int_{x_0}^x (x-t)^{n-1}y_j(t)dt \quad (j = 1, 2, \ldots, m).$$

Of course, it is well known that $Y_j(x)$ *is the n-fold integral of* $y_j(x)$; *that is to say,*

$$Y_j(x) = \int_{x_0}^x \int_{x_0}^{u_{n-1}} \int_{x_0}^{u_{n-2}} \cdots \int_{x_0}^{u_1} y_j(t)dtdu_1du_2\ldots du_{n-1}.$$

For a specific example, a basis of solutions of the n^{th} *derivative of Airy's equation*

$$\ell[y](x) = (y''(x) - xy(x))^{(n)} \quad (x \in \mathbb{R})$$

is $\{Ai(x), Bi(x), y_3(x), y_4(x), \ldots, y_{n+2}(x)\}$ *where, for* $k = 1, 2, \ldots, n$,

$$\begin{aligned} y_{k+2}(x) &= Bi(x)\int_0^x t^{k-1}Ai(t)dt - Ai(x)\int_0^x t^{k-1}Bi(t)dt \\ &= x^k\left\{\frac{1}{k\Gamma(2/3)}\left[\frac{Bi(x)}{3^{2/3}} - \frac{Ai(x)}{3^{1/6}}\right]{}_1F_2\left(\frac{k}{3};\frac{2}{3},\frac{k+3}{3};\frac{x^3}{9}\right) - \right. \\ &\left. \frac{x}{(k+1)\Gamma(1/3)}\left[\frac{Bi(x)}{3^{1/3}} + 3^{1/6}Ai(x)\right]{}_1F_2\left(\frac{k+1}{3};\frac{4}{3},\frac{k+4}{3};\frac{x^3}{9}\right)\right\}. \end{aligned}$$

Here, we have used the fact that Green's function $G_2(x,t)$ *for Airy's equation can be taken to be*

$$G_2(x,t) = Ai(t)Bi(x) - Ai(x)Bi(t).$$

Example 3.5. *In this example, we find a basis for solutions to* $\ell^2[y] = 0$ *on* $(0,\infty)$*, where* $\ell[\cdot]$ *is the second-order Bessel expression given by*

$$\ell[y](x) = y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{\nu^2}{x^2}\right)y(x) \quad (x \in (0,\infty)).$$

The square of the Bessel equation has been considered at length in the literature because of its importance in spectral theory; for example, see [8]*. With basis* $\{y_1(x) = J_\nu(x), y_2(x) = Y_\nu(x)\}$ *of* $\ell[y] = 0$ *on* $(0,\infty)$*, we see from Lemma 2.1 and (3.3) that a basis of solutions to* $\ell^2[y] = 0$ *on* $(0,\infty)$*, where*

$$\begin{aligned} \ell^2[y](x) = y^{(4)}(x) &+ \frac{2}{x}y'''(x) + \left(2 - \left(\frac{2\nu^2+1}{x^2}\right)\right)y''(x) + \left(\frac{2}{x} + \left(\frac{2\nu^2+1}{x^3}\right)\right)y'(x) \\ &+ \left(\left(1 - \frac{\nu^2}{x^2}\right)^2 - \frac{4\nu^2}{x^4}\right)y(x), \end{aligned}$$

is given by $\{y_1, y_2, y_3, y_4\}$*, where*

$$\begin{aligned} y_3(x) &= \int_0^x t[J_\nu(t)Y_\nu(x) - J_\nu(x)Y_\nu(t)]J_\nu(t)dt \\ &= A_\nu(x)Y_\nu(x) - B_\nu(x)J_\nu(x), \end{aligned}$$

$$\begin{aligned} y_4(x) &= \int_0^x t[J_\nu(t)Y_\nu(x) - J_\nu(x)Y_\nu(t)]Y_\nu(t)dt \\ &= B_\nu(x)Y_\nu(x) - C_\nu(x)J_\nu(x), \end{aligned}$$

and where

$$A_\nu(x) := \int_0^x tJ_\nu^2(t)dt = \frac{x^2}{2}[J_\nu^2(x) - J_{\nu-1}(x)J_{\nu+1}(x)],$$

$$\begin{aligned} B_\nu(x) &:= \int_0^x tJ_\nu(t)Y_\nu(t)dt \\ &= \frac{\nu}{\pi}\left[{}_1F_2(-1/2;\nu,-\nu;-x^2) - 1\right] + \frac{x^{2\nu+2}\Gamma(-\nu-1)}{2\sqrt{\pi}\Gamma(1/2-\nu)(2\nu)!}\,{}_1F_2(\nu+1/2;\nu+2,2\nu+1;-x^2), \end{aligned}$$

$$C_\nu(x) := \int_0^x tY_\nu^2(t)dt = \frac{x^2}{2}[Y_\nu^2(x) - Y_{\nu-1}(x)Y_{\nu+1}(x)].$$

If $\nu \in \mathbb{N}$ *or* $\nu = 0$*, the expression for* $B_\nu(x)$ *must be interpreted by taking appropriate limits.*

