LEFT-DEFINITE THEORY WITH APPLICATIONS TO ORTHOGONAL POLYNOMIALS

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Dedicated to Professor Jesus Dehesa on the occasion of his 60th birthday

Abstract. In the past several years, there has been considerable progress made on a general left-definite theory associated with a self-adjoint operator A that is bounded below in a Hilbert space H; the term 'left-definite' has its origins in differential equations but Littlejohn and Wellman [25] generalized the main ideas to a general abstract setting. In particular, it is known that such an operator A generates a continuum $\{H_r\}_{r>0}$ of Hilbert spaces and a continuum of $\{A_r\}_{r>0}$ of self-adjoint operators. In this paper, we review the main theoretical results in [25]; moreover, we apply these results to several specific examples, including the classical orthogonal polynomials of Laguerre, Hermite, and Jacobi.

1. Introduction

In this paper, we bring together several recent results, and examples, concerning the theory of self-adjoint operators that are bounded below in a Hilbert space. The mathematical literature has numerous examples of self-adjoint differential operators that are bounded below in a Hilbert space that generate a 'left-definite' study; a specific explanation of this terminology is given below in Section 2. The origins of left-definite theory (the ideas can be traced to fundamental work of Hermann Weyl [41] on his landmark study of second-order differential equations) and the term *left-definite* (actually, the German *Links-definit*) first appeared in the literature in 1965 in a paper by Schäfke and Schneider [35]. Over the past forty years, there has been a resurgence in this study by several authors. In fact, there are other interpretations of 'left-definite' in the literature; we refer to the paper [18] and the references cited therein for another viewpoint on 'left-definite' problems.

A general left-definite theory for arbitrary self-adjoint operators A that are bounded below in a Hilbert space was developed by Littlejohn and Wellman in [25]; we refer the reader to this contribution for a detailed list of references. Since the publication of [25] in 2002, there have been several papers written on the applications of this theory to specific operators studied in mathematical physics and functional analysis. In particular, this theory can be applied to the second-order differential equations of Laguerre, Hermite, and Jacobi which have classical orthogonal polynomial solutions. Furthermore, as a consequence of this general theory, there are new applications to combinatorics as well as new information on various powers of A.

Date: Februrary 16, 2008 (C:\SWDocs\Papers\Leftdef_surveyFV.tex).

1991 *Mathematics Subject Classification.* Primary 34B30, 47B25, 47B65; Secondary 33C65, 34B20.

Key words and phrases. self-adjoint operator, Hilbert space, Sobolev space, Dirichlet inner product, left-definite Hilbert space, left-definite self-adjoint operator, Laguerre polynomials, Stirling numbers of the second kind, Legendre-Stirling numbers, Jacobi-Stirling numbers.

This paper is based on a plenary lecture given by L. L. Littlejohn in honor of Professor J. S. Dehesa at the conference entitled "Special functions, Information theory, Mathematical physics" in Granada, Spain from September 17-19, 2007.

The contents of this paper are as follows. In Section 2, we motivate this left-definite theory from the original viewpoint of differential equations. In Section 3, we review the main results appearing in [25]. Lastly, in Section 4, we consider several examples to illustrate this theory.

The main aim of this paper is to 'draw' further mathematicians into the subject of left-definite theory. Indeed, there are numerous self-adjoint ordinary, partial and difference equations for which this theory may be applied. As the reader will see in this paper, considerable more information can be derived about the original self-adjoint operator by considering the associated left-definite theory.

2. Left-Definite Theory from the Viewpoint of Differential Equations

As mentioned in the Introduction, the left-definite theory has its origins in differential equations; we now discuss this connection in more detail. For the sake of clarity and simplicity, we consider differential expressions that have smooth coefficients but note that more general expressions can be considered. We will also limit our discussion to motivating *first* left-definite spaces and *first* left-definite operators; indeed, until the paper [25] appeared, there was no mention of other left-definite spaces or operators in the literature.

Let $\ell[\cdot]$ be the differential expression

$$\ell[y](x) = \frac{1}{w(x)} \sum_{j=0}^{N} (-1)^j \left(a_j(x) y^{(j)}(x) \right)^{(j)} \quad (x \in I), \tag{2.1}$$

where

$$\begin{cases} \text{(i) } I = (a,b) \text{ is an interval of the real line } \mathbb{R}, \\ \text{(ii) each } a_j : I \to \mathbb{R} \text{ is a non-negative, } j\text{-times continuously differentiable function,} \\ \text{(iii) } w : I \to \mathbb{R} \text{ is a positive, continuous function,} \\ \text{(iv) } a_0(x) \geq k w(x) \text{ for all } x \in I, \text{ where } k \text{ is some fixed, positive constant.} \end{cases} \tag{2.2}$$

Notation-wise, we say that $\ell[\cdot]$ is Lagrangian symmetrizable with symmetry factor w; see the papers [23], [24] and, most recently, [13] for further information on Lagrangian symmetrizability, Lagrangian symmetry, and symmetry factors.

Central to the Glazman-Krein-Naimark theory, differential operators generated by the Lagrangian symmetrizable expression $\ell[\cdot]$ are studied in the weighted Hilbert space $L^2(I;w)$, where

$$L^2(I;w) = \{f : I \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_a^b |f(x)|^2 w(x) dx < \infty\},$$

with inner product

$$(f,g)_w = \int_a^b f(x)\overline{g}(x) w(x) dx \quad (f,g \in L^2(I;w)). \tag{2.3}$$

The maximal operator $\mathcal{L}$, generated by $\ell[\cdot]$, in $L^2(I;w)$ is defined to be

$$\begin{aligned} \mathcal{L}f &= \ell[f] \\ f &\in \mathcal{D}, \end{aligned}$$

where the dense set $\mathcal{D}$ is the so-called maximal domain defined by

$$\mathcal{D} := \{f : I \to \mathbb{C} \mid f, f', \ldots, f^{(2n-1)} \in AC_{\text{loc}}(I), f, \ell[f] \in L^2(I;w)\}.$$

The adjoint operator $\mathcal{L}_0 := \mathcal{L}^*$, called the minimal operator, is a closed symmetric operator in $L^2(I;w)$. Furthermore, since the expression $\ell[\cdot]$ has real coefficients, the deficiency indices of $\mathcal{L}_0$ are equal; this guarantees the existence of self-adjoint extensions A of $\mathcal{L}_0$ in $L^2(I;w)$. If A is such a self-adjoint extension with domain $\mathcal{D}(A)$, then it is the case that $\mathcal{D}(A) \subseteq \mathcal{D}$ and $Af = \ell[f]$ for each $f \in \mathcal{D}(A)$.

In the setting $L^2(I;w)$, there are two classic identities that play an important role in the construction and study of differential operators generated by $\ell[\cdot]$. The first such identity is *Green's formula*, given by

$$\int_c^d \ell[f](x)\overline{g}(x)w(x)dx = \int_c^d f(x)\overline{\ell[g]}(x)w(x)dx + [f,g](d) - [f,g](c) \quad (f,g \in \mathcal{D}); \tag{2.4}$$

here $[c,d]$ is any compact subinterval of $I = (a,b)$ and $[\cdot,\cdot]$ is the associated skew-symmetric sesquilinear (symplectic) form obtained through integration by parts. From the definition of $\mathcal{D}$, it follows that, in (2.4), we may take $d \to b^-$ and $c \to a^+$; in this limit case, Green's formula yields

$$(\ell[f], g)_w = (f, \ell[g])_w + [f,g](b) - [f,g](a), \tag{2.5}$$

where $(\cdot,\cdot)_w$ is the inner product defined in (2.3). The second, classic identity - crucial to the study of left-definite theory - is *Dirichlet's identity*, given by

$$\int_c^d \ell[f](x)\overline{g}(x)w(x)dx = \sum_{j=0}^{N} \int_c^d a_j(x)f^{(j)}(x)\overline{g}^{(j)}(x)dx + \{f,g\}(d) - \{f,g\}(c) \quad (f,g \in \mathcal{D}); \tag{2.6}$$

here, the form $\{\cdot,\cdot\}$ is 'half' of the sesquilinear form $[\cdot,\cdot]$. One stark difference between Green's formula and Dirichlet's formula is that, in general, we are not guaranteed finiteness, individually, of any of the three terms

$$\sum_{j=0}^{N} \int_c^d a_j(x)f^{(j)}(x)\overline{g}^{(j)}(x)dx, \ \{f,g\}(c), \ \{f,g\}(d),$$

on the right-hand side of (2.6) as $d \to b^-$ or as $c \to a^+$.

Let $\mathcal{D}'$ be a subspace of $\mathcal{D}$ in $L^2(I;w)$. The expression $\ell[\cdot]$ is

(i) *strong limit-point* at $x = a$ in $\mathcal{D}'$ (respectively, $x = b$ in $\mathcal{D}'$) if $\lim_{x\to a^+}\{f,g\}(x) = 0$ (respectively, $\lim_{x\to b^-}\{f,g\}(x) = 0$) for all $f,g \in \mathcal{D}'$;

(ii) *Dirichlet at* $x = a$ (respectively, at $x = b$) in $\mathcal{D}'$ if $\sum_{j=0}^{N}\int_a^{x_0} a_j(x)\left|f^{(j)}(x)\right|^2 dx$ exists and is finite for some $x_0 \in (a,b)$ (respectively, $\sum_{j=0}^{N}\int_{x_0'}^{b} a_j(x)\left|f^{(j)}(x)\right|^2 dx$ exists and is finite for some $x_0' \in (a,b)$) for all $f \in \mathcal{D}'$.

We note that if $\ell[\cdot]$ is Dirichlet at both $x = a$ and $x = b$ in $\mathcal{D}'$, then

$$\sum_{j=0}^{N} \int_a^b a_j(x)\left|f^{(j)}(x)\right|^2 dx$$

exists and is finite for all $f \in \mathcal{D}'$.

We are now in position to motivate the origins of left-definite theory from the viewpoint of differential equations. Suppose A is a self-adjoint operator with (dense) domain $\mathcal{D}(A)$ in $L^2(I;w)$ generated by the differential expression $\ell[\cdot]$ defined in (2.1). In addition, suppose that $\ell[\cdot]$ is strong limit-point and Dirichlet at both $x = a$ and $x = b$ in $\mathcal{D}(A)$. We then see, from (2.6), by letting $c \to a^+$ and $d \to b^-$ that

$$(Af, g)_w = \int_a^b \ell[f](x)\overline{g}(x)w(x)dx = \sum_{j=0}^{N} \int_a^b a_j(x)f^{(j)}(x)\overline{g}^{(j)}(x)dx \quad (f, g \in \mathcal{D}(A)).$$

Moreover, from the coefficient conditions in (2.2), we see that

$$\begin{aligned}(Af, f)_w = (\ell[f], f)_w &= \sum_{j=0}^{N} \int_a^b a_j(x) \left| f^{(j)}(x) \right|^2 dx \geq \int_a^b a_0(x) \left| f(x) \right|^2 dx \\ &\geq k \int_a^b \left| f(x) \right|^2 w(x)dx = k(f, f)_w \geq 0 \quad (f \in \mathcal{D}(A)).\end{aligned}$$

In particular, we see that the bilinear form $(\cdot, \cdot)_1 : \mathcal{D}(A) \times \mathcal{D}(A) \to \mathbb{C}$ defined by

$$(f, g)_1 := (Af, g)_w \quad (f, g \in \mathcal{D}(A))$$

is an inner product; we call $(\cdot, \cdot)_1$ the first left-definite inner product associated with the pair $(L^2(I; w), A)$. To be correct, the adjective 'first' is not mentioned in the earlier literature on left-definite theory. As we shall see in the following sections, there is a continuum of left-definite inner products $(\cdot, \cdot)_r$ $(r > 0)$ associated with $(L^2(I; w), A)$.

