A FUNDAMENTAL THEOREM OF LEFT-DEFINITE OPERATOR THEORY

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Dedicated to our colleague, mentor, and friend A. M. Krall (1936-2008)

Abstract. If A is a self-adjoint operator that is bounded below in a Hilbert space H, Littlejohn and Wellman [12] showed that, for each $r > 0$, there exists a unique Hilbert space H_r and a unique self-adjoint operator A_r in H_r satisfying certain conditions dependent on H and A. The space H_r and the operator A_r are called, respectively, the r^{th} left-definite space and r^{th} left-definite operator associated with (H, A). In this paper, we show that the operators A, A_r, and A_s $(r, s > 0)$ are isometrically isomorphically equivalent and that the spaces H, H_r, and H_s $(r, s > 0)$ are isometrically isomorphic. These results are then used to *reproduce* the left-definite spaces and left-definite operators. Furthermore, we will see that our new results imply that the spectra of A and A_r are equal, giving us another proof of this phenomenon that was first established in [12].

1. Introduction

Left-definite spectral theory has its origins in the theory of differential equations, dating back to fundamental work of Hermann Weyl in [21]; see, for example, the paper [18] (where the terminology 'Links-definit' is first introduced), and the recent treatise [22, Chapters 5 and 12] where an excellent discussion, as well as a comprehensive bibliography, is given of left-definite theory applied to second-order Sturm-Liouville boundary value problems.

There are at least two different, but related, definitions of 'left-definite' in the literature; we refer the reader to [22, Chapters 5 and 12] for a discussion of a left-definite theory that is different from the one that we consider in this paper. In [22] (see also [7], [8]), the authors deal with a left-definite theory discussing the study of the Sturm-Liouville equation

$$My := -(py')' + qy = \lambda wy \text{ on } J = (a, b), \quad -\infty \leq a < b \leq \infty,$$

where the weight function w may change sign on J. Extensions of this theory to higher-order equations or systems of equations can be found in [14], [15], [16], [19], and [20]. Our notion of 'left-definite' is discussed in [12]; in this paper the present authors develop a general, and abstract, left-definite theory for self-adjoint operators that are bounded below by a positive constant in a Hilbert space. In applications to differential equations, this theory is best applied to formally Lagrangian symmetric differential expressions of the form

$$\ell[y](x) = \sum_{j=0}^{n} (-1)^j (a_j(x) y^{(j)}(x))^{(j)} \quad (x \in I)$$

and spectral problems of the form

$$\ell[y] = \lambda y, \tag{1.1}$$

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where each coefficient $a_j : I \to \mathbb{R}$ is positive on the open interval I. In this case, the (first) left-definite space is a Sobolev-Hilbert function space H_1 generated by the positive-definite inner product

$$(f,g) = \sum_{j=0}^{n} \int_I a_j(x) f^{(j)}(x) \overline{g}^{(j)}(x) dx. \tag{1.2}$$

Since the inner product (1.2) is generated by $\ell[\cdot]$, the left-hand side of (1.1), it has been customary to call H_1 the (first) left-definite setting for the functional analytic study of $\ell[\cdot]$.

Prior to the publication of [12], the mathematical literature on left-definite theory and differential equations dealt exclusively with *first* left-definite spaces and *first* left-definite operators. However, in [12], the authors construct a continuum of left-definite spaces $\{H_r\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$ associated with an arbitrary self-adjoint operator A that is bounded below by a positive constant in a Hilbert space. This left-definite theory was subsequently applied to the classical second order differential equations of Hermite [4], Legendre [5], Jacobi [6], Laguerre [12], and Fourier [13]; see also the recent survey paper [2] and the important left-definite contributions of A. M. Krall [9] and [10]. The construction of these spaces and operators in these examples were relatively routine and provided significantly new information about the original operator A; more details on why this is the case will be given in Section 2 below.

One of the main results in [12] shows the spectrum of A and each of its left-definite operators A_r are equal. This result was completely unexpected at first and suggested that deeper general results remained to be discovered. Could it be that A and each A_r, in some sense, are similar operators? If so, this would explain their spectra being equal. In this paper, we show that A and each A_r are isometrically isomorphically (unitarily) equivalent which implies that their spectra are identical. In discovering this new proof, we are also able to simplify some of the original results given in [12]. In particular, we see that these isometric isomorphisms *reproduce* the left-definite spaces and operators.

The contents of this paper are as follows. In Section 2, we review the abstract left-definite theory developed in [12]. Since the Hilbert space spectral theorem is a key tool in developing results in both [12] and in this paper, we discuss this result and its consequences in Section 3. Section 4 deals with some key preliminary results that are needed for the main results of this paper which are presented in Section 5.

2. Abstract Left-Definite Theory

The results in this section are discussed, in full, in [12].

Let V denote a vector space (over the complex field $\mathbb{C}$) and suppose that $(\cdot,\cdot)$ is an inner product with norm $\|\cdot\|$, generated from $(\cdot,\cdot)$, such that $H = (V, (\cdot,\cdot))$ is a Hilbert space. Suppose V_r (the subscripts will be made clear shortly) is a linear manifold (subspace) of the vector space V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote an inner product and its associated norm, respectively, over V_r (quite possibly different from $(\cdot,\cdot)$ and $\|\cdot\|$). We denote the resulting inner product space by $H_r = (V_r, (\cdot,\cdot)_r)$.

Throughout this section, we assume that $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI, for some $k > 0$; that is,

$$(Ax, x) \geq k(x,x) \quad (x \in \mathcal{D}(A)).$$

We now define an r^{th} left-definite space associated with (H, A).

