ON PROPERTIES OF THE LEGENDRE DIFFERENTIAL EXPRESSION

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Dedicated to Professor A. M. Krall

ABSTRACT. This paper surveys the known properties and presents many new results for the Legendre differential expression and the associated linear differential operators in its right-definite and left-definite Hilbert function spaces. The main tool for obtaining the best-possible form of these results is an integral operator inequality due to Chisholm, Everitt, and Littlejohn. Many of the results follow from an unpublished manuscript of Everitt and Marić and a forthcoming paper of Arvesú, Littlejohn, and Marcellán. A recent paper on left-definite differential operators by Vonhoff provides for an interesting comparison with the properties and results in this paper.

1. Introduction

Notations: $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex number fields respectively; $\mathbb{N}$ and $\mathbb{N}_0$ denote the positive and non-negative integers respectively; L denotes Lebesgue integration and AC absolute continuity with respect to Lebesgue measure; (a,b) and $[\alpha,\beta]$ represent open and compact intervals of $\mathbb{R}$, respectively; other notations are introduced in the sections below.

This paper surveys the existing literature and presents many new results concerning the classical Legendre differential expression, defined by

$$\ell[f] := -\left((1-x^2)f'\right)' + kf \quad (x \in (-1,1)), \tag{1.1}$$

where $k \in \mathbb{R}$ is a fixed, positive constant, and its related operators.

The framework for these properties and results is the classical Lebesgue space $L^2(-1,1)$ and the Sobolev space $H_1^2(-1,1)$ defined by

$$H_1^2(-1,1) := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}(-1,1);\ f, (1-x^2)^{1/2} f' \in L^2(-1,1)\}$$

with inner product $(\cdot,\cdot)_1$ given by

$$(f,g)_1 := \int_{-1}^{1} \left((1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\right) dx \quad (f,g \in H_1^2(-1,1)).$$

The differential expression $\ell[\cdot]$ is Lagrangian symmetric (formally self-adjoint) and generates unbounded, self-adjoint differential operators in both the Hilbert spaces $L^2(-1,1)$ and $H_1^2(-1,1)$.

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In the space $L^2(-1,1)$, the expression $\ell[\cdot]$ generates a continuum of self-adjoint operators, the so-called right-definite operators, with domains determined by the general GKN (Glazman, Krein, Naimark) theory of symmetric boundary conditions for quasi-differential expressions; see [13, Chapter V]. Amongst these operators there is a distinguished operator called the Legendre self-adjoint differential operator, denoted in this paper by T. The spectrum $\sigma(T)$ of T is simple and discrete, and given explicitly by

$$\sigma(T) = \{\lambda_n := n(n+1) + k \mid n \in \mathbb{N}_0\}; \tag{1.2}$$

the associated eigenfunctions, or eigenvectors in the space $L^2(-1,1)$, are the classical Legendre polynomials $\{P_n\}_{n=0}^{\infty}$.

In the Sobolev space $H_1^2(-1,1)$, the expression $\ell[\cdot]$ generates a unique, self-adjoint differential operator, denoted in this paper by S, the so-called left-definite Legendre operator. There is an extensive literature devoted to the study of left-definite differential operators (for example, see the paper by Littlejohn and Wellman [11]) but the results closest to the GKN structure are to be found in the work of Vonhoff (see [19] and the references cited therein), based on the Niessen-Schneider theory of S-Hermitian eigenvalue problems; see Section 11 below for further historical comments on the left-definite theory associated with Legendre's equation. In this present paper, the operator S is defined in $H_1^2(-1,1)$ through the properties of the right-definite operator T in $L^2(-1,1)$; the spectrum $\sigma(S)$ of S mirrors the spectral properties of T in that it is simple, discrete and given by

$$\sigma(S) = \sigma(T) = \{\lambda_n \mid n \in \mathbb{N}_0\},$$

where λ_n is defined in (1.2); again, the associated eigenvectors are the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$.

The results in this paper are best-possible, particularly in describing the properties of the two domains $\mathcal{D}(T)$ and $\mathcal{D}(S)$ of the operators T and S. The methods used may be described as classical within Hilbert space theory but require repeated use of the sharp integral operator results in the Chisholm, Everitt, Littlejohn paper [4].

In writing this article it is important to have in mind a view of the significant and relevant results in the existing literature. In this respect we comment here on seven papers listed in the references; the papers [15] and [16] by Pleijel , the paper of Everitt [5] and an unpublished manuscript [8] by Everitt and Marić, the paper [10] by A. M. Krall and Littlejohn, the paper [19] by Vonhoff, and the paper [2] by Arvesú, Littlejohn and Marcellán.