Endow $\mathcal{D}(A)$ with the inner product $(\cdot, \cdot)_1$ and let

$$H_1 = \overline{\mathcal{D}(A)},$$

where the closure is with respect to the topology induced from the inner product $(\cdot, \cdot)_1$. The Hilbert space $(H_1, (\cdot, \cdot)_1)$ is called the (first) left-definite space associated with the pair $(L^2(I; w), A)$. For $f \in \mathcal{D}(A) \subseteq H_1$, it is generally the case that $Af \notin H_1$; that is to say, the operator A is not a linear operator *in* H_1. A natural question to ask is whether there is a self-adjoint operator in H_1 that is generated by the differential expression $\ell[\cdot]$. We show in the upcoming sections that, in fact, there is a unique self-adjoint operator A_1 in H_1 that is a restriction of the operator A; this operator A_1 is called the (first) left-definite operator associated with $(L^2(I; w), A)$.

Before proceeding to the next section, we make one final remark that explains the terminology 'left-definite'. Our use of the notation, and the companion terminology 'right-definite' has a very simple explanation. Indeed, the spectral equation $\ell[y] = \lambda y$, where $\ell[\cdot]$ is defined in (2.1), can be rewritten as

$$\sum_{j=0}^{N} (-1)^j \left(a_j(x)y^{(j)}(x) \right)^{(j)} = \lambda w(x)y(x) \quad (x \in I). \tag{2.7}$$

Since the (first) left-definite inner product is generated by the *left* hand side of this spectral equation, the terminology 'left-definite' seems rather natural; similarly, since the weight $w(x)$ appears on the *right* hand side of (2.7), it is natural to call the classical $L^2(I; w)$ setting the 'right-definite' setting.

3. Definitions and Main Results

Let V be a vector space (over the complex field $\mathbb{C}$) with inner product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$; the resulting inner product space is denoted by $(V, (\cdot, \cdot))$. Suppose V_r (the subscript will be made clear shortly) is a subspace of V and let $(\cdot,\cdot)_r$ be an inner product (quite possibly different from $(\cdot,\cdot)$) on V_r. We begin with the following definition.

Definition 3.1. *Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number k; i.e.*

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

Let $H_1 = (V_1, (\cdot,\cdot)_1)$, where V_1 is a subspace of V and $(\cdot,\cdot)_1$ is an inner product on V_1. Then H_1 is said to be a left-definite (Hilbert) space associated with the pair (H, A) if each of the following conditions hold:

1. *H_1 is a Hilbert space,*
2. *$\mathcal{D}(A)$ is a subspace of V_1,*
3. *$\mathcal{D}(A)$ is dense in H_1,*
4. *$(x,x)_1 \geq k\,(x,x) \quad (x \in V_1)$, and*
5. *$(x,y)_1 = (Ax, y) \quad (x \in \mathcal{D}(A),\ y \in V_1)$.*

If A is a self-adjoint operator in H that is bounded below by a positive number k then, from the Hilbert space spectral theorem (see [34]), we see that A^r is a self-adjoint operator bounded below by $k^r I$ for each $r > 0$. Consequently, we can generalize Definition 3.1.

Definition 3.2. *Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number k; i.e.*

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

Let $r > 0$. If there exists a subspace V_r of V and an inner product $(\cdot,\cdot)_r$ on V_r such that $H_r = (V_r, (\cdot,\cdot)_r)$ is a left-definite space associated with the pair (H, A^r), we call H_r an r^{th} left-definite space associated with the pair (H, A). Specifically, H_r is an r^{th} left-definite space associated with the pair (H, A) if each of the following conditions hold:

1. *H_r is a Hilbert space,*
2. *$\mathcal{D}(A^r)$ is a subspace of V_r,*
3. *$\mathcal{D}(A^r)$ is dense in H_r,*
4. *$(x,x)_r \geq k^r\,(x,x) \quad (x \in V_r)$, and*
5. *$(x,y)_r = (A^r x, y) \quad (x \in \mathcal{D}(A^r),\ y \in V_r)$.*

Given an arbitrary self-adjoint operator A that is bounded below by a positive constant, it is not clear that, for $r > 0$, a left-definite space H_r exists for the pair (H, A). However, as we see in the statement of Theorem 3.1 below, such a space not only exists but it is unique.

In the case of A being generated by a Lagrangian symmetric expression $\ell[\cdot]$ (as in (2.1)), part (5) of the above definition shows that it is essential to know explicit powers of $\ell[\cdot]$ in order to determine the r^{th} left-definite inner product $(\cdot,\cdot)_r$; furthermore, these powers must be known in their Lagrangian symmetric form. To this end, we refer the reader to the recent contribution [13] where, in particular, it is shown that if $\ell[\cdot]$ is Lagrangian symmetrizable with symmetry factor w, then each integral power $\ell^n[\cdot]$ $(n \in \mathbb{N})$ is also Lagrangian symmetrizable with symmetry factor w. Determining these powers turns out to be a difficult, but interesting, combinatorial problem; we consider several examples in Section 4 below. It is also important to note that, in many of our examples of self-adjoint operators generated from Lagrangian symmetric differential expressions $\ell[\cdot]$, it will not be possible to explicitly determine the r^{th} left-definite space H_r when r is non-integral. Indeed, when r is not a positive integer, it is difficult to get meaningful information about the expression $\ell^r[\cdot]$.

Theorem 3.1. *Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot, \cdot))$ that is bounded below by kI, where $k > 0$. Let $r > 0$. Define $H_r = (V_r, (\cdot, \cdot)_r)$ by*

$$V_r = \mathcal{D}(A^{r/2}), \tag{3.1}$$

and

$$(x, y)_r = (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r). \tag{3.2}$$

Then H_r is an r^{th} left-definite space associated with the pair (H, A) in the sense of Definition 3.2. Moreover, suppose $H_r = (V_r, (\cdot, \cdot)_r)$ and $H_r' = (V_r', (\cdot, \cdot)_r')$ are r^{th} left-definite spaces associated with the pair (H, A). Then $V_r = V_r'$ and $(x, y)_r = (x, y)_r'$ for all $x, y \in V_r = V_r'$; i.e. $H_r = H_r'$. Consequently $H_r = (V_r, (\cdot, \cdot)_r)$, as defined in (3.1) and (3.2), is the unique r^{th} left-definite Hilbert space associated with (H, A).

Proof. see [25, Theorem 1]. □

We are now in position to define a left-definite operator associated with A.

Definition 3.3. *Let $H = (V, (\cdot, \cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number k. Let $r > 0$ and let H_r be the r^{th} left-definite space associated with (H, A). If there exists a self-adjoint operator $A_r : H_r \to H_r$ that is a restriction of A; that is to say,*

$$A_r x = Ax \tag{3.3}$$
$$x \in \mathcal{D}(A_r) \subset \mathcal{D}(A),$$

we call such an operator an r^{th} left-definite operator associated with (H, A).

As the following theorem shows, there exists a unique left-definite operator A_r in H_r associated with (H, A).

Theorem 3.2. *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI for some $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot, \cdot)_r)$ be the r^{th} left-definite space associated with (H, A). Then there exists a unique left-definite operator A_r in H_r associated with (H, A). More specifically, if there exists a self-adjoint operator $\widetilde{A}_r : H_r \to H_r$ such that $\widetilde{A}_r x = Ax$ for all $x \in \mathcal{D}(\widetilde{A}_r) \subset \mathcal{D}(A)$, then $A_r = \widetilde{A}_r$. Furthermore,*

$$\mathcal{D}(A_r) = V_{r+2}, \tag{3.4}$$

and A_r is bounded below by kI in H_r.

Proof. see [25, Theorem 2]. □

The following corollary is an immediate consequence of Theorems 3.1 and 3.2.

Corollary 3.3. *Suppose A is a self-adjoint operator in the Hilbert space H that is bounded below by kI, where $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot, \cdot)_r)$ and A_r denote, respectively, the r^{th} left-definite space and r^{th} left-definite operator associated with (H, A). Then*

1. $\mathcal{D}(A^r) = V_{2r}$; *in particular,* $\mathcal{D}(A^{1/2}) = V_1$ *and* $\mathcal{D}(A) = V_2$;
2. $\mathcal{D}(A_r) = \mathcal{D}(A^{(r+2)/2})$; *in particular,* $\mathcal{D}(A_1) = \mathcal{D}(A^{3/2}) = V_3$ *and* $\mathcal{D}(A_2) = \mathcal{D}(A^2) = V_4$.

In the next theorem, we see that when A is a bounded, self-adjoint operator that is bounded below by a positive constant k, then the left-definite theory is trivial. However, the situation is very different when A is unbounded; indeed, left-definite theory in this case is rich.

Theorem 3.4. *Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI for some $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ and A_r denote the r^{th} left-definite space and r^{th} left-definite operator, respectively, associated with (H, A).*

(1) *Suppose A is bounded. Then, for each $r > 0$,*
   (i) $V = V_r$;
   (ii) *the inner products $(\cdot,\cdot)$* and *$(\cdot,\cdot)_r$ are equivalent*;
   (iii) $A = A_r$.
(2) *Suppose A is unbounded. Then*
   (i) *V_r is a proper subspace of V;*
   (ii) *V_s is a proper subspace of V_r whenever $0 < r < s$;*
   (iii) *the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_s$ are not equivalent for any $s > 0$;*
   (iv) *the inner products $(\cdot,\cdot)_r$ and $(\cdot,\cdot)_s$ are not equivalent for any $r, s > 0$, $r \neq s$;*
   (v) *$\mathcal{D}(A_r)$ is a proper subspace of $\mathcal{D}(A)$ for each $r > 0$;*
   (vi) *$\mathcal{D}(A_s)$ is a proper subspace of $\mathcal{D}(A_r)$ whenever $0 < r < s$.*

Proof. see [25, Section 8]. □

In addition, we list the following two theorems concerning the spectra of the left-definite operators $\{A_r\}_{r>0}$.