Definition 2.1. *Let $r > 0$ and suppose V_r is a linear subspace of the Hilbert space $H = (V, (\cdot,\cdot))$ and $(\cdot,\cdot)_r$ is an inner product on $V_r \times V_r$. Let $H_r = (V_r, (\cdot,\cdot)_r)$. We say that H_r is an r^{th} left-definite*

space associated with the pair (H, A) if each of the following conditions hold:
(1) H_r *is a Hilbert space,*
(2) $\mathcal{D}(A^r)$ *is a linear subspace of* V_r,
(3) $\mathcal{D}(A^r)$ *is dense in* H_r,
(4) $(x, x)_r \geq k^r (x, x) \quad (x \in V_r)$, *and*
(5) $(x, y)_r = (A^r x, y) \quad (x \in \mathcal{D}(A^r),\ y \in V_r)$.

Remark 2.1. As mentioned in the Section 1, all discussions of left-definite theory in the literature, prior to the publication of [12], was restricted to the case $r = 1$. As evidenced in Theorem 2.1 below, if we know explicitly the other left-definite spaces then we know significantly more information about the original operator A and its powers.

It is not clear, from Definition 2.1, if such a self-adjoint operator A generates a left-definite space for a given $r > 0$. However, in [12], the authors prove the following theorem; the Hilbert space spectral theorem plays a significant role in establishing this result.

Theorem 2.1. (*see* [12, Theorem 3.1]) *Suppose* $A : \mathcal{D}(A) \subset H \to H$ *is a self-adjoint operator that is bounded below by* kI, *for some* $k > 0$. *Let* $r > 0$. *Define* $H_r = (V_r, (\cdot, \cdot)_r)$ *by*

$$V_r = \mathcal{D}(A^{r/2}),$$

and

$$(x, y)_r = (A^{r/2} x, A^{r/2} y) \quad (x, y \in V_r). \tag{2.1}$$

Then H_r *is a left-definite space associated with the pair* (H, A). *Moreover, suppose* $H_r' := (V_r', (\cdot, \cdot)_r')$ *is another* r^{th} *left-definite space associated with the pair* (H, A). *Then* $V_r = V_r'$ *and* $(x, y)_r = (x, y)_r'$ *for all* $x, y \in V_r = V_r'$; *i.e.* $H_r = H_r'$. *That is to say,* $H_r = (V_r, (\cdot, \cdot)_r)$ *is the unique left-definite space associated with* (H, A).

Remark 2.2. Although all five conditions in Definition 2.1 are necessary in the proof of Theorem 2.1, the most important property, in a sense, is the one given in part (5). Indeed, this property asserts that the r^{th} left-definite inner product is generated from the r^{th} power of A. In particular, if A is generated from a Lagrangian symmetric differential expression $\ell[\cdot]$, the r^{th} left-definite inner product $(\cdot, \cdot)_r$ is determined by the r^{th} (composite) power of $\ell[\cdot]$. Even though these left-definite spaces and left-definite inner products exist for all $r > 0$, quite often we can only *explicitly* obtain these spaces and inner products when r is a positive integer. This is especially the case if we do not explicitly know the spectral resolution of the identity of A, as is the case, for example, of the classical differential equations of Laguerre, Hermite, and Jacobi. In these cases, computing the r^{th} power ($r \in \mathbb{N}$) of these second-order differential expressions yields some interesting combinatorics. Indeed, the classical Stirling numbers of the second kind appear in the integral powers of the Laguerre and Hermite expressions (see [12] and [4]) while the Jacobi-Stirling numbers, a new set of combinatorial numbers, are the coefficients of the integral powers of the Jacobi expression (see [6]). We refer the reader to [12] where another example is discussed in which the *entire* continuum of left-definite spaces and inner products for a well-known self-adjoint operator in the classical Hilbert space $\ell^2(\mathbb{N})$ are explicitly constructed.

Remark 2.3. A statement is in order regarding the apparent ambiguity between part (v) of Definition 2.1 and the explicit inner product $(\cdot, \cdot)_r$ given in (2.1) of Theorem 2.1. From part (2)(ii) of Theorem 2.3, we see that $\mathcal{D}(A^r) = V_{2r} \subset V_r$. Consequently, if $x \in \mathcal{D}(A^r)$ and $y \in V_r$, we see from the self-adjointness of $A^{r/2}$ that

$$(x, y)_r = (A^r x, y) = (A^{r/2}(A^{r/2} x), y) = (A^{r/2} x, A^{r/2} y).$$

The fact that $A^{r/2}x \in \mathcal{D}(A^{r/2})$, when $x \in \mathcal{D}(A^r)$, follows from [12, Theorem 4.3, equation (4.3) and Lemma 5.3, equations (5.8) and (5.9)]; we give similar arguments to this fact later in this paper.

Definition 2.2. *For $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ denote the r^{th} left-definite space associated with (H, A). If there exists a self-adjoint operator $A_r : \mathcal{D}(A_r) \subset H_r \to H_r$ satisfying*

$$A_r f = Af \quad (f \in \mathcal{D}(A_r) \subset \mathcal{D}(A)),$$

we call such an operator an r^{th} left-definite operator associated with (H, A).

Again, it is not immediately clear that such an A_r exists for a given $r > 0$; in fact, however, as the next theorem shows, A_r exists and is unique.

Theorem 2.2. (*see* [12, Theorem 3.2]) *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI, for some $k > 0$. For any $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H, A). Then there exists a unique left-definite operator A_r in H_r associated with (H, A). Moreover,*

$$\mathcal{D}(A_r) = V_{r+2} \subset \mathcal{D}(A).$$

We note that this r^{th} left-definite operator A_r is special in the sense that if the subscript $r+2$ in the above domain is altered, the resulting transformation fails to be a mapping from H_r into H_r.