1. *The Pleijel papers* [15] *and* [16]
   - (i) The initial study of Legendre's equation in the left-definite setting is due to Pleijel; see [15] and [16]. In particular, we owe to Pleijel the important obervation that the Legendre equation is limit-point at both endpoints ± 1 in the left-definite setting $H_1^2(-1,1)$.
2. *The Everitt paper* [5] *and the Everitt-Marić unpublished manuscript* [8]
   - (i) In [5], there is a detailed account of both the right-definite and left-definite analysis of the Legendre differential equation. In particular, the approach for the right-definite study of this equation is in the spirit of the approach taken by Titchmarsh in [18], who is generally credited with the first analytical study of the Legendre

equation in $L^2(-1,1)$. In the left-definite setting, Everitt first shows that the resolvent operator $\mathcal{R}(T) = T^{-1}$ is injective and self-adjoint in the left-definite space, which he denotes by $H^2(-1,1)$; consequently its inverse, which as we will see in Section 11 below is the same as our left-definite operator S in this present paper, is self-adjoint. He further determines the spectrum of his left-definite operator and shows that the Legendre polynomials form a complete orthogonal set in $H^2(-1,1)$.

(*ii*) The unpublished Everitt-Marić manuscript [8] contains the first proof of the smoothness property

$$f \in \mathcal{D}(T) \Rightarrow f' \in L^2(-1,1) \tag{1.3}$$

of the domain $\mathcal{D}(T)$ of the Legendre differential operator; see the remark at the end of Section 7 below. This manuscript also contains the definition of the left-definite operator S that we use in this present paper (see Definition 11.1). Furthermore, the manuscript [8] includes the first proof of the characterization of $\mathcal{D}(S)$; we give this proof in Theorem 12.1.

3. *The Krall-Littlejohn paper* [10]
   (*i*) In [10], the left-definite operator (which they denote by $\mathcal{L}$) has the same definition as our operator S (see Definition 11.1 below).
   (*ii*) The first proof of the self-adjointness of the left-definite operator is given in [10, Theorem 4.6]; we give a different proof in this paper (see Theorem 11.2).
4. *The Vonhoff paper* [19]
   (*i*) Within the general left-definite theory developed in [19], the Legendre differential expression $\ell[\cdot]$ has deficiency index 0, see [19, Lemma 3.3], in the space $H_1^2(-1,1)$ (here, $H_1^2(-1,1)$ is the notation of our paper; the notation in [19, Section 3] is $W(-1,1)$); compare with Theorem 11.6 below. The Legendre maximal operator $S_{\max}$, in [19, Section 3], (for the general definition of $S_{\max}$, see [19, Section 2]), is self-adjoint in $H_1^2(-1,1)$, see [19, Section 3, Theorem 3.4]; as we will see in Section 11, this operator $S_{\max}$ is identical with our left-definite operator S.
   (*ii*) There is an extensive and valuable list of references on the left-definite theory of differential operators, and the application to the corresponding properties of orthogonal polynomials. There are detailed assessments of the results from these referenced papers in [19, Section 1], [19, Section 3.2], and [19, Section 4.2].
   (*iii*) Some, but not the sharp and best-possible results, of the properties of the domain $\mathcal{D}(S)$, as proved in Theorem 12.1 in Section 12 below, are given in [19].
5. *The Arvesú, Littlejohn, Marcellán paper* [2]
   (*i*) This paper [2] uses quite different methods to the techniques used in this present paper; in particular no reference is made to the results in [4]. Nevertheless, this paper obtains the same sharp, best-possible properties for the domain $\mathcal{D}(T)$ of the right-definite Legendre operator, see [2, Corollaries 7.1 and 7.2], and for the domain $\mathcal{D}(S)$, see [2, Theorem 8.1]; we refer again to these results in the sections below.
   (*ii*) In [2, Theorem 8.2] there is a new proof of the main result (1.3) obtained by Everitt and Marić in the unpublished manuscript [8].

The contents of the paper are as follows: the Legendre differential expression and equation are considered in Sections 2 and 3; the Hilbert function spaces $L^2(-1,1)$ and $H_1^2(-1,1)$ are discussed in Section 4; the integral inequality given in [4] is discussed in Section 5; the definition and properties of the right-definite Legendre differential operator T with domain $\mathcal{D}(T)$ are given in Sections 6,7, and 8; Sections 9 and 10 concern the maximal differential operator T_1 and its self-adjoint restrictions; finally the definition and properties of the left-definite Legendre operator S and its domain $\mathcal{D}(S)$ are given in Sections 11 and 12.

2. The Legendre differential expression

The *Legendre differential equation* is defined by

$$\ell[y](x) = \lambda y(x), \tag{2.1}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue parameter and $\ell[\cdot]$ is the *Legendre differential expression* defined by

$$\begin{aligned} \ell[f](x) :&= -\left((1-x^2)f'(x)\right)' + kf(x) \\ &= -(1-x^2)f''(x) + 2xf'(x) + kf(x), \end{aligned} \tag{2.2}$$

for $f \in \mathcal{D}(\ell)$, where

$$\mathcal{D}(\ell) = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1)\};$$

here k is a fixed, non-negative constant. In most textbooks, the classical Legendre expression (2.2) is written with $k = 0$; however, for spectral reasons, it is essential to have $k > 0$. We note that this expression is Lagrange symmetric (formally self-adjoint).