Theorem 3.5. *For each $r > 0$, let A_r denote the r^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by kI where $k > 0$. Then*

(a) *the point spectra of A and A_r coincide; i.e. $\sigma_p(A) = \sigma_p(A_r)$;*
(b) *the continuous spectra of A and A_r coincide; i.e. $\sigma_c(A) = \sigma_c(A_r)$;*
(c) *the resolvents of A and A_r coincide; i.e. $\rho(A) = \rho(A_r)$.*

Proof. see [25, Section 10]. □

The last result in this section is the following theorem; it is precisely this theorem that is interesting from the viewpoint of orthogonal polynomials.

Theorem 3.6. *Suppose A, H, $\{H_r\}_{r>0}$, and $\{A_r\}_{r>0}$ are as in Theorems* 3.1 *and* 3.2 *above. If $\{\varphi_m\}_{m=0}^{\infty}$ is a (complete) orthogonal set of eigenfunctions of A in H, then for each $r > 0$, $\{\varphi_m\}_{m=0}^{\infty}$ is a (complete) set of orthogonal eigenfunctions of the r^{th} left-definite operator A_r in the r^{th} left-definite space H_r.*

4. Examples

4.1. **A Well Known Example in ℓ^2.** Specific details of this example can be found in [25, Section 11]. Let ℓ^2 denote the usual Hilbert space of square-summable sequences of complex numbers with inner product

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

for $x = (x_n)_{n=1}^{\infty} = (x_1, x_2, \ldots, x_n, \ldots)$ and $y = (y_n)_{n=1}^{\infty} = (y_1, y_2, \ldots, y_n, \ldots) \in \ell^2$. Define $A : \ell^2 \to \ell^2$ by

$$Ax = (x_1, 2x_2, \ldots, nx_n, \ldots),$$

for

$$x \in \mathcal{D}(A) = \{x = (x_n)_{n=1}^{\infty} \in \ell^2 \mid \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty\}.$$

It is not difficult to show that A is an unbounded, self-adjoint operator with spectrum $\sigma(A) = \mathbb{N}$ and corresponding eigenfunctions $\{e_m\}_{m=1}^{\infty}$, where $e_m = \{\delta_{mj}\}_{j=1}^{\infty}$ and δ_{mj} denotes the standard Kronecker delta symbol. Moreover,

$$(Ax, x) = \sum_{n=1}^{\infty} n |x_n|^2 \geq \sum_{n=1}^{\infty} |x_n|^2 = (x, x),$$

so A is bounded below by $1I$ in ℓ^2.

In this example, it is possible to explicitly determine, for all $r > 0$, each left-definite space H_r and each left-definite operator A_r. Indeed, we can determine these spaces and operators for all $r > 0$ since we can explicitly determine the spectral resolution of the identity of A in this case. With details in [25], the r^{th} left-definite space is given by $H_r = (V_{r,}(\cdot, \cdot)_r)$, where

$$V_r = \{x = (x_n)_{n=1}^{\infty} \in \ell^2 \mid \sum_{n=1}^{\infty} n^r |x_n|^2 < \infty\},$$

and $(\cdot, \cdot)_r : V_r \times V_r \to \mathbb{C}$ is given by

$$(x, y)_r = \sum_{n=1}^{\infty} n^r x_n \overline{y_n} \quad (x = (x_n)_{n=1}^{\infty}, y = (y_n)_{n=1}^{\infty} \in V_r).$$

The r^{th} left-definite operator $A_r : H_r \to H_r$ is explicitly given by

$$A_r x = (x_1, 2x_2, \ldots, nx_n, \ldots) \quad (x = (x_n)_{n=1}^{\infty} \in \mathcal{D}(A_r)),$$

where

$$\mathcal{D}(A_r) = V_{r+2} = \{x = (x_n)_{n=1}^{\infty} \in \ell^2 \mid \sum_{n=1}^{\infty} n^{r+2} |x_n|^2 < \infty\}.$$

We remark that the sequence $\{e_m\}_{m=1}^{\infty}$ is a complete set of vectors in each space H_r; in fact, they also form a complete set of eigenfunctions of each left-definite operator A_r.

4.2. **Laguerre's Differential Equation and Laguerre Polynomials for $\alpha > -1$.** For complete details on this example, see [25, Section 12]. The Laguerre differential expression is defined to be

$$\ell_{\text{Lag}}[y](x) := \frac{1}{x^{\alpha} e^{-x}} \left(-(x^{\alpha+1} e^{-x} y'(x))' + k x^{\alpha} e^{-x} y(x) \right) \quad (x \in (0, \infty)), \tag{4.1}$$

where, in this example, we assume that $\alpha > -1$ (in the next example, we will consider this expression when $-\alpha \in \mathbb{N}$). In various areas of mathematics and mathematical physics, the Laguerre differential equation

$$\ell_{\text{Lag}}[y](x) = \lambda y(x) \quad (x \in (0, \infty)), \tag{4.2}$$

or, equivalently,

$$-xy''(x) + (x - 1 - \alpha)y'(x) + ky(x) = \lambda y(x) \quad (x \in (0, \infty)),$$

is important. Part of the importance of this equation is that the Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$ are eigenfunctions of (4.2). Specifically, $y = L_m^\alpha(x)$ is a solution of (4.2) when $\lambda = m + k$. The right-definite setting for this differential expression is the Hilbert space $L_\alpha^2(0, \infty) = L^2((0, \infty); x^\alpha e^{-x})$ with inner product

$$(f, g) = \int_0^\infty f(x)\overline{g}(x)x^\alpha e^{-x}dx \quad (f, g \in L_\alpha^2(0, \infty));$$

in this space, the Laguerre polynomials are well known to form a complete orthonormal set when appropriately normalized; we refer the reader to [32, Chapter 12] or [36, Chapter V] for various properties of the Laguerre polynomials.

With the maximal domain Δ of $\ell_{\mathrm{Lag}}[\cdot]$ in $L_\alpha^2(0, \infty)$ defined to be

$$\Delta = \{f \in L_\alpha^2(0, \infty) \mid f, f' \in AC_{\mathrm{loc}}(0, \infty); \ell_{\mathrm{Lag}}[f] \in L_\alpha^2(0, \infty)\},$$

we define the operator $A : L_\alpha^2(0, \infty) \to L_\alpha^2(0, \infty)$ by

$$Af(x) = \ell_{\mathrm{Lag}}[f](x) \quad (f \in \mathcal{D}(A)), \tag{4.3}$$

where the domain of A is given by

$$\mathcal{D}(A) = \begin{cases} \{f \in \Delta \mid \ \lim_{x \to 0^+} x^{\alpha+1}e^{-x}f'(x) = 0\} & \text{if } -1 < \alpha < 1 \\ \Delta & \text{if } \alpha \geq 1. \end{cases} \tag{4.4}$$

Then, as an application of the Glazman-Krein-Naimark theory [28, Theorem 4, Section 18.1], A is a self-adjoint operator and has the Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$ as a complete set of eigenfunctions; moreover, the spectrum of A is given by $\sigma(A) = \{m + k \mid m \in \mathbb{N}_0\}$. For further details on the spectral theory of the Laguerre equation and other second-order classical differential equations, the reader is referred to [2, Appendix II, Section 9] and the account in [29].

It is well-known that

$$(Af, f) = \int_0^\infty \left[t^{\alpha+1}e^{-t}\left|f'(t)\right|^2 + kt^\alpha e^{-t}\left|f(t)\right|^2\right]dt \geq k(f, f) \qquad (f \in \mathcal{D}(A));$$

that is, A is bounded below in $L_\alpha^2(0, \infty)$ by kI. Consequently, the left-definite theory applies to this particular operator.

In order to compute the left-definite spaces, left-definite inner products, and left-definite operators, we need to compute powers of $\ell_{\mathrm{Lag}}[\cdot]$. Furthermore, as noted in the last section, since the spectral resolution identity of A is not explicitly known, we are limited to computing only *integral* powers of $\ell_{\mathrm{Lag}}[\cdot]$. However, even with this limitation, the results are interesting from several points of view. Indeed, the classical Stirling numbers of the second kind appear in these integral powers of $\ell_{\mathrm{Lag}}[\cdot]$. Specifically, for each $n \in \mathbb{N}$,

$$\ell_{\mathrm{Lag}}^n[y] = \frac{1}{x^\alpha e^{-x}} \sum_{j=0}^n (-1)^j \left(c_j(n, k)x^{\alpha+j}e^{-x}y^{(j)}(x)\right)^{(j)},$$

where

$$c_0(n, k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases} \tag{4.5}$$

and, for $j \in \{1, 2, \ldots n\}$,

$$c_j(n,k) = \begin{cases} S_n^{(j)} & \text{if } k = 0 \\ \sum_{m=0}^{n-1} \binom{n}{m} S_{n-m}^{(j)} k^m & \text{if } k > 0, \end{cases} \tag{4.6}$$

where $S_n^{(j)}$ is the Stirling number of the second kind, defined by

$$S_n^{(j)} := \sum_{i=0}^{j} \frac{(-1)^{i+j}}{j!} \binom{j}{i} i^n \quad (n, j \in \mathbb{N}_0). \tag{4.7}$$

By definition, $S_n^{(j)}$ is the number of ways of partitioning n elements into j non-empty subsets (in particular, $S_0^j = 0$ for any $j \in \mathbb{N}$); we refer the reader to [1, pp. 824-825] and [7] for various properties of these numbers. It is important to note that the numbers $c_j(n,k)$ $(j = 0, 1, \ldots n)$ are all non-negative; in fact, when $k > 0$, each $c_j(n,k) > 0$.

With the details explicitly given in [25, Section 12], we note that, for each $n \in \mathbb{N}$, the n^{th} left-definite space $H_n = (V_n, (\cdot,\cdot)_n)$ associated with $(L^2((0,\infty); x^\alpha e^{-x}), A)$ is given by

$$\begin{aligned} V_n &= \{f : (0,\infty) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\text{loc}}(0,\infty); f^{(j)} \in L^2_{\alpha+j}(0,\infty) \ (j = 0, 1, \ldots n)\} \\ &= \{f : (0,\infty) \to \mathbb{C} \mid f^{(j)} \in AC_{\text{loc}}(0,\infty) \ (j = 0, 1, \ldots n-1); f^{(n)} \in L^2_{n+\alpha}(0,\infty)\}, \end{aligned} \tag{4.8}$$

and

$$(f,g)_n := \sum_{j=0}^{n} c_j(n,k) \int_0^\infty f^{(j)}(t) \overline{g}^{(j)}(t) t^{\alpha+j} e^{-t} dt \quad (f, g \in V_n),$$

where $L^2_{\alpha+j}(0,\infty) = L^2((0,\infty); x^{\alpha+j} e^{-x})$. The second identity in (4.8) follows from the Chisholm-Everitt-Littlejohn inequality (see [6]). It is the case that the Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$ form a complete orthogonal set in each H_n; in fact,

$$(L_m^\alpha, L_r^\alpha)_n = \sum_{j=0}^{n} c_j(n,k) \int_0^\infty \frac{d^j L_m^\alpha(t)}{dt^j} \frac{d^j L_r^\alpha(t)}{dt^j} t^{\alpha+j} e^{-t} dt = (m+k)^n \delta_{m,r}.$$

Define $A_n : \mathcal{D}(A_n) \subset H_n \to H_n$ by

$$A_n f(x) = \ell_{\text{Lag}}[f](x) \quad (\text{a.e. } x \in (0,\infty))$$

for

$$f \in \mathcal{D}(A_n) = \{f : (0,\infty) \to \mathbb{C} \mid f, f', \ldots, f^{(n+1)} \in AC_{\text{loc}}(0,\infty); f^{(n+2)} \in L^2_{\alpha+n+2}(0,\infty)\}. \tag{4.9}$$

Then A_n is the n^{th} left-definite operator associated with the pair $(L^2_\alpha(0,\infty), A)$. Furthermore, the Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$ are eigenfunctions of A_n and the spectrum of A_n is explicitly given by $\sigma(A_n) = \{m + k \mid m \in \mathbb{N}_0\}$.