The next theorem gives further explicit information regarding the left-definite spaces and left-definite operators associated with (H, A).

Theorem 2.3. (*see* [12, Theorem 3.4]) *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI, for some $k > 0$. Let $\{H_r = (V_r, (\cdot,\cdot)_r)\}_{r>0}$ and $\{A_r\}_{r>0}$ be the left-definite spaces and left-definite operators, respectively, associated with (H, A). Then the following results are true.*

(1) *Suppose A is bounded. Then, for each $r > 0$,*
  - (i) $V = V_r$;
  - (ii) *the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_r$ are equivalent;*
  - (iii) $A = A_r$.

(2) *Suppose A is unbounded. Then*
  - (i) *V_r is a proper subspace of V;*
  - (ii) *V_s is a proper subspace of V_r whenever $0 < r < s$;*
  - (iii) *the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_r$ are not equivalent for any $r > 0$;*
  - (iv) *the inner products $(\cdot,\cdot)_r$ and $(\cdot,\cdot)_s$ are not equivalent for any $r, s > 0$, $r \neq s$;*
  - (v) *$\mathcal{D}(A_r)$ is a proper subspace of $\mathcal{D}(A)$ for each $r > 0$;*
  - (vi) *$\mathcal{D}(A_r)$ is a proper subspace of $\mathcal{D}(A_s)$ whenever $0 < s < r$.*
  - (vii) *If $\{\phi_\alpha\}_{\alpha\in I}$ is a complete set of eigenfunctions of A, then $\{\phi_\alpha\}_{\alpha\in I}$ is a complete set of eigenfunctions of A_r in H_r for each $r > 0$.*

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of a self-adjoint, bounded below operator A and each of its associated left-definite operators A_r $(r > 0)$ are identical; see [11, Section 7.2] for the definitions concerning the various components of the spectrum listed below and the resolvent set of a linear operator.

Theorem 2.4. (*see* [12, Theorem 3.6]) *For each $r > 0$, let A_r denote the r^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by kI in H, for some $k > 0$. Then*

(a) *the point spectra of A and A_r coincide; that is, $\sigma_p(A_r) = \sigma_p(A)$;*
(b) *the continuous spectra of A and A_r coincide; that is, $\sigma_c(A_r) = \sigma_c(A)$;*

(c) *the resolvent sets of A and A_r are equal; that is,* $\rho(A_r) = \rho(A)$.

We give another, more direct, proof of Theorem 2.4 in Section 5.

3. The Spectral Theorem

Let $\mathcal{B}$ denote the σ-algebra of Borel sets of $\mathbb{R}$ and let $B(H)$ be the Banach algebra of bounded linear operators on a Hilbert space H. A *resolution of the identity* is a mapping $E : \mathcal{B} \to B(H)$ defined by the following properties:

(3.1)
(i) $E(\emptyset) = 0$ and $E(\mathbb{R}) = I$,
(ii) $E(\Delta)$ is idempotent, that is, $(E(\Delta))^2 = E(\Delta)$, for all $\Delta \in \mathcal{B}$,
(iii) $E(\Delta)$ is self-adjoint in H for all $\Delta \in \mathcal{B}$,
(iv) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2) = E(\Delta_2)E(\Delta_1)$ for all $\Delta_1, \Delta_2 \in \mathcal{B}$,
(v) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{B}$ with $\Delta_1 \cap \Delta_2 = \emptyset$,
(vi) For each $x, y \in H$, the mapping $E_{x,y} : \mathcal{B} \to \mathbb{C}$ defined by $E_{x,y}(\Delta) := (E(\Delta)x, y)$ is a complex, regular Borel measure.

In this case, we also say that the collection of sets $\{E(\Delta)\}_{\Delta \in \mathcal{B}}$ is a resolution of the identity.

It is well known (see, for example, [17, Chapters 12 and 13]) that every self-adjoint operator A in H, bounded or unbounded, induces a unique resolution of the identity E, in this case called a *spectral resolution* of A, on the Borel subsets of the spectrum $\sigma(A)$ of A. In fact, the Hilbert space spectral theorem, which we state later in this section, asserts that such an A can be reconstructed from E through integration.

Remark 3.1. For later purposes, we remark that a *spectral family* (see [11, Section 9.7] or [1, Section 67]) is a one-parameter family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of bounded operators in H satisfying

(3.2)
(1) E_λ is self-adjoint and idempotent for each $\lambda \in \mathbb{R}$,
(2) For $\lambda < \mu$, $E_\mu - E_\lambda$ is a positive operator,
(3) $\lim_{\lambda \to \infty} E_\lambda x = x$ for each $x \in H$,
(4) $\lim_{\lambda \to -\infty} E_\lambda x = 0$ for each $x \in H$,
(5) $E_{\lambda+0}x := \lim_{\mu \to \lambda^+} E_\mu x = E_\lambda x$ for each $\lambda \in \mathbb{R}$ and $x \in H$.

It is not difficult to show that if E is a spectral resolution of the identity for A, then $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is a spectral family, where

$$E_\lambda := E(-\infty, \lambda] \quad (\lambda \in \mathbb{R}).$$

We are now in position to state the spectral theorem in a Hilbert space (see [17, Theorems 13.24 and 13.30]).