Example 3.6. *The classical second-order Legendre differential equation is defined by*

$$\ell[y](x) = y''(x) - \frac{2x}{1-x^2}y'(x) + \frac{n(n+1)}{1-x^2}y(x) = 0 \quad (x \in (-1,1)).$$

Two linearly independent solutions of $\ell[y] = 0$ are given by

$$y_1(x) = P_n(x) \text{ and } y_2(x) = Q_n(x),$$

where $P_n(x)$ is the Legendre polynomial of degree n and $Q_n(x)$ is the Legendre function of the second kind. The Green's function associated with $\ell[\cdot]$ in this case can be taken to be

$$G(x,t) = (1-t^2)[P_n(t)Q_n(x) - P_n(x)Q_n(t)].$$

Then, an additional two linearly independent solutions of $\ell^2[y] = 0$, where

$$\begin{aligned}\ell^2[y](x) &= y^{(4)}(x) - \frac{4x}{1-x^2}y'''(x) + \left(\frac{(-2n^2-2n)x^2 + (2n^2+2n-4)}{(1-x^2)^2}\right)y''(x) \\ &\quad - \frac{8x}{(1-x^2)^3}y'(x) + \left(\frac{n(n+1)(n^2+n+2) - (n-1)(n+2)x}{(1-x^2)^3}\right)y(x) \quad (x \in (-1,1)),\end{aligned}$$

are

$$y_3(x) = \int_0^x (1-t^2)[P_n(t)Q_n(x) - P_n(x)Q_n(t)]P_n(t)dt = A(x)Q_n(x) - B(x)P_n(x),$$

and

$$y_4(x) = \int_0^x (1-t^2)[P_n(t)Q_n(x) - P_n(x)Q_n(t)]Q_n(t)dt = B(x)Q_n(x) - C(x)P_n(x),$$

where

$$A(x) := \int_0^x (1-t^2)P_n^2(t)dt,$$

$$B(x) := \int_0^x (1-t^2)P_n(t)Q_n(t)dt,$$

and

$$C(x) := \int_0^x (1-t^2)Q_n^2(t)dt.$$

In the particular case of $n = 0$, two solutions of $\ell^2[y] = 0$ on $(-1,1)$ are given by

$$y_1(x) = 1 = P_0(x) \text{ and } y_2(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = Q_0(x).$$

In this special case, the Green function is

$$G(x,t) = \frac{1}{2}(1-t^2)\left[\ln\left(\frac{1+x}{1-x}\right) - \ln\left(\frac{1+t}{1-t}\right)\right].$$

Two additional linearly independent solutions of $\ell^2[y] = 0$ on $(-1,1)$ are given by

$$\begin{aligned}y_3(x) &= \int_0^x G(x,t)y_1(t)dt = \int_0^x (1-t^2)\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)\right]dt \\ &= \frac{x^2}{6} - \frac{\ln(1-x^2)}{3},\end{aligned}$$

and

$$y_4(x) = \int_0^x G(x,t)y_2(t)dt = \frac{1}{4}\int_0^x (1-t^2)\left[\ln\left(\frac{1+x}{1-x}\right) - \ln\left(\frac{1+t}{1-t}\right)\right]\ln\left(\frac{1+t}{1-t}\right)dt;$$

after some simplifications, we see that we can take

$$y_4(x) = 4x + x^2 \ln\left(\frac{1+x}{1-x}\right) + 4\mathrm{Li}_2\left(\frac{1-x}{2}\right) - 4\mathrm{Li}_2\left(\frac{1+x}{2}\right)$$

where $Li_2(\cdot)$ is the dilogarithm (polylogarithm) function, defined by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (x \in (-1,1)).$$

For a good reference on the dilogarithm function, see [2, Chapter 2.6].

Example 3.7. *Consider the second-order differential equation*

$$\ell[y] = x^2 y'' + (1+x)y'(x) = 0 \quad (x \in (0,\infty)).$$

Whereas all of our previous examples deal with either regular or regular singular endpoints, this equation has an irregular singular point at $x = 0$. This equation is obtained by transforming a special case of the confluent hypergeometric equation

$$xy''(x) + (1-x)y'(x) = 0$$

through the transformation $x \to 1/x$. Two independent solutions of $\ell[y] = 0$ on $(0,\infty)$ are given by

$$y_1(x) = 1 \text{ and } y_2(x) = \Gamma\left(0, -\frac{1}{x}\right),$$

where $\Gamma(\cdot,\cdot)$ is the incomplete upper gamma function defined by

$$\Gamma(s,x) := \int_x^{\infty} t^{s-1}e^{-t}dt;$$

see [1, Chapter 6]*. The Green's function associated with $\ell[\cdot]$ is given by*

$$G(x,t) = te^{-1/t}\left[\Gamma\left(0,-\frac{1}{x}\right) - \Gamma\left(0,-\frac{1}{t}\right)\right].$$