The Laguerre example was the first detailed application of the general left-definite theory developed in [25]. It was of great surprise to both authors that explicit information could be obtained - and in such a relatively straightforward manner - about the left-definite spaces $\{H_n\}_{n=1}^\infty$ in this case. Moreover, it was unexpected to see the connection between the powers of the second-order classical Laguerre differential expression and the Stirling numbers of the second kind; this is a new application of these combinatorial numbers. As a consequence, as can be seen from Corollary 3.3, we obtain explicit characterizations of the domain of each power $A^{n/2}$ for $n \in \mathbb{N}$. In particular, the characterizations

$$\mathcal{D}(A^{1/2}) = \{f : (0,\infty) \to \mathbb{C} \mid f \in AC_{\mathrm{loc}}(0,\infty); f' \in L^2_{\alpha+1}(0,\infty)\},$$

and

$$\mathcal{D}(A) = \{f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\mathrm{loc}}(0,\infty); f'' \in L^2_{\alpha+2}(0,\infty)\} \tag{4.10}$$

are new; compare the representation in (4.10) with (4.4).

4.3. **Laguerre's Differential Equation and Laguerre Polynomials for $-\alpha \in \mathbb{N}$.** The complete details on the analysis for this example may be found in the contributions [15] and [17].

In this section, we fix $-\alpha \in \mathbb{N}$; in this case, the Laguerre polynomials $\{L_m^\alpha\}_{m=-\alpha}^\infty$ form a complete orthogonal set in the Hilbert space $L^2_\alpha(0,\infty) = L^2((0,\infty); x^\alpha e^{-x})$; this is a consequence of the remarkable identity (see [36, p. 102])

$$L_n^\alpha(x) = (-1)^\alpha \frac{(n+\alpha)!}{n!} x^{-\alpha} L_{n+\alpha}^{-\alpha}(x)$$

and the classical orthogonality of $\{L_n^{-\alpha}\}_{n=0}^\infty$ with respect to the positive measure $d\mu = x^{-\alpha}e^{-x}dx$. However $L_n^\alpha \notin L^2_\alpha(0,\infty) \quad (n = 0, 1, \ldots, -\alpha - 1)$ as can easily be seen by a direct calculation.

From the Glazman-Krein-Naimark theory, the operator $A : \mathcal{D}(A) \subset L^2_\alpha(0,\infty) \to L^2_\alpha(0,\infty)$ defined by

$$\begin{aligned} Af &= \ell[f] = \frac{1}{x^\alpha e^{-x}} \left(-(x^{\alpha+1}e^{-x}y'(x))' + kx^\alpha e^{-x}y(x)\right) \\ f &\in \mathcal{D}(A), \end{aligned}$$

where

$$\mathcal{D}(A) = \{f : (0,\infty) \to C \mid f, f' \in AC_{\mathrm{loc}}(0,\infty); f, \ell[f] \in L^2_\alpha(0,\infty)\}$$

is self-adjoint with discrete spectrum $\sigma(A) = \{m + k \mid m \in \mathbb{N}, m \geq -\alpha\}$; the Laguerre polynomials $\{L_m^\alpha\}_{m=-\alpha}^\infty$ are eigenfunctions, and A is bounded below in $L^2_\alpha(0,\infty)$ by kI. Consequently, there is a continuum of left-definite spaces $\{H_r\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$ associated with $(L^2_\alpha(0,\infty), A)$. As in the last example, we can compute these spaces only for $r \in \mathbb{N}$; in fact, for $n \in \mathbb{N}$, the n^{th} left-definite space associated with $(L^2_\alpha(0,\infty), A)$ is given by $H_n = (V_n, (\cdot,\cdot)_n)$, where

$$V_n := \{f : (0,\infty) \to \mathbb{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(0,\infty) \ (j = 0, 1, \ldots, n-1); f^{(n)} \in L^2_{n+\alpha}(0,\infty)\}, \tag{4.11}$$

and

$$(f,g)_n := \sum_{j=0}^n c_j(n,k) \int_0^\infty f^{(j)}(t)\overline{g}^{(j)}(t)t^{\alpha+j}e^{-t}dt \quad (f, g \in V_n); \tag{4.12}$$

where $c_0(n,k)$ and $c_j(n,k)$ are given in (4.5) and (4.6), respectively. In particular, the $(-\alpha)^{th}$ left-definite space $H_{-\alpha} = (V_{-\alpha}, (\cdot,\cdot)_{-\alpha})$ associated with $(L^2_\alpha(0,\infty), A)$ is given by

$$V_{-\alpha} := \{f : (0,\infty) \to \mathbb{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(0,\infty) \ (j = 0, 1, \ldots, -\alpha-1); f^{(-\alpha)} \in L^2((0,\infty); e^{-x})\}, \tag{4.13}$$

and

$$(f,g)_{-\alpha} := \sum_{j=0}^{-\alpha} c_j(-\alpha,k) \int_0^\infty f^{(j)}(t)\overline{g}^{(j)}(t)t^{\alpha+j}e^{-t}dt \quad (f, g \in V_{-\alpha}). \tag{4.14}$$

The proofs of (4.13) and (4.14) follow *mutatis mudandis* for similar results given in the last section. The difference, however, between this example and the classical Laguerre case in the previous example lies in the results that we now discuss.

Kwon and Littlejohn showed that the *entire* set of Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$ are orthonormal with respect to the inner product

$$\begin{aligned}(f,g)_{\mathrm{Lag},\alpha} := &\sum_{r=0}^{-\alpha-1}\sum_{j=0}^{r} B_{r,j}(\alpha)\left[f^{(r)}(0)\overline{g}^{(j)}(0)+f^{(j)}(0)\overline{g}^{(r)}(0)\right]\\ &+\int_0^\infty f^{(-\alpha)}(t)\overline{g}^{(-\alpha)}(t)e^{-t}dt\end{aligned}$$

where the numbers $B_{r,j}(\alpha)$ are given by

$$B_{r,j}(\alpha)=\begin{cases}\sum_{p=0}^{j}(-1)^{r+j}\binom{-\alpha-1-p}{r-p}\binom{-\alpha-1-p}{j-p} & \text{if } 0\le j<r\le-\alpha-1\\ \frac{1}{2}\sum_{p=0}^{r}\binom{-\alpha-1-p}{r-p}^2 & \text{if } 0\le j=r\le-\alpha-1.\end{cases}$$

In fact, the Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$, as defined in [36], form a complete orthonormal sequence in the Hilbert-Sobolev space $(W_\alpha,(\cdot,\cdot)_{\mathrm{Lag},\alpha})$ where

$$W_\alpha=\{f:[0,\infty)\to\mathbb{C}\mid f^{(j)}\in AC_{loc}[0,\infty)\ (j=0,1,\ldots-\alpha-1); f^{(-\alpha)}\in L^2((0,\infty);e^{-x})\}.$$

It is natural to ask the following question: is there a self-adjoint operator T, generated by the Laguerre differential expression, in W_α that has the *entire* sequence of Laguerre polynomials $\{L_m^{-\alpha}\}_{m=0}^\infty$ as eigenfunctions? The answer is yes. We briefly describe how the authors in [17] construct T.

Notice that if p is a polynomial of degree $\le-\alpha-1$ and q is a smooth function satisfying $q^{(j)}(0)=0$ for $j=0,1,\ldots,-\alpha-1$ and $q^{(-\alpha)}\in L^2((0,\infty);e^{-x})$, then $(p,q)_{\mathrm{Lag},\alpha}=0$. This observation leads to the orthogonal decomposition

$$W_\alpha=W_{\alpha,1}\oplus W_{\alpha,2},$$

where

$$W_{\alpha,1}=\{f\in W_\alpha\mid f^{(j)}(0)=0\ (j=0,1,\ldots,-\alpha-1)\},$$

and

$$W_{\alpha,2}=\{f\in W_\alpha\mid f \text{ is a polynomial of degree } \le-\alpha-1\}.$$

It is the case that $W_{\alpha,1}=\overline{\mathrm{span}\{L_m^\alpha\}_{m=-\alpha}^\infty}$ and $W_{\alpha,2}=\overline{\mathrm{span}\{L_m^\alpha\}_{m=0}^{-\alpha-1}}$, where the closures are with respect to the norm induced by the Sobolev inner product $(\cdot,\cdot)_{\mathrm{Lag},\alpha}$. To construct the self-adjoint operator T in W_α, we construct self-adjoint operators $T_{\alpha,1}$ in $W_{\alpha,1}$ and $T_{\alpha,2}$ in $W_{\alpha,2}$, both generated by the Laguerre differential expression $\ell[\cdot]$, and then we take $T=T_{\alpha,1}\oplus T_{\alpha,2}$.

Constructing $T_{\alpha,2}$ is easy; indeed, simply take $T_{\alpha,2}f=\ell[f]$ for $f\in\mathcal{D}(T_{\alpha,2})=W_{\alpha,2}$; it is easy to check that $T_{\alpha,2}$ is self-adjoint in $W_{\alpha,2}$ and the spectrum of $T_{\alpha,2}$ is given by

$$\sigma(T_{\alpha,2})=\{m+k\mid m=0,1,\ldots,-\alpha-1\}.$$

The more difficult question is how do we construct the self-adjoint operator $T_{\alpha,1}$? Remarkably, as it turns out, the self-adjoint operator $T_{\alpha,1}$ is the $(-\alpha)^{th}$ left-definite operator associated with $(L_\alpha^2(0,\infty),A)$! Indeed, the operator

$$\begin{gathered}T_{\alpha,1}f=\ell[f]\\ f\in\mathcal{D}(T_{\alpha,1})=V_{-\alpha+2}=\{f:(0,\infty)\to\mathbb{C}\mid f^{(j)}\in AC_{\mathrm{loc}}(0,\infty)\ (j=0,1,\ldots,-\alpha+1);\\ f^{(-\alpha+2)}\in L_2^2(0,\infty)\}\end{gathered}$$

is self-adjoint in $W_{\alpha,1}$; furthermore, the Laguerre polynomials $\{L_m^\alpha\}_{m=-\alpha}^\infty$ are eigenfunctions of $T_{\alpha,1}$ and the spectrum of $T_{\alpha,1}$ is given by $\sigma(T_{\alpha,1}) = \{m+k \mid m \in \mathbb{N} \text{ and } m \geq -\alpha\}$.