Theorem 3.1. (***The Spectral Theorem***) *Let A be a self-adjoint operator (bounded or unbounded) in a Hilbert space $H = (V, (\cdot, \cdot))$. Let E be the spectral resolution of the identity associated with A. Then, for each $r > 0$, the self-adjoint operator A^r has (densely defined) domain $\mathcal{D}(A^r)$ given by*

$$\mathcal{D}(A^r) = \{x \in H \mid \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} < \infty\},$$

and is characterized by the identities

$$(A^r x, y) = \int_{\mathbb{R}} \lambda^r dE_{x,y} \quad (x \in \mathcal{D}(A^r),\ y \in H),$$

and

$$\|A^r x\|^2 = \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} \qquad (x \in \mathcal{D}(A^r)).$$

Conversely, suppose $F : \mathcal{B} \to B(H)$ is a spectral resolution of the identity. Then there exists a unique self-adjoint operator $\widetilde{A}$ in H with (densely defined) domain

$$\mathcal{D}(\widetilde{A}) = \{x \in H \mid \int_{\mathbb{R}} \lambda^2 dF_{x,x} < \infty\}$$

that is characterized by

$$(\widetilde{A}x, y) = \int_{\mathbb{R}} \lambda dF_{x,y} \quad (x \in \mathcal{D}(\widetilde{A}),\ y \in H),$$

and

$$\left\|\widetilde{A}x\right\|^2 = \int_{\mathbb{R}} \lambda^2 dF_{x,x} \qquad (x \in \mathcal{D}(\widetilde{A})).$$

We emphasize that this particular version of the spectral theorem is crucial in obtaining the results listed in the previous section. Indeed, most discussions of the spectral theorem deal with a spectral family (see, for example, [1, Chapter 6, Section 67] or [11, Section 9.7]) instead of the resolution of the identity; it is doubtful that we could have obtained the results, summarized in Section 2, with any 'weaker' form of the spectral theorem that we are using.

Key to several results in [12], as well as this paper, are the following fundamental identities:

$$dE_{A^r x,y} = \lambda^r dE_{x,y} \quad (r \in \mathbb{R}, x \in \mathcal{D}(A^r), y \in H) \tag{3.3}$$

and

$$dE_{x,A^r y} = \lambda^r dE_{x,y} \quad (r \in \mathbb{R}, x \in H, y \in \mathcal{D}(A^r)). \tag{3.4}$$

These identities are proven in [12, Equations (5.8) and (5.9)] for $r > 0$ but similar arguments show these results also hold for $r \leq 0$.

4. Preliminary Results

Throughout this section, we suppose that $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI in H, where k is some positive constant; that is to say,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

Proposition 4.1. *Let $r > 0$. Then each of the powers $A^r : \mathcal{D}(A^r) \to H$ is bounded below by $k^r I$.*

Proof. From the Hilbert space spectral theorem, A^r is a self-adjoint operator. Furthermore, for $x \in \mathcal{D}(A^r)$,

$$(A^r x, x) = \int_{\mathbf{R}} \lambda^r dE_{x,x} = \int_{[k,\infty)} \lambda^r dE_{x,x} \geq k^r \int_{[k,\infty)} dE_{x,x} = k^r \int_{\mathbf{R}} dE_{x,x} = k^r (x, x).$$

□

Proposition 4.2. *Let $r > 0$. Then $A^r : \mathcal{D}(A^r) \to H$ is injective.*

Proof. Suppose $x, y \in \mathcal{D}(A^r)$ and $A^r x = A^r y$. Then

$$0 = (A^r(x - y), x - y) \geq k^r \|x - y\|^2$$

in which case it is clear that $x = y$. □

Proposition 4.3. *Let $r > 0$. Then $A^r : \mathcal{D}(A^r) \to H$ is onto H.*

Proof. Since A^r is injective we see that $0 \in \rho(A^r)$, the resolvent set of A^r. Furthermore, A^r is closed since A^r is self-adjoint. From [11, Lemma 7.2-3], we see that the domain of $R_0(A^r) = A_r^{-1}$ is H and, consequently, A^r is onto H. □

Recall (see [1, Chapter 6, Section 87]) that a linear mapping $U: H_1 \to H_2$, where H_1 and H_2 are Hilbert spaces, with respective inner products $(\cdot,\cdot)_{H_1}$ and $(\cdot,\cdot)_{H_2}$, is an isometric isomorphism (also called a unitary transformation) if U is a bijection of H_1 onto H_2 and

$$(x,y)_{H_1} = (Ux,Uy)_{H_2} \quad (x,y \in H_1);$$

in this case, we say that H_1 and H_2 are unitarily equivalent. In this case, we have

$$(Ux,Uy)_{H_2} = (x,U^*Uy)_{H_1}, \tag{4.1}$$

from which it follows that

$$U^* = U^{-1}. \tag{4.2}$$

Proposition 4.4. *Let $r>0$. Then $A^{r/2}:\mathcal{D}(A^{r/2}) = V_r \to H$ is an isometric isomorphism between the Hilbert spaces $H_r = (V_r,(\cdot,\cdot)_r)$ and $H = (V,(\cdot,\cdot))$.*

Proof. This follows immediately from Propositions 4.2 and 4.3 and the identity in (2.1), namely

$$(x,y)_r = (A^{r/2}x, A^{r/2}y) \quad (x,y \in V_r).$$

□

The results of this section can be used to give another proof, different from that given in [12], of the completeness of each H_r. The main technique used in this proof is well-known but we give the proof below for the sake of completeness. It is interesting that we can reproduce the r^{th} left-definite space H_r through the isometric isomorphism $A^{r/2}$; this construction is considerably simpler than the original construction in [12].