From Corollary 2.1, two additional linearly independent solutions of $\ell^2[y] = 0$ on $(0,\infty)$ are:

$$\begin{aligned} y_3(x) &= \int^x te^{-1/t}\left[\Gamma\left(0,-\frac{1}{x}\right) - \Gamma\left(0,-\frac{1}{t}\right)\right]dt \\ &= \frac{1}{2}\left[x(x-1)e^{-1/x} + \Gamma\left(0,\frac{1}{x}\right)\right]\Gamma\left(0,-\frac{1}{x}\right) - B(x), \end{aligned}$$

and

$$y_4(x) = \int^x te^{-1/t}\left[\Gamma\left(0,-\frac{1}{x}\right) - \Gamma\left(0,-\frac{1}{t}\right)\right]\Gamma\left(0,-\frac{1}{t}\right)dt = B(x)\Gamma\left(0,-\frac{1}{x}\right) - C(x)$$

where

$$B(x) := \int^x te^{-1/t}\Gamma\left(0,-\frac{1}{t}\right)dt,$$

and

$$C(x) := \int^x te^{-1/t}\Gamma^2\left(0,-\frac{1}{t}\right)dt.$$

The asympotics of y_2 are well known and can be found in [1, Section 6.5.32]*; in fact,*

$$y_2(x) = \Gamma\left(0,-\frac{1}{x}\right) \sim -xe^{1/x}\sum_{n=0}^{\infty} n!x^n \text{ as } x \to 0.$$

Lastly, we briefly describe the asymptotic behavior of y_3 *and* y_4 *near* $x = 0$*. Since*

$$B'(x) = xe^{-1/x}\Gamma(0, -1/x),$$

we see that

$$B(x) \sim -\sum_{n=0}^{\infty} \frac{n!}{n+3} x^{n+3} \text{ as } x \to 0.$$

Moreover,

$$\Gamma^2\left(0, -\frac{1}{x}\right) \sim x^2 e^{2/x} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} k!(n-k)!\right] x^n \text{ as } x \to 0.$$

Hence

$$C(x) \sim \sum_{n=0}^{\infty} (-1)^n \left[\sum_{k=0}^{n} k!(n-k)!\right] \Gamma\left(-n-4, -\frac{1}{x}\right) \text{ as } x \to 0.$$

Consequently, from the complete asymptotic expansions for $\Gamma(0, \pm 1/x)$, $B(x)$ *and* $C(x)$*, we find that a first-order approximation at* $x = 0$ *is*

$$B(x) \sim -\frac{x^3}{3}, \qquad \Gamma(0, \frac{1}{x}) \sim xe^{-1/x}, \qquad C(x) \sim \Gamma(-4, -\frac{1}{x}) \sim -x^5 e^{1/x}.$$

Therefore, when $x \to 0$*, we get that*

$$y_3(x) \sim \frac{x^3}{3} \text{ and } y_4(x) \sim \frac{x^4}{3} e^{1/x}.$$

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References

[1] M. Abramowitz and I. Stegun (editors), *Handbook of Mathematical Functions,* Dover Publications, New York, 1972.

[2] George E. Andrews, Richard Askey, and Ranjan Roy, *Special Functions,* Encyclopedia of Mathematics and its Applications, Volume 71, Cambridge University Press, Cambridge, United Kingdom, 1999.

[3] M. Bronstein, *An improved algorithm for factoring linear ordinary differential operators,* Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISAAC 94 - 7/94, Association for Computing Machinery, Oxford, United Kingdom, 336-340.

[4] M. Bronstein, *On the factorisation of linear ordinary differential operators,* Math. Comput. Simulation, 42(1996), no. 4-6, 387-389.

[5] G. J. Etgen, G. D. Jones, and W. E. Taylor, Jr., *On the factorizations of linear differential operators,* Trans. Amer. Math. Soc., 297(2)(1986), 717-728.

[6] W. N. Everitt, C. Markett, and L. L. Littlejohn, *Properties of the solutions of the fourth-order Bessel-type differential equation*, J. Math. Anal. Appl., 359 (2009), 252-264.

[7] Ross L. Finney and Donald R. Ostberg, *Elementary Differential Equations with Linear Algebra*, Addison-Wesley Publishers, Reading, MA., 1981.

[8] Charles T. Fulton, *The Bessel-squared equation in the lim-2, lim-3, and lim-4 cases,* Quart. J. Math. Oxford Ser. (2) 40 (1989), no. 160, 423-456.

[9] Mark van Hoeij, *Factorization of Linear Differential Operators,* Ph.D. thesis, Katholieke Universiteit Nijmegen, 1996.

[10] Mark van Hoeij, *Factorization of Differential Operators with Rational Functions Coefficients,* J. Symbolic Computation, 24(1997), 537-561.

[11] F. Schwarz, *A factorization algorithm for linear differential equations,* Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISAAC 89, 17-25.

[12] A. Zettl, *Factorization of differential operators,* 27(2)(1971), 425-426.
[13] A. Zettl, *Explicit conditions for the factorization of nth order linear differential operators,* Proc. Amer. Math. Soc., 41(1)(1973), 137-145.
[14] A. Zettl, *General theory of the factorization of ordinary differential operators,* Trans. Amer. Math. Soc., 197(1974), 341-353.
[15] Daniel Zwillinger, *Handbook of Differential Equations* (third edition), Academic Press, New York, 1998.

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