We now define the operator $T : \mathcal{D}(T) \subset W_\alpha \to W_\alpha$ by

$$Tf = T_{\alpha,1}f_1 + T_{\alpha,2}f_2 \quad (f = f_1 + f_2 \in \mathcal{D}(T_{\alpha,1}) \oplus \mathcal{D}(T_{\alpha,2}))$$
$$\mathcal{D}(T) = \mathcal{D}(T_{\alpha,1}) \oplus \mathcal{D}(T_{\alpha,2}).$$

Observe that $Tf = T_{\alpha,1}f_1 + T_{\alpha,2}f_2 = \ell[f_1] + \ell[f_2] = \ell[f_1+f_2] = \ell[f]$. Furthermore, since we explicitly know the domains of both $T_{\alpha,1}$ and $T_{\alpha,2}$, we can precisely determine $\mathcal{D}(T)$; indeed,

$$\begin{aligned}\mathcal{D}(T) = \{f : [0,\infty) \to \mathbb{C} \mid\ & f^{(j)} \in AC_{\text{loc}}[0,\infty)\ (j = 0, 1, \ldots, -\alpha-1);\\ & f^{(-\alpha+j)} \in AC_{\text{loc}}(0,\infty)\ (j = 0,1);\\ & f^{(-\alpha+j)} \in L^2((0,\infty); x^j e^{-x})\ (j = 0,1,2)\}.\end{aligned}$$

The entire set of Laguerre polynomials $\{L_m^\alpha\}_{m=0}^\infty$ are eigenfunctions of T in W_α and the spectrum of T is given by

$$\sigma(T) = \{m + k \mid m \in \mathbb{N}_0\}.$$

4.4. **Hermite's Differential Equation and Hermite Polynomials.** Complete details of this example may be found in [14]; as shown in this paper, there is close similarity between the Hermite differential expression and the Laguerre case in Example 4.2. The second-order Hermite differential expression is defined to be

$$\begin{aligned}\ell_H[y](x) &:= -y''(x) + 2xy'(x) + ky(x)\\ &= \exp(x^2)\left(-(\exp(-x^2)y'(x))' + k\exp(-x^2)y(x)\right) \quad (x \in \mathbb{R} = (-\infty,\infty)).\end{aligned} \tag{4.15}$$

Again, the parameter k is a fixed, non-negative constant. One solution of the equation

$$\ell_H[y](x) = (2n+k)y(x) \quad (x \in \mathbb{R})$$

is $y = H_m(x)$, the m^{th} degree Hermite polynomial. Properly normalized, the sequence $\{H_m\}_{m=0}^\infty$ is orthonormal in the Hilbert space $H = L^2((-\infty,\infty); \exp(-x^2))$ with inner product

$$(f,g) := \int_{-\infty}^{\infty} f(t)\overline{g}(t)\exp(-t^2)dt \quad (f,g \in H).$$

This space is the appropriate right-definite setting for the study of $\ell_H[\cdot]$. In fact, there is a unique self-adjoint operator $A : \mathcal{D}(A) \subseteq H \to H$ generated by $\ell_H[\cdot]$; indeed, it is given by

$$\begin{aligned} Af &= \ell_H[f]\\ \mathcal{D}(A) &= \{f : \mathbb{R} \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(\mathbb{R}); f, \ell_H[f] \in H\}.\end{aligned} \tag{4.16}$$

The Hermite polynomials $\{H_m\}_{m=0}^\infty$ form a complete orthonormal set of eigenfunctions of A; moreover, the spectrum of A is discrete and given by $\sigma(A) = \{2m + k \mid m \in \mathbb{N}_0\}$. Furthermore, A is bounded below by kI in H. Consequently, there is a sequence of left-definite spaces $\{H_n\}_{n=1}^\infty$ associated with $(L^2((-\infty,\infty); \exp(-x^2)), A)$ which we can compute from the integral powers of the Hermite expression $\ell_H[\cdot]$. These computations are similar to those in Section 4.2; indeed, for $n \in \mathbb{N}$, it is the case that

$$\ell_H^n[y](x) = \frac{1}{\exp(-x^2)}\sum_{j=0}^{n}(-1)^j\left(c_j(n,k)\exp(-x^2)y^{(j)}(x)\right)^{(j)} \quad (x \in \mathbb{R}), \tag{4.17}$$

where

$$c_0(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases} \tag{4.18}$$

and, for $j = 1, 2, \ldots n$,

$$c_j(n,k) = \begin{cases} 2^{n-j} S_n^{(j)} & \text{if } k = 0 \\ 2^{n-j} \sum_{m=0}^{n-1} \binom{n}{m} S_{n-m}^{(j)} \left(\frac{k}{2}\right)^m & \text{if } k > 0, \end{cases} \tag{4.19}$$

where $\{S_n^{(j)}\}$ are the Stirling numbers of the second kind defined in (4.7). Furthermore, for each $n \in \mathbb{N}$, the n^{th} left-definite Hilbert space $H_n = (V_n, (\cdot,\cdot)_n)$ is given by

$$V_n = \{f : \mathbb{R} \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\text{loc}}(\mathbb{R});\, f^{(n)} \in L^2(\mathbb{R}; \exp(-t^2))\}, \tag{4.20}$$

and

$$(f,g)_n = \sum_{j=0}^{n} c_j(n,k) \int_{-\infty}^{\infty} f^{(j)}(t)\overline{g}^{(j)}(t) \exp(-t^2)dt \quad (f, g \in V_n).$$

In particular, we obtain the following new characterization of the domain of the operator A, defined in (4.16):

$$\mathcal{D}(A) = V_2 = \{f : \mathbb{R} \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(\mathbb{R});\, f'' \in L^2(\mathbb{R}; \exp(-t^2))\}.$$

We note that the Hermite polynomials $\{H_m\}_{m=0}^{\infty}$ are a complete orthogonal set in each H_n; in fact,

$$(H_m, H_r)_n = \sum_{j=0}^{n} c_j(n,k) \int_{-\infty}^{\infty} \frac{d^j(H_m(t))}{dt^j} \frac{d^j(H_r(t))}{dt^j} \exp(-t^2)dt = (2m+k)^n \delta_{m,r}.$$

Lastly, we note that the n^{th} left-definite self-adjoint operator $A_n : \mathcal{D}(A_n) \subseteq H_n \to H_n$ is given explicitly by

$$A_n f = \ell_H[f]$$
$$f \in \mathcal{D}(A_n) = V_{n+2} = \{f : \mathbb{R} \to \mathbb{C} \mid f, f', \ldots, f^{(n+1)} \in AC_{\text{loc}}(\mathbb{R});\, f^{(n+2)} \in L^2(\mathbb{R}; \exp(-t^2))\}.$$

4.5. **Legendre's Differential Equation and Legendre Polynomials.** For this example, details can be found in the papers [4] and [16]. The first discussion of the Legendre case in the left-definite setting can be traced to Pleijel ([30] and [31]). The Legendre differential expression is defined to be

$$\begin{aligned} \ell_{\text{Leg}}[y](x) &:= -\left((1-x^2)y'(x)\right)' + ky(x) \\ &= -(1-x^2)y''(x) + 2xy'(x) + ky(x) \quad (x \in (-1,1)); \end{aligned} \tag{4.21}$$

here k is a fixed, non-negative constant. When $\lambda = m(m+1) + k$, the m^{th} Legendre polynomial $y = P_m(x)$ is a solution of

$$\ell_{\text{Leg}}[y](x) = \lambda y(x) \quad (x \in (-1,1)). \tag{4.22}$$

The Legendre polynomials $\{P_m\}_{m=0}^{\infty}$, properly normalized, form a complete orthonormal set in the classical Hilbert space $L^2(-1,1)$. We refer the reader to [32] or [36] for various properties of the Legendre polynomials.

With the maximal domain Δ of $\ell_{\text{Leg}}[\cdot]$ in $L^2(-1,1)$ defined to be

$$\Delta = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1);\; f, \ell_{\text{Leg}}[f] \in L^2(-1,1)\},$$

we define the operator $A : L^2(-1,1) \to L^2(-1,1)$ by

$$Af(x) = \ell_{\mathrm{Leg}}[f](x) \quad (f \in \mathcal{D}(A);\ \text{a.e. } x \in (-1,1)), \tag{4.23}$$

where the domain of A is given by

$$\mathcal{D}(A) = \{f \in \Delta \mid \lim_{x\to\pm 1}(1-x^2)f'(x) = 0\}. \tag{4.24}$$

As an application of the Glazman-Krein-Naimark theory [28, Theorem 4, Section 18.1], A is self-adjoint in $L^2(-1,1)$ and has the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ as a complete set of eigenfunctions. Moreover, the spectrum of A is given by $\sigma(A) = \{m(m+1)+k \mid m \in \mathbb{N}_0\}$ and A is bounded below by kI in $L^2(-1,1)$. For further details on the analysis of Legendre's equation, the reader is referred to [2, Appendix II, Section 9], and the accounts in [27] and [29].

The integral powers of the Legendre expression are given by

$$\ell^n_{\mathrm{Leg}}[y] = \sum_{j=0}^{n}(-1)^j \left(c_j(n,k)x^{\alpha+j}e^{-x}y^{(j)}(x)\right)^{(j)} \quad (n \in \mathbb{N}), \tag{4.25}$$

where

$$c_0(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases} \tag{4.26}$$

and, for $j \in \{1,2,\ldots n\}$,

$$c_j(n,k) = \begin{cases} PS_n^{(j)} & \text{if } k = 0 \\ \sum_{m=0}^{n-1}\binom{n}{m}PS_{n-m}^{(j)}k^m & \text{if } k > 0; \end{cases} \tag{4.27}$$

here $PS_n^{(j)}$ is the Legendre-Stirling number, defined by

$$PS_n^{(j)} = \sum_{m=1}^{j}(-1)^{m+j}\frac{(2m+1)(m^2+m)^n}{(m+j+1)!(j-m)!}. \quad (n,j \in \mathbb{N}). \tag{4.28}$$

As with the Stirling numbers of the second kind, it is the case that $PS_n^{(j)} > 0$ for $n \geq j$.

Recently, George Andrews [3] has obtained a combinatorial interpretation of the Legendre-Stirling numbers; details will be forthcoming. It is interesting that the Legendre-Stirling numbers and the Stirling numbers of the second kind share many similar properties; for further details see [16].