Theorem 4.1. *For each $r>0$, $H_r = A^{-r/2}H$. In particular, H_r is a Hilbert space.*

Proof. The identity $H_r = A^{-r/2}H$ is clear from Proposition 4.4. Let $\{x_n\} \subset H_r$ be a Cauchy sequence and let $y_n = A^{r/2}x_n$. Since

$$\left\| A^{r/2}(x_n - x_m)\right\| = \|x_n - x_m\|_r \to 0 \text{ as } n,m\to\infty,$$

we see that $\{A^{r/2}x_n\}$ is a Cauchy sequence in H. From the completeness of H, there exists $y\in H$ such that $A^{r/2}x_n \to y$ in H. Let $x = A^{-r/2}y$; since $A^{-r/2}$ is a bijection of H onto H_r, we see that $x \in H_r$ and $y = A^{r/2}x$. Moreover

$$\|x_n - x\|_r = \left\|A^{r/2}(x_n - x)\right\| = \left\|A^{r/2}x_n - y\right\| \to 0$$

so H_r is complete. □

We now give a new proof of the symmetry of each left-definite operator A_r; in the next section, we provide a new proof that A_r is self-adjoint. The first step is in showing that $V_{r+2} = \mathcal{D}(A_r)$ is dense in H_r.

Theorem 4.2. *For each $r>0$, $V_{r+2} = \mathcal{D}(A_r)$ is dense in H_r.*

Proof. Let $x\in H_r$ and let $\varepsilon>0$. Since $\mathcal{D}(A)$ is dense in H, there exists $y\in\mathcal{D}(A)$ such that

$$\left\|A^{r/2}x - y\right\| < \varepsilon.$$

Let $x^* = A^{-r/2}y$ so $x^* \in V_r$ and $y = A^{r/2}x^*$. In fact, $x^* \in V_{r+2}$ since

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{r+2} dE_{x^*,x^*} &= \int_{\mathbb{R}} \lambda^{r+2} dE_{A^{-r/2}y,A^{-r/2}y} \\ &= \int_{\mathbb{R}} \lambda^{r+2}\lambda^{-r/2}\lambda^{-r/2} dE_{y,y} \text{ by (3.3) and (3.4)} \\ &= \int_{\mathbb{R}} \lambda^2 dE_{y,y} \\ &< \infty \text{ since } y \in \mathcal{D}(A).\end{aligned}$$

Finally, we see that

$$\|x - x^*\|_r = \left\|A^{r/2}(x - x^*)\right\| = \left\|A^{r/2}x - y\right\| < \varepsilon$$

so that V_{r+2} is dense in H_r. □

Theorem 4.3. *For each $r > 0$, A_r is a symmetric operator in H_r.*

Proof. Let $x, y \in \mathcal{D}(A_r) = V_{r+2} \subset V_r = \mathcal{D}(A^{r/2})$. We first note that

$$A^{r/2}x, A^{r/2}y \in \mathcal{D}(A); \tag{4.3}$$

indeed

$$\begin{aligned}&\int_{\mathbb{R}} \lambda^2 dE_{A^{r/2}x,A^{r/2}x} \\ &= \int_{\mathbb{R}} \lambda^2\lambda^{r/2}\lambda^{r/2} dE_{x,x} \text{ by (3.3) and (3.4)} \\ &= \int_{\mathbb{R}} \lambda^{r+2} dE_{x,x} < \infty.\end{aligned}$$

It follows that

$$\begin{aligned}(A_r x, y)_r &= (A^{r/2}A_r x, A^{r/2}y) \\ &= (A^{r/2}Ax, A^{r/2}y) \\ &= (AA^{r/2}x, A^{r/2}y) \\ &= (A^{r/2}x, AA^{r/2}y) \text{ from (4.3) and since } A \text{ is self-adjoint} \\ &= (A^{r/2}x, A^{r/2}Ay) \\ &= (A^{r/2}x, A^{r/2}A_r y) \\ &= (x, A_r y)_r.\end{aligned}$$

□

The self-adjointness of A_r will follow immediately from Theorems 5.1 and 5.2 below.

5. Main Results

We begin this section with the following fundamental result which is key to establishing our main result in Theorem 5.2 below.

Theorem 5.1. *Suppose that $U : H_1 \to H_2$ is an isometric isomorphism of the Hilbert space H_1, with inner product $(\cdot,\cdot)_{H_1}$, onto the Hilbert space H_2, with inner product $(\cdot,\cdot)_{H_2}$. Suppose that $E = \{E(\Delta\}_{\Delta\in\mathcal{B}}$ is a resolution of the identity in H_1. Then*

(a) $UEU^{-1} := \{UE(\Delta)U^{-1}\}_{\Delta\in\mathcal{B}}$ *is a resolution of the identity in* H_2;

(b) $\{UE(\lambda)U^{-1}\}_{\lambda\in\mathbb{R}}$ *is a spectral family in* H_2, *where* $E(\lambda) = E(-\infty, \lambda]$ *for each* $\lambda \in \mathbb{R}$; *more generally, if* $\{F(\lambda)\}_{\lambda\in\mathbb{R}}$ *is a spectral family in* H_1, *then* $\{UF(\lambda)U^{-1}\}_{\lambda\in\mathbb{R}}$ *is a spectral family in* H_2;

(c) *Suppose* $S = UTU^{-1}$ *and* $\mathcal{D}(S) = U\mathcal{D}(T)$. *Then* S *is a self-adjoint operator in* H_2. *Furthermore, if* E *is the spectral resolution of the self-adjoint operator* T *in* H_1, *then* UEU^{-1} *is the spectral resolution of* S *in* H_2. *Moreover,*
   (α) $\rho(T) = \rho(S)$; *that is to say, the resolvent sets of* T *and* S *are equal;*
   (β) $\sigma_p(T) = \sigma_p(S)$; *that is to say, the point spectra of* T *and* S *are equal;*
   (γ) $\sigma_c(T) = \sigma_c(S)$; *that is to say, the continuous spectra of* T *and* S *are equal.*

Proof. Proof of part (a): Since $E(\varnothing) = 0$ and $E(\mathbb{R}) = I$, it is clear that

$$UE(\varnothing)U^{-1} = 0 \text{ and } UE(\mathbb{R})U^{-1} = I$$

and hence property (i) of (3.1) holds. Let $\Delta \in \mathcal{B}$; since $E(\Delta)$ is idempotent we see that

$$(UE(\Delta)U^{-1})^2 = (UE(\Delta)U^{-1})(UE(\Delta)U^{-1}) = U(E(\Delta))^2U^{-1} = UE(\Delta)U^{-1},$$

establishing property (ii). To show property (iii), let $x, y \in H_2$. From (4.2) and the self-adjointness of $E(\Delta)$, we find that

$$\begin{aligned}(UE(\Delta)U^{-1}x, y)_{H_2} &= (x, (UE(\Delta)U^{-1})^*y)_{H_2} \\ &= (x, (U^{-1})^*(E(\Delta))^*U^*y)_{H_2} \\ &= (x, U^{**}E(\Delta)U^*y)_{H_2} \\ &= (x, UE(\Delta)U^{-1}y)_{H_2}\end{aligned}$$

and hence $UE(\Delta)U^{-1}$ is a bounded, self-adjoint operator. Let $\Delta_1, \Delta_2 \in \mathcal{B}$; since $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$, we see that

$$\begin{aligned}UE(\Delta_1 \cap \Delta_2)U^{-1} &= UE(\Delta_1)E(\Delta_2)U^{-1} \\ &= (UE(\Delta_1)U^{-1})(UE(\Delta_2)U^{-1})\end{aligned}$$

from which property (iv) of (3.1) holds. Now suppose that $\Delta_1 \cap \Delta_2 = \varnothing$; since $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$, we see that

$$\begin{aligned}UE(\Delta_1 \cup \Delta_2)U^{-1} &= U(E(\Delta_1) + E(\Delta_2))U^{-1} \\ &= UE(\Delta_1)U^{-1} + UE(\Delta_2)U^{-1},\end{aligned}$$

establishing property (v). To show property (vi), let $\{\Delta_j\}_{j=1}^{\infty} \subset \mathcal{B}$ be a partition of Δ. From [17, Section 12.17], we know that

$$\sum_{j=1}^{\infty} E(\Delta_j)z = E(\Delta)z \quad (z \in H_1)$$

and, in particular,

$$\sum_{j=1}^{\infty} E(\Delta_j)U^{-1}x = E(\Delta)U^{-1}x.$$

Furthermore, since U is bounded,

$$UE(\Delta)U^{-1}x = U\left(\sum_{j=1}^{\infty} E(\Delta_j)U^{-1}x\right) = \sum_{j=1}^{\infty} UE(\Delta_j)U^{-1}x. \tag{5.1}$$

It follows then, for $x, y \in H_2$, that the set function $(UE(\cdot)U^{-1})_{x,y}$ defined by

$$(UE(\Delta)U^{-1})_{x,y} := (UE(\Delta)U^{-1}x, y)_{H_2} \quad (\Delta \in \mathcal{B}) \tag{5.2}$$

is a complex, regular Borel measure. Indeed, we see from (5.1) and the continuity of the inner product that, if $\{\Delta_j\}_{j=1}^{\infty}$ is a partition of Δ, then

$$\begin{aligned}
(UE(\Delta)U^{-1})_{x,y} &= (UE(\Delta)U^{-1}x, y)_{H_2} \\
&= (UE\left(\cup_{j=1}^{\infty}\Delta_j\right)U^{-1}x, y)_{H_2} \\
&= (U\left(\sum_{j=1}^{\infty} E(\Delta_j)U^{-1}x\right), y)_{H_2} \\
&= \sum_{j=1}^{\infty} (UE(\Delta_j)U^{-1}x, y)_{H_2}.
\end{aligned}$$

This establishes property (vi) and completes the proof of part (a) of the theorem.

Proof of part (b): The first part of the statement in (b) follows from Remark 3.1; the second part can be proven in a similar manner to part (a) above; alternatively, there is a proof in [1, Chapter 6; Section 87].

Proof of part (c): From (5.2), we see from (3.1) that, for any Borel set Δ and any $x, y \in H_2$,

$$\begin{aligned}
(UE(\Delta)U^{-1})_{x,y} &= (UE(\Delta)U^{-1}x, y)_{H_2} \\
&= (E(\Delta)U^{-1}x, U^{*}y)_{H_1} \\
&= (E(\Delta)U^{-1}x, U^{-1}y)_{H_1} \\
&= E_{U^{-1}x, U^{-1}y}(\Delta)
\end{aligned}$$

and, hence,

$$d(UEU^{-1})_{x,y} = dE_{U^{-1}x, U^{-1}y}.$$

On the other hand, for $x \in \mathcal{D}(S)$ and $y \in H_2$, we see from Theorem 3.1 that

$$\begin{aligned}
(Sx, y)_2 &= (UTU^{-1}x, y)_2 \\
&= (TU^{-1}x, U^{-1}y)_1 \\
&= \int_{\mathbb{R}} \lambda dE_{U^{-1}x, U^{-1}y} \\
&= \int_{\mathbb{R}} \lambda d(UEU^{-1})_{x,y}
\end{aligned}$$

and, consequently, it follows that UEU^{-1} is the spectral resolution for S. The self-adjointness of S follows from the spectral theorem. Finally, the proofs of (α), (β), and (γ) are standard and can be found in other sources; for example, see [1, Chapter 6; Section 87]. □

We are now in position to prove the main result of this paper.