With the details explicitly given in [15], we note that, for each $n \in \mathbb{N}$, the n^{th} left-definite space $H_n = (V_n, (\cdot,\cdot)_n)$ associated with $(L^2((-1,1)), A)$ is given by

$$\begin{aligned} V_n &= \{f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\mathrm{loc}}(-1,1); f^{(j)} \in L^2_j(-1,1)\ (j = 0,1,\ldots n)\} \\ &= \{f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\mathrm{loc}}(-1,1); f^{(n)} \in L^2_n(-1,1)\}, \end{aligned} \tag{4.29}$$

where, for $j \in \mathbb{N}_0$,

$$L^2_j(-1,1) = L^2((-1,1);(1-x^2)^j) \tag{4.30}$$

and, assuming $k > 0$, the n^{th} left-definite inner product is given by

$$(f,g)_n := \sum_{j=0}^{n} c_j(n,k)\int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t^2)^j dt \quad (f,g \in V_n),$$

where $c_0(n,k)$ and $c_j(n,k)$ are given in (4.26) and (4.27), respectively. In particular the domain of A, given in (4.24), has the new characterization

$$\mathcal{D}(A) = V_2 = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); (1-t^2)f'' \in L^2(-1,1)\}. \tag{4.31}$$

We refer the reader to [12] for another proof of this characterization and for an in-depth study of other equivalent determinations of this domain. It is the case that the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ form a complete orthogonal set in each H_n; in fact,

$$(P_m, P_r)_n = \sum_{j=0}^{n} c_j(n,k) \int_0^{\infty} \frac{d^j P_m(t)}{dt^j} \frac{d^j P_r(t)}{dt^j} (1-t^2)^j dt = (m(m+1)+k)^n \delta_{m,r}.$$

Define $A_n : \mathcal{D}(A_n) \subset H_n \to H_n$by

$$A_n f(x) = \ell_{\text{Leg}}[f](x) \quad (\text{a.e. } x \in (-1,1))$$

for

$$f \in \mathcal{D}(A_n) := \{f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n+1)} \in AC_{\text{loc}}(-1,1); f^{(n+2)} \in L^2_{n+2}(-1,1)\}.$$

Then A_n is the n^{th} left-definite self-adjoint operator associated with the pair $(L^2(-1,1), A)$. Furthermore, the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ are eigenfunctions of each A_n and the spectrum of A_n is explicitly given by $\sigma(A_n) = \{m(m+1)+k \mid m \in \mathbb{N}_0\}$. In particular, the domain of the first left-definite operator A_1 is given by

$$\mathcal{D}(A_1) = V_3 = \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{\text{loc}}(-1,1); (1-t^2)^{3/2} f''' \in L^2(-1,1)\};$$

this characterization answers a question posed by Everitt in [8]; see also [12].

4.6. **Jacobi's Differential Equation and Jacobi Polynomials for $\alpha, \beta > -1$.** The full details for this example can be found in [10]. For this section, we assume that $\alpha, \beta > -1$ are fixed. This is the classical 'positive-definite' Jacobi case; the choice $\alpha = \beta = 0$ gives the Legendre case discussed in the previous section. The classical second-order Jacobi differential expression $\ell_{\alpha,\beta}[\cdot]$ is defined by

$$\begin{aligned} \ell_{\alpha,\beta}[y](x) &= \frac{1}{w_{\alpha,\beta}(x)} \left(\left(-(1-x)^{\alpha+1}(1+x)^{\beta+1}) y'(x) \right)' + k(1-x)^{\alpha}(1+x)^{\beta} y(x) \right) \\ &= -(1-x^2)y'' + (\alpha - \beta + (\alpha+\beta+2)x)y'(x) + ky(x) \quad (x \in (-1,1)), \end{aligned} \tag{4.32}$$

where k is a fixed non-negative parameter, and

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta} \quad (x \in (-1,1)). \tag{4.33}$$

With

$$\lambda_m^{(\alpha,\beta)} = m(m+\alpha+\beta+1)+k \quad (m \in \mathbb{N}_0), \tag{4.34}$$

the Jacobi equation

$$\ell_{\alpha,\beta}[y](x) = \lambda_m^{(\alpha,\beta)} y(x) \quad (x \in (-1,1))$$

has $y = P_m^{(\alpha,\beta)}(x)$ as a solution, where $P_m^{(\alpha,\beta)}(x)$ is the m^{th} Jacobi polynomial. Properly normalized, $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ forms a complete orthonormal set in the right-definite setting $L^2((-1,1); w_{\alpha,\beta}(x)) := L^2_{\alpha,\beta}(-1,1)$ with inner product

$$(f,g)_{\alpha,\beta} = \int_{-1}^{1} f(t)\overline{g}(t) w_{\alpha,\beta}(t) dt \quad (f, g \in L^2_{\alpha,\beta}(-1,1)). \tag{4.35}$$

The maximal domain Δ of $\ell_{\alpha,\beta}[\cdot]$ in $L^2_{\alpha,\beta}(-1,1)$ is defined to be

$$\Delta = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1);\; f, \ell_{\alpha,\beta}[f] \in L^2_{\alpha,\beta}(-1,1)\}. \tag{4.36}$$

Applying the GKN theory [28], the operator $A : \mathcal{D}(A) \subset L^2_{\alpha,\beta}(-1,1) \to L^2_{\alpha,\beta}(-1,1)$ defined by

$$Af = \ell_{\alpha,\beta}[f] \tag{4.37}$$

for $f \in \mathcal{D}(A)$, where

$$\mathcal{D}(A_k) = \begin{cases} \Delta & \text{if } \alpha, \beta \geq 1 \\ \{f \in \Delta \mid \lim\limits_{t\to 1}(1-t)^{\alpha+1} f'(t) = 0\} & \text{if } |\alpha| < 1 \text{ and } \beta \geq 1 \\ \{f \in \Delta \mid \lim\limits_{t\to -1}(1+t)^{\beta+1} f'(t) = 0\} & \text{if } |\beta| < 1 \text{ and } \alpha \geq 1 \\ \{f \in \Delta \mid \lim\limits_{t\to \pm 1}(1-t)^{\alpha+1}(1+t)^{\beta+1} f'(t) = 0\} & \text{if } -1 < \alpha, \beta < 1, \end{cases} \tag{4.38}$$

is self-adjoint in $L^2_{\alpha,\beta}(-1,1)$; see [28] and [29]. The Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ form a (complete) set of eigenfunctions of A in $L^2_{\alpha,\beta}(-1,1)$ and the spectrum of A is given by

$$\sigma(A) = \{\lambda_m^{(\alpha,\beta)} \mid m \in \mathbb{N}_0\},$$

where $\lambda_m^{(\alpha,\beta)}$ is defined in (4.34). In particular, we see that $\sigma(A) \subset [k, \infty)$, from which it follows (see [34, Chapter 13]) that A is bounded below by kI in $L^2_{\alpha,\beta}(-1,1)$; that is to say,

$$(Af, f)_{\alpha,\beta} \geq k(f,f)_{\alpha,\beta} \quad (f \in \mathcal{D}(A)). \tag{4.39}$$

Consequently, the left-definite theory discussed in Section 3 can be applied to this self-adjoint operator.

The integral powers of the Jacobi differential expression are given by

$$\ell^n_{\alpha,\beta}[y](x) = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=0}^{n} (-1)^j \left(c_j^{(\alpha,\beta)}(n,k)(1-x)^{\alpha+j}(1+x)^{\beta+j} y^{(j)}(x) \right)^{(j)} \quad (n \in \mathbb{N}_0),$$

where

$$c_0^{(\alpha,\beta)}(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases} \tag{4.40}$$

and

$$c_j^{(\alpha,\beta)}(n,k) = \begin{cases} P^{(\alpha,\beta)}S_n^{(j)} & \text{if } k = 0 \\ \sum_{s=0}^{n-j} \binom{n}{s} P^{(\alpha,\beta)}S_{n-s}^{(j)} k^s & \text{if } k > 0 \end{cases} \quad (j \in \{1, \ldots, n\}). \tag{4.41}$$

In this case, $P^{(\alpha,\beta)}S_n^{(j)}$ is given explicitly by

$$P^{(\alpha,\beta)}S_n^{(j)} = \sum_{r=0}^{j} (-1)^{r+j} \frac{\Gamma(\alpha+\beta+r+1)\Gamma(\alpha+\beta+2r+2)\,[r(r+\alpha+\beta+1)]^n}{r!(j-r)!\Gamma(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+j+r+2)} \tag{4.42}$$

for each $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, n\}$; we call the numbers $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ the Jacobi-Stirling numbers. We note that $P^{(0,0)}S_n^{(j)} = PS_n^{(j)}$, the Legendre-Stirling number defined in (4.28). Developing various properties of these numbers is currently in progress; recent work of Andrews [3] gives a combinatorial interpretation of these numbers.

Define, for each $n \in \mathbb{N}$,

$$\begin{aligned} V_n^{(\alpha,\beta)} &= \{f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\text{loc}}(-1,1); f^{(j)} \in L^2_{\alpha+j,\beta+j}(-1,1) \ (j = 0, 1, \ldots n)\} \\ &= \{f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\text{loc}}(-1,1); (1-t^2)^{n/2} f^{(n)} \in L^2_{\alpha,\beta}(-1,1)\}, \end{aligned}$$

where $L^2_{\alpha+j,\beta+j}(-1,1)$ is defined in (4.30), and let $(\cdot,\cdot)_n^{(\alpha,\beta)}$ be the inner product

$$(f,g)_n^{(\alpha,\beta)} = \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \quad (f,g \in V_n^{(\alpha,\beta)}).$$

Finally, let

$$H_n^{(\alpha,\beta)} = (V_n^{(\alpha,\beta)}, (\cdot,\cdot)_n^{(\alpha,\beta)}) \quad (n \in \mathbb{N}).$$

Then, as shown in [10], $H_n^{(\alpha,\beta)}$ is the n^{th} left-definite space associated with $(L^2_{\alpha,\beta}(-1,1), A)$. Furthermore, the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ form a complete orthogonal set in the space $H_n^{(\alpha,\beta)}$; in fact,

$$\begin{aligned} (P_m^{(\alpha,\beta)}, P_r^{(\alpha,\beta)})_n^{(\alpha,\beta)} &= \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} \frac{d^j(P_m^{(\alpha,\beta)}(t))}{dt^j} \frac{d^j(P_r^{(\alpha,\beta)}(t))}{dt^j}(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \\ &= (m(m+\alpha+\beta+1)+k)^n \delta_{m,r}. \end{aligned}$$

In particular, we note that

$$\mathcal{D}(A) = H_2^{(\alpha,\beta)} = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); (1-t^2)f'' \in L^2_{\alpha,\beta}(-1,1)\};$$

compare the simplicity to this formula to the original one given in (4.38)!