Theorem 5.2. (***Fundamental Theorem of Left-Definite Theory***) *Suppose* $A : \mathcal{D}(A) \subseteq H \to H$ *is a self-adjoint operator in the Hilbert space* $H = (V, (\cdot,\cdot))$ *that is bounded below by* kI *for some* $k > 0$. *Let* $\{H_r = (V_r, (\cdot,\cdot)_r)\}_{r>0}$ *and* $\{A_r\}_{r>0}$ *be, respectively, the continua of left-definite Hilbert spaces and left-definite operators associated with* (H, A). *Write* $H_0 = H$, $V_0 = V$, $(\cdot,\cdot)_0 = (\cdot,\cdot)$, *and* $A_0 = A$. *Let* $r, s \geq 0$. *Then we have the following:*

- (i) *The operator* $A^{(r-s)/2} : H_r \to H_s$ *is an isometric isomorphism* (*unitary transformation*) *of the Hilbert space* H_r *onto the Hilbert space* H_s;
- (ii) *For each* $m \geq 0$, *the operator* $A^{(r-s)/2} : H_{r+m} \to H_{s+m}$ *is an isometric isomorphism* (*unitary transformation*) *of the Hilbert space* H_{r+m} *onto* H_{s+m}. *In particular,* $A^{(r-s)/2}$ *is a bijection of* V_{r+2} (*the domain of the* r^{th} *left-definite operator* A_r) *onto* V_{s+2} (*the domain of the* s^{th} *left-definite operator* A_s).
- (iii) *The following operator identities are valid:*

$$A^{-r/2} A A^{r/2} x = A_r x \quad (x \in \mathcal{D}(A_r)), \tag{5.3}$$

$$A^{r/2} A_r A^{-r/2} x = Ax \quad (x \in \mathcal{D}(A)), \tag{5.4}$$

and

$$A^{(r-s)/2} A_r A^{(s-r)/2} x = A_s x \quad (x \in \mathcal{D}(A_s)). \tag{5.5}$$

In particular, each of the operators A, A_r *and* A_s *are similar; more specifically,* A_r *and* A_s *are isometrically isomorphically* (*unitarily*) *equivalent.*

- (iv) *The following domain relationship is valid:*

$$\mathcal{D}(A_s) = A^{(r-s)/2} \mathcal{D}(A_r), \tag{5.6}$$

or equivalently

$$V_{s+2} = A^{(r-s)/2} V_{r+2}.$$

In particular,

$$\mathcal{D}(A) = A^{r/2} \mathcal{D}(A_r) \tag{5.7}$$

and

$$\mathcal{D}(A_r) = A^{-r/2} \mathcal{D}(A). \tag{5.8}$$

- (v) *Each* A_r *is a self-adjoint operator in* H_r;
- (vi) *If* E *is the spectral resolution of the identity for* A, *then* $A^{-r/2} E A^{r/2}$ *is the spectral resolution of the identity for* A_r;
- (vii) *The following results concerning the spectra and resolvent sets are valid*:
  - (a) $\sigma_p(A_r) = \sigma_p(A_s)$; *that is to say, the point spectra of* A_r *and* A_s *are equal; in particular,* $\sigma_p(A) = \sigma_p(A_r)$;
  - (b) $\sigma_c(A_r) = \sigma_c(A_s)$; *that is to say, the continuous spectra of* A_r *and* A_s *are equal; in particular,* $\sigma_c(A) = \sigma_c(A_r)$;
  - (c) $\rho(A_r) = \rho(A_s)$; *that is to say, the resolvent sets of* A_r *and* A_s *are equal; in particular,* $\rho(A) = \rho(A_r)$.

Proof. Let $r, s \geq 0$.

Proof of (i): Notice that if $s = 0$, then (i) is contained in Proposition 4.4. By Proposition 4.4, we see from the diagram

$$H_r \quad \xrightarrow{A^{r/2}} \quad H \quad \xrightarrow{A^{-s/2}} \quad H_s$$

that $A^{-s/2}A^{r/2} = A^{(r-s)/2} : H_r \to H_s$ is a bijection of H_r onto H_s. Moreover, for $x, y \in H_r$, we have

$$(A^{(r-s)/2}x, A^{(r-s)/2}y)_s = (A^{s/2}A^{(r-s)/2}x, A^{s/2}A^{(r-s)/2}y) = (A^{r/2}x, A^{r/2}y) = (x, y)_r,$$

which completes the proof of part (i).

Proof of (ii): This proof follows immediately from (i) since

$$A^{(r-s)/2} = A^{((r+m)-(s+m))/2}.$$

Proof of (iii): Let $x \in \mathcal{D}(A_r) = V_{r+2} = \{y \in H \mid \int_{\mathbb{R}} \lambda^{r+2} dE_{y,y} < \infty\}$. Since $V_{r+2} \subseteq V_r = \mathcal{D}(A^{r/2})$, we see that $x \in \mathcal{D}(A^{r/2})$. From (3.3) and (3.4), we see that

$$\int_{\mathbb{R}} \lambda^2 dE_{A^{r/2}x, A^{r/2}x} = \int_{\mathbb{R}} \lambda^{r+2} dE_{x,x} < \infty,$$

so $A^{r/2}x \in \mathcal{D}(A)$. Since $\mathcal{D}(A^{-r/2}) = H$, it is clear that $AA^{r/2}x \in \mathcal{D}(A^{-r/2})$ and

$$A^{-r/2}AA^{r/2}x = A^{-r/2}A^{r/2+1}x = Ax = A_r x,$$

establishing (5.3). To prove (5.4), let $x \in \mathcal{D}(A) = \{y \in H \mid \int_{\mathbb{R}} \lambda^2 dE_{y,y} < \infty\}$. Again, since $\mathcal{D}(A^{-r/2}) = H$, it is clear that $x \in \mathcal{D}(A^{-r/2})$. Another application of (3.3) and (3.4) shows that

$$\int_{\mathbb{R}} \lambda^{r+2} dE_{A^{-r/2}x, A^{-r/2}x} = \int_{\mathbb{R}} \lambda^{r+2}\lambda^{-r/2}\lambda^{-r/2} dE_{x,x} = \int_{\mathbb{R}} \lambda^2 dE_{x,x} < \infty$$

and, hence, $A^{-r/2}x \in \mathcal{D}(A_r)$. In a similar fashion, we see that

$$\int_{\mathbb{R}} \lambda^r dE_{A^{-r/2+1}x, A^{-r/2+1}x} = \int_{\mathbb{R}} \lambda^r \lambda^{-r/2+1}\lambda^{-r/2+1} dE_{x,x} = \int_{\mathbb{R}} \lambda^2 dE_{x,x} < \infty$$

so that $A^{-r/2+1}x = A_r A^{-r/2}x \in \mathcal{D}(A^{r/2})$. Finally,

$$A^{r/2}A_r A^{-r/2}x = A^{r/2}A^{-r/2+1}x = Ax = A_r x,$$

completing the proof of (5.4). To prove (5.5), we suppose that $s \le r$; the case $s > r$ can be proved in a similar fashion. Let $x \in \mathcal{D}(A_s) = V_{s+2} = \{y \in H \mid \int_{\mathbb{R}} \lambda^{s+2} dE_{y,y} < \infty\}$. Since $(s-r)/2 \le 0$, we see that $x \in \mathcal{D}(A^{(s-r)/2}) = H$. Furthermore, from (3.3) and (3.4), we see that

$$\int_{\mathbb{R}} \lambda^{r+2} dE_{A^{(s-r)/2}x, A^{(s-r)/2}x} = \int_{\mathbb{R}} \lambda^{r+2}\lambda^{(s-r)/2}\lambda^{(s-r)/2} dE_{x,x} = \int_{\mathbb{R}} \lambda^{s+2} dE_{x,x} < \infty$$

which shows that $A^{(s-r)/2}\, x \in \mathcal{D}(A_r)$ and $A_r A^{(s-r)/2}x = A^{(s-r+2)/2}x$. Moreover, since

$$\begin{aligned}
&\int_{\mathbb{R}} \lambda^{r-s} dE_{A^{(s-r+2)/2}x, A^{(s-r+2)/2}x} \\
&= \int_{\mathbb{R}} \lambda^{r-s}\lambda^{(s-r+2)/2}\lambda^{(s-r+2)/2} dE_{x,x} \text{ by (3.3) and (3.4)} \\
&= \int_{\mathbb{R}} \lambda^2 dE_{x,x} < \infty \text{ since } x \in \mathcal{D}(A_s) \subseteq \mathcal{D}(A),
\end{aligned}$$

we see that $A_r A^{(s-r)/2}x \in \mathcal{D}(A^{(r-s)/2})$. Lastly, we see that

$$A^{(r-s)/2}A_r A^{(s-r)/2}x = A^{(r-s)/2}AA^{(s-r)/2}x = Ax = A_s x,$$

completing the proof of (ii).

Proof of (iv): The domain identity in (5.6) follows immediately from (ii) since $\mathcal{D}(A_r) = V_{r+2}$

for all $r \geq 0$.

Proofs of (v), (vi), and (vii): From Proposition 4.4, $A^{-r/2}$ is an isometric isomorphism from H onto H_r. Moreover, from (5.3) and (5.8), we have

$$A^{-r/2}AA^{r/2} = A_r$$

and

$$\mathcal{D}(A_r) = A^{-r/2}\mathcal{D}(A).$$

Hence, with $U = A^{-r/2}$, $T = A$, and $S = A_r$, the conditions of Theorem 5.1 are satisfied. Consequently, A_r is a self-adjoint operator in H_r, $A^{-r/2}E$ is the spectral resolution of the identity for A_r, and the spectral results listed in (a), (b), and (c) of part (vii) are valid. This completes the proof of the theorem.

□

Remark 5.1. The operator identities, given in (5.3), (5.4), and (5.5), clearly connect the operators A, A_r, and A_s. As mentioned in Section 1, based on the general theory built in [12], this *similarity* connection was completely unexpected. Furthermore, the operator domain identities given in (5.6), (5.7), and (5.8) give a remarkable relationship between the domains of the operator A and the left-definite operators $\{A_r\}_{r>0}$. We note that it is possible to prove these identities directly using the spectral theorem; however, the proofs are immediate once it is shown that $A^{(r-s)/2}$ is an isometric isomorphism between the spaces H_r and H_s (part (i) of the above theorem) as well as between the spaces H_{r+2} and H_{s+2} (part (ii)).

Remark 5.2. From part (iv) of Theorem 5.2, we get a new characterization of the domain of the r^{th} left-definite operator A_r, namely

$$\mathcal{D}(A_r) = V_{r+2} = \{A^{-r/2}x \mid x \in \mathcal{D}(A)\}.$$

Remark 5.3. Lastly, we remark that (5.7) is intriguing in the sense that, for any $r \geq 0$, $A^{r/2}\mathcal{D}(A_r)$ is always equal to $\mathcal{D}(A)$; again, this is a new characterization of the domain of A.

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