The n^{th} left-definite operator associated with $(L^2_{\alpha,\beta}(-1,1), A)$ is given by

$$A_n : \mathcal{D}(A_n) \subset H_n^{(\alpha,\beta)} \to H_n^{(\alpha,\beta)}$$

by

$$A_n f = \ell_{\alpha,\beta}[f] \quad (f \in \mathcal{D}(A_n) = V_{n+2}^{(\alpha,\beta)}).$$

Then A_n is the n^{th} left-definite operator associated with the pair $(L^2_{\alpha,\beta}(-1,1), A)$. The spectrum of A_n is given by $\sigma(A_n) = \{m(m+\alpha+\beta+1)+k \mid m \in \mathbb{N}_0\}$ and the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ form a complete set of eigenfunctions of each left-definite operator A_n.

4.7. **The Fourier BVP.** One of the most classic, and oft-studied, examples of a two-point self-adjoint boundary value problem is the Fourier boundary value problem

$$\begin{aligned} \ell[y](x) = -y''(x) + ky(x) = \lambda y(x) \quad (x \in [a,b]) \\ y(a) = y(b);\ y'(a) = y'(b); \end{aligned} \tag{4.43}$$

here $[a,b]$ is a compact interval of $\mathbb{R}$ and k is a fixed, non-negative constant. Indeed, the eigenfunction expansion in this case produces the classical Fourier series expansion for $f \in H = L^2[a,b]$ with inner product

$$(f,g) = \int_a^b f(x)\overline{g}(x)dx \quad (f,g \in H).$$

The left-definite theory for this example produces some new, and interesting, information that we discuss in this example; full details may be found in the recent contribution [26].

Fix $k > 0$ and let $\ell[\cdot]$ denote the regular differential expression defined by

$$\ell[f](x) = -f''(x) + kf(x) \quad (x \in [a, b]). \tag{4.44}$$

The operator A that we specifically discuss in this section is defined to be

$$\begin{aligned} &Af = \ell[f] \\ &f \in \mathcal{D}(A) = \{f : [a, b] \to \mathbb{C} \mid f, f' \in AC[a, b]; f'' \in H; f(a) = f(b); f'(a) = f'(b)\}. \end{aligned} \tag{4.45}$$

It is well known (see [28]) that A is self-adjoint in H and has a discrete spectrum $\sigma(A)$. A routine calculation shows that the eigenvalues of A are given by

$$\lambda_m = \left(\frac{2m\pi}{b-a}\right)^2 + k \quad (m \in \mathbb{N}_0). \tag{4.46}$$

The eigenvalue $\lambda_0 = k$ is simple and each nonzero constant is an eigenfunction; we let

$$y_0(x) = 1/\sqrt{2}. \tag{4.47}$$

For $m \in \mathbb{N}$, the two independent solutions of $\ell[f](x) = \lambda_m f(x)$ on $[a, b]$ are

$$\begin{cases} y_{m,1}(x) = \cos\left(\frac{2m\pi}{b-a}x\right) & (m \in \mathbb{N}) \\ y_{m,2}(x) = \sin\left(\frac{2m\pi}{b-a}x\right) & (m \in \mathbb{N}). \end{cases} \tag{4.48}$$

It is well known (see [34, Chapter 4]) that the collection of eigenfunctions

$$E = \{y_0\} \cup \{y_{m,1}\}_{m\in\mathbb{N}} \cup \{y_{m,2}\}_{m\in\mathbb{N}} \tag{4.49}$$

forms a complete orthogonal set of functions in $L^2[a, b]$.

For $f \in \mathcal{D}(A)$, we see from integration by parts and the boundary conditions in (4.45) that

$$\begin{aligned} (Af, f) &= \int_a^b \left[-f''(x) + kf(x)\right] \overline{f}(x)dx = -f'(x)\overline{f}(x) \mid_a^b + \int_a^b \left[\left|f'(x)\right|^2 + k\left|f(x)\right|^2\right] dx \\ &= \int_a^b \left[\left|f'(x)\right|^2 + k\left|f(x)\right|^2\right] dx \geq k \int_a^b \left|f(x)\right|^2 dx = k(f, f); \end{aligned}$$

that is, A is bounded below by kI in H. Consequently, the left-definite theory discussed in the last section can be applied to this operator A.

For each $n \in \mathbb{N}$, a straightforward calculation shows that

$$\ell^n[y](x) = \sum_{j=0}^n (-1)^j \binom{n}{j} k^{n-j} y^{(2j)}(x) \qquad (x \in [a, b]). \tag{4.50}$$

For $n \in \mathbb{N}$, the n^{th} left-definite space $H_n = (V_n, (\cdot,\cdot)_n)$ is given by

$$V_n = \{f : [a, b] \to \mathbb{C} \mid f^{(j)} \in AC[a, b],\ f^{(j)}(a) = f^{(j)}(b)\ (j = 0, 1, \ldots, n-1); f^{(n)} \in L^2[a, b]\}, \tag{4.51}$$

and

$$(f, g)_n = \sum_{j=0}^n \binom{n}{j} k^{n-j} \int_a^b f^{(j)}(x)\overline{g}^{(j)}(x)dx \quad (f, g \in V_n). \tag{4.52}$$

In (4.51), observe that, as n increases, so does the number of boundary conditions in each $V_n = \mathcal{D}(A^{n/2})$. And, in particular, note that the domain of $A^{1/2}$, the positive square root of A, involves only the single separated boundary condition $f(a) = f(b)$.

Lastly, for $n \in \mathbb{N}$, the n^{th} left-definite (self-adjoint) operator $A_n : \mathcal{D}(A_n) \subset H_n \to H_n$, associated with the pair (H, A), is given by

$$\begin{aligned} A_n f &= \ell[f] \\ f &\in \mathcal{D}(A_n), \end{aligned}$$

where

$$\mathcal{D}(A_n) = \{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b],\ f^{(j)}(a) = f^{(j)}(b)\ (j = 0, 1, \ldots, n+1); f^{(n+2)} \in L^2[a,b]\}.$$

Moreover, $\sigma(A_n) = \sigma(A) = \left\{(2m\pi/(b-a))^2 + k \mid m \in \mathbb{N}_0\right\}$ and the set E, defined in (4.49), forms a complete orthogonal set in each H_n.

4.8. **Jacobi's Differential Equation and Jacobi Polynomials For $\alpha, \beta = -1$.** Details of this example can be found in the thesis [5]. For $\alpha = \beta = -1$, the Jacobi differential expression, defined in (4.32) simplifies to

$$\ell_{-1,-1}[y](x) = (1-x^2)\left(-(y'(x))' + k(1-x^2)^{-1}y(x)\right) \quad (x \in (-1,1));$$

here, we assume that k is a fixed, non-negative constant. The right-definite setting for this expression is $L^2\left((-1,1);(1-x^2)^{-1}\right)$; the maximal domain associated with $\ell_{-1,-1}[.]$ is defined to be

$$\Delta^{(-1,-1)} = \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell_{-1,-1}[f] \in L^2\left((-1,1);(1-x^2)^{-1}\right)\};$$

From the GKN theory, the operator

$$A : \mathcal{D}(A) \subset L^2\left((-1,1);(1-x^2)^{-1}\right) \longrightarrow L^2\left((-1,1);(1-x^2)^{-1}\right),$$

defined by

$$\begin{aligned} Af &= \ell_{-1,-1}[f] \\ f &\in \mathcal{D}(A) = \Delta^{(-1,-1)}, \end{aligned} \tag{4.53}$$

is self-adjoint and bounded below by kI in $L^2\left((-1,1);(1-x^2)^{-1}\right)$. However, unlike in the classical case discussed in Section 4.6, the Jacobi polynomial of degree 1 as defined in, say, [36], is degenerate. This is not a serious problem since any polynomial of degree 1 will be a solution of $\ell_{-1,-1}[y](x) = 0$. On the other hand, again unlike the classical case, the Jacobi polynomial of degree 0 (namely $P_0^{(-1,-1)}(x) = 1$) and any choice of $P_1^{(-1,-1)}(x)$ will not be in $L^2\left((-1,1);(1-x^2)^{-1}\right)$ due to the singularities in the weight function $w_{-1,-1}(x) = (1-x^2)^{-1}$. In this case, the Jacobi polynomials of degree ≥ 2, that is $\{P_m^{(-1,-1)}\}_{m=2}^{\infty}$, form a complete orthogonal set of polynomial eigenfunctions of A in $L^2\left((-1,1);(1-x^2)^{-1}\right)$. We note that, by Favard's theorem, it is not possible for the entire sequence $\{P_m^{(-1,-1)}\}_{m=0}^{\infty}$ to be orthogonal on the real line with respect to any bilinear form of the type

$$(f,g) = \int_{\mathbb{R}} f\overline{g}d\mu,$$

for any choice of signed or positive measure μ.

In [22], the authors prove that, with the choice $P_1^{(-1,-1)}(x) = x/\sqrt{3}$, the *entire* sequence of Jacobi polynomials $\{P_m^{(-1,-1)}\}_{m=0}^{\infty}$ can be normalized so that they form an orthonormal set with respect to the Sobolev inner product

$$\phi(f,g) = \frac{1}{2}f(-1)\overline{g}(-1) + \frac{1}{2}f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx. \tag{4.54}$$

In fact, they form a complete orthonormal sequence in the Sobolev space

$$W^{(-1,-1)} = \{f : [-1,1] \to \mathbb{C} \mid f \in AC[-1,1]; f' \in L^2(-1,1)\} \tag{4.55}$$

with inner product $\phi(\cdot,\cdot)$. Is there a self-adjoint operator in $W^{(-1,-1)}$, generated by the Jacobi differential expression $\ell_{-1,-1}[\cdot]$, that has the Jacobi polynomials $\{P_m^{(-1,-1)}\}_{m=0}^{\infty}$ as eigenfunctions? We show that the answer is yes; indeed, the left-definite theory associated with A, given in (4.53), is instrumental in this construction.

The integral powers of $\ell_{-1,-1}[\cdot]$, the coefficients $c_j^{(-1,-1)}(n,k)$, the left-definite vector spaces $V_n^{(-1,-1)}$, and the left-definite inner products $(\cdot,\cdot)_n^{(-1,-1)}$ can be found in exactly the same fashion as in [10]; indeed, by letting $\alpha = \beta = -1$ in the formulas in Section 4.6, we obtain the necessary expressions, combinatorial numbers, spaces, and inner products. Indeed, for $n \in \mathbb{N}$, we see that the n^{th} left-definite Hilbert space associated with the pair $(L^2\left((-1,1);(1-x^2)^{-1}\right), A)$ is given by $H_n^{(-1,-1)} = (V_n^{(-1,-1)}, (\cdot,\cdot)_n^{(-1,-1)})$, where

$$\begin{aligned} V_n^{(-1,-1)} = \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\mathrm{loc}}(-1,1); \\ f^{(j)} \in L^2\left((-1,1);(1-x^2)^{j-1}\right), j = 0,1,..,n\} \end{aligned} \tag{4.56}$$

and

$$(f,g)_n^{(-1,-1)} = \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t^2)^{j-1}dt.$$

Moreover, the Jacobi polynomials $\{P_m^{(-1,-1)}\}_{m=2}^{\infty}$ form a complete orthogonal set in each $H_n^{(-1,-1)}$ and they satisfy the orthogonality relation

$$(P_m^{(-1,-1)}, P_r^{(-1,-1)})_n = (m(m-1)+k)^n \delta_{m,r.}$$

Furthermore, for each $n \in \mathbb{N}$, define $A_n : \mathcal{D}(A_n) \subset H_n^{(-1,-1)} \longrightarrow H_n^{(-1,-1)}$ by

$$A_n f = \ell_{-1,-1}[f] \qquad (f \in \mathcal{D}(A_n) = V_{n+2}^{(-1,-1)}).$$

Then A_n is the n^{th} left-definite operator associated with $\left(L^2\left((-1,1);(1-x^2)^{-1}\right), A\right)$, the spectrum of A_n is given by

$$\sigma(A_n) = \{m(m-1)+k \mid m \in \mathbb{N}_0\} = \sigma(A),$$

and the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ form a complete set of eigenfunctions of A_n in $H_n^{(-1,-1)}$.

To construct a self-adjoint operator T that is a realization of the Jacobi differential expression having the full sequence of Jacobi polynomials as eigenfunctions in the space $W^{(-1,1)}$, defined in (4.55), we first note the orthogonal decomposition

$$W^{(-1,-1)} = W_{1,1} \oplus W_{1,2},$$

where

$$W_{1,1} = \left\{ f \in W^{(-1,-1)} \mid f(\pm 1) = 0 \right\}$$
$$W_{1,2} = \left\{ f \in W^{(-1,-1)} \mid f''(x) = 0 \right\}.$$

It is the case that $\{P_m^{(-1,-1)}\}_{m=2}^{\infty}$ is a complete orthonormal set in $W_{1,1}$ and $\{P_m^{(-1,-1)}\}_{m=0}^{1}$ is a complete orthonormal set in the two-dimensional space $W_{1,2}$. Furthermore, as is shown in [5],

$$W_{1,1} = V_1^{(-1,-1)},$$

where $V_1^{(-1,-1)}$ denotes the first left-definite space defined in (4.56). Moreover, the inner products inner products $(\cdot,\cdot)_1^{(-1,-1)}$ and $\phi(\cdot,\cdot)$, where $\phi(\cdot,\cdot)$ is defined in (4.54), are equivalent on $W_{1,1} = V_1^{(-1,-1)}$.

Details are given in [5] that show that the first left-definite operator

$$T_1 : \mathcal{D}(T_1) \subset W_{1,1} \longrightarrow W_{1,1}$$

given by

$$T_1 f = A_1 f = \ell_{-1,-1}[f]$$
$$f \in \mathcal{D}(T_1) := V_3^{(-1,-1)}$$

is self-adjoint in $(W_{1,1}, \phi(\cdot,\cdot))$. It is easy to construct a self-adjoint operator T_2 in $W_{1,2}$ generated by $\ell_{-1,-1}[\cdot]$:

$$T_2 f = \ell_{-1,-1}[f],$$
$$\mathcal{D}(T_2) = \mathcal{P}_1,$$

where $\mathcal{P}_1$ is the space of all polynomials of degree less than two.

For each $f \in W^{(-1,-1)}$, write $f = f_1 + f_2$ where $f_i \in W_{1,i}, (i = 1, 2)$. Define

$$T : \mathcal{D}(T) \subset W^{(-1,-1)} \to W^{(-1,-1)}$$

by

$$Tf = T_1 f_1 + T_2 f_2 = \ell[f_1] + \ell[f_2] = \ell[f],$$

for

$$f \in \mathcal{D}(T) = \mathcal{D}(T_1) \oplus \mathcal{D}(T_2).$$

Then T is self-adjoint in $\left(W^{(-1,-1)}, \phi(\cdot,\cdot)\right)$ and has the entire sequence of Jacobi polynomials $\{P_m^{(-1,-1)}\}_{m=0}^{\infty}$ as eigenfunctions. From the explicit determination of $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$, it is not difficult to obtain the following characterization of $\mathcal{D}(T)$:

$$\begin{aligned}\mathcal{D}(T) = \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{\text{loc}}(-1,1); (1-x^2)f''', \\ (1-x^2)^{1/2} f'', f' \in L^2(-1,1)\} \\ = \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{\text{loc}}(-1,1); (1-x^2)f''' \in L^2(-1,1)\}.\end{aligned}$$

Furthermore, the spectrum of T is given by $\sigma(T) = \{m(m-1)+k \mid m \in \mathbb{N}_0\}$ and T is bounded below by kI in $\left(W^{(-1,-1)}, \phi(\cdot,\cdot)\right)$.

4.9. **The Fourth Order Legendre Type Differential Equation and Legendre Type Polynomials.** Details of this example can be found in [9], [11], [20], [27], [38], [39], and [37]. In particular, the recent thesis [37] of Tuncer deals with a complete left-definite analysis of the Legendre type expression; we report on the results contained in this thesis.

The fourth-order Legendre type expression is defined to be

$$\ell_{\mathrm{LT}}[y](x) = ((1-x^2)^2 y''(x))'' - ((8+4A(1-x^2)y'(x))' + ky(x) \quad (x \in (-1,1)); \tag{4.57}$$

here, A is a fixed positive constant and k is a fixed, non-negative parameter. The earliest occurrence of this expression in the mathematical literature is attributed to H. L. Krall [19] in 1938; indeed, Krall showed that when

$$\lambda_m = m(m+1)(m^2+m-2+4A)+k \quad (m \in \mathbb{N}_0),$$

there is a real polynomial solution $y = P_{m,A}(x)$ to $\ell_{\mathrm{LT}}[y](x) = \lambda_m y(x)$ for each non-negative integer m. These polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ are called the Legendre type polynomials. They form a complete orthogonal set in the right-definite Hilbert space

$$L^2_\mu[-1,1] = \{f : [-1,1] \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \|f\|_\mu^{1/2} = \int_{[-1,1]} |f|^2 \, d\mu < \infty\},$$

where μ is the positive Borel measure defined through the inner product

$$(f,g)_\mu = \frac{f(1)\overline{g}(1)}{2} + \frac{A}{2}\int_{-1}^{1} f(x)\overline{g}(x)dx + \frac{f(-1)\overline{g}(-1)}{2}. \tag{4.58}$$

In fact, the Legendre type polynomials satisfy the orthogonality relation

$$(P_{m,A}, P_{r,A})_\mu = A(A + \frac{m(m-1)}{2})(A + \frac{(m+1)(m+2)}{2})/(2m+1)\delta_{m,r} \quad (m,r \in \mathbb{N}_0). \tag{4.59}$$

In [9] and [11], the authors show that the operator T in $L^2_\mu[-1,1]$, defined by

$$(Tf)(x) = \begin{cases} -8Af'(-1)+kf(-1) & \text{if } x=-1 \\ \ell[f](x) & \text{if } -1<x<1 \\ 8Af'(1)+kf(1) & \text{if } x=1, \end{cases} \tag{4.60}$$

with

$$\mathcal{D}(T) = \{f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\mathrm{loc}}(-1,1); f, \ell_{\mathrm{LT}}[f] \in L^2(-1,1)\} \tag{4.61}$$

is self-adjoint with discrete spectrum $\sigma(T) = \{\lambda_m \mid m \in \mathbb{N}_0\}$ and has the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ as eigenfunctions. Moreover,

$$(Tf,g)_\mu = \frac{A}{2}\int_{-1}^{+1} \left((1-x^2)^2 f''(x)\overline{g}''(x) + (8+4A(1-x^2))f'(x)\overline{g}'(x)\right) dx + k(f,g)_\mu \quad (f,g \in \mathcal{D}(T));$$

in particular, note that

$$(Tf,f)_\mu \geq k(f,f)_\mu \quad (f \in \mathcal{D}(T))$$

so that T is bounded below by kI in $L^2_\mu[-1,1]$. Consequently, the general left-definite theory applies to this operator T.

The n^{th} power of the Legendre type expression is given by

$$\ell^n_{\mathrm{LT}}[y](x) = \sum_{j=0}^{2n} (-1)^j \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1} \right) y^{(j)}(x) \right)^{(j)},$$

where the sequences $\{a_j(n,k)\}$ and $\{b_j(n,k)\}$ are defined by

$$a_0(n,k) = \begin{cases} 0 & \text{if } k=0 \\ k^n & \text{if } k>0 \end{cases},$$

and, for $j = 1, 2, \ldots, 2n$,

$$a_j(n,k) = \begin{cases} a_{n,j} & \text{if } k=0 \\ \sum_{r=0}^{n-1} \binom{n}{r} a_{n-r,j} k^r & \text{if } k>0 \end{cases},$$

where

$$a_{n,j} = \sum_{m=0}^{j} \frac{(-1)^{m+j} (2m+1) \left(m^2+m\right)^n \left(m^2+m-2+4A\right)^n}{(m+j+1)!\,(j-m)!},$$

$b_0(n,k) = 0$, and for $j = 1, 2, \ldots, 2n$,

$$b_j(n,k) = \begin{cases} b_{n,j} & \text{if } k=0 \\ \sum_{r=0}^{n-1} \binom{n}{r} b_{n-r,j} k^r & \text{if } k>0 \end{cases},$$

where

$$b_{n,j} = \sum_{m=0}^{j} \frac{(-1)^{m+j}\, 4(2m+1) \left(2Aj + (j+1)\left(j+m^2+m\right)\right) \left(m^2+m\right)^n \left(m^2+m-2+4A\right)^n}{(m+j+1)!\,(j-m)!\,(2A+m(m-1))\,(2A+(m+1)(m+2))}.$$

Define, for each $n \in \mathbb{N}$, $H_n = (V_n, (\cdot,\cdot)_n)$, where

$$V_n = \{f : [-1,1] \to \mathbb{C} \mid f \in AC[-1,1]; f^{(j)} \in AC_{\mathrm{loc}}(-1,1); (1-x^2)^{(2n-1)/2} f^{(2n-1)} \in L^2(-1,1)\},$$

and, for $f, g \in V_n$

$$(f,g)_n = \sum_{j=1}^{2n} \int_{-1}^{1} \left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1} \right) f^{(j)}(x) \overline{g}^{(j)}(x) dx + k^n (f,g)_\mu.$$

Then H_n is the n^{th} left-definite space associated with $(L^2_\mu[-1,1], T)$. Lastly, the n^{th} left-definite operator $T_n : \mathcal{D}(T_n) \subset H_n \to H_n$ is given by

$$T_n f = \ell_{\mathrm{LT}}[f]$$
$$f \in \mathcal{D}(T_n) = V_{n+2}.$$

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