For $-1 < a < b < +1$, *Green's formula* and *Dirichlet's formula* for $\ell[\cdot]$ are given, respectively by

$$\int_a^b \left\{\ell[f](x)\overline{g}(x) - \overline{\ell[g]}(x)f(x)\right\} dx = [f,g](x) \mid_a^b \quad (f, g \in \mathcal{D}(\ell)), \tag{2.3}$$

and

$$\begin{aligned} \int_a^b \ell[f](x)\overline{g}(x)dx &= -(1-x^2)f'(x)\overline{g}(x) \mid_a^b \\ &\quad + \int_a^b \left((1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\right) dx \quad (f, g \in \mathcal{D}(\ell)); \end{aligned} \tag{2.4}$$

here $[\cdot,\cdot](\cdot) : \mathcal{D}(\ell) \times \mathcal{D}(\ell) \times (-1,1) \to \mathbb{C}$ is the sesquilinear (symplectic) form defined by

$$[f,g](x) := (1-x^2)\left(f(x)\overline{g}'(x) - f'(x)\overline{g}(x)\right) \quad (f, g \in \mathcal{D}(\ell);\ -1 < x < 1). \tag{2.5}$$

3. The Legendre differential equation

The complex form of the Legendre differential equation (2.1) is given by

$$(z^2-1)w''(z) + 2zw'(z) + (k-\lambda)w(z) = 0 \quad (z \in \mathbb{C}). \tag{3.1}$$

In this latter form, there are two finite regular singularities at ± 1 so that the Frobenius method can be applied to obtain series solutions. The only Frobenius indicial root is $\sigma = 0$, a double root at both $z = \pm 1$.

At $z=1$, the series solutions take the form

$$w_{1,+}(z)=1+a_1(\lambda,k)(z-1)+a_2(\lambda,k)(z-1)^2+\dots, \tag{3.2}$$

and

$$w_{2,+}(z)=w_{1,+}(z)\log(z-1)+\sum_{n=0}^{\infty}b_n(\lambda,k)(z-1)^n. \tag{3.3}$$

Both these infinite series are absolutely and locally uniformly convergent in $|z-1|<2$. We briefly explain the factor $\log(z-1)$ in (3.3); write $z-1=\rho\exp(i\varphi)$, where $\rho=|z-1|$ and, on cutting the z complex plane from 1 to ∞ on $\mathbb{R}\subset\mathbb{C}$, the amplitude φ is confined to $0\leq\varphi<2\pi$. Then define

$$\log(z-1):=\ln(\rho)+i\varphi. \tag{3.4}$$

The infinite series for $w_{1,+}$ and $w_{2,+}$ are differentiable and give

$$w'_{1,+}(z)=a_1(\lambda,k)+2a_2(\lambda,k)(z-1)+\dots \tag{3.5}$$

$$\begin{aligned} w'_{2,+}(z) &= \frac{w_{1,+}(z)}{z-1}+w'_{1,+}(z)\log(z-1)+\sum_{n=0}^{\infty}nb_n(\lambda,k)(z-1)^{n-1} \\ &= \frac{1}{z-1}+\sum_{n=1}^{\infty}a_n(\lambda,k)(z-1)^{n-1}+\log(z-1)\sum_{n=1}^{\infty}na_n(\lambda,k)(z-1)^n \\ &\quad +\sum_{n=0}^{\infty}nb_n(\lambda,k)(z-1)^{n-1}. \end{aligned} \tag{3.6}$$

There are corresponding solutions of (3.1) in respect of the regular singularity at -1; that is,

$$w_{1,-}(z)=1+\sum_{n=1}^{\infty}c_n(\lambda,k)(z+1)^n, \tag{3.7}$$

and

$$w_{2,-}(z)=w_{1,-}(z)\log(z+1)+\sum_{n=0}^{\infty}d_n(\lambda,k)(z+1)^n, \tag{3.8}$$

convergent in $|z+1|<2$. Here we define the factor $\log(z+1)=\ln(\rho)+i\varphi$, where $z+1=\rho\exp(i\varphi)$ and now, on cutting the z-plane from -1 to $-\infty$ on $\mathbb{R}\subset\mathbb{C}$, we take the amplitude φ to satisfy $-\pi<\varphi\leq\pi$.

There are results similar to (3.5) and (3.6) for the derivatives of $w_{1,-}(z)$ and $w_{2,-}(z)$.

We note that when $\lambda\in\mathbb{R}$ all the coefficients $\{a_n(\lambda,k)\}$, $\{b_n(\lambda,k)\}$, $\{c_n(\lambda,k)\}$, and $\{d_n(\lambda,k)\}$ are real-valued.

Taking into account our discussion above on the relationship between the complex logarithm function and the natural logarithm, we use these four Frobenius solutions to define solutions of (2.2) on the interval $(-1,1)$ with the same notation:

$$y_{1,+}(x,\lambda,k) \equiv y_{1,+}(x,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(\lambda,k)(x-1)^n \tag{3.9}$$

$$y_{2,+}(x,\lambda) = y_{1,+}(x,\lambda)\ln(1-x) + \sum_{n=0}^{\infty} b_n(\lambda,k)(x-1)^n \tag{3.10}$$

$$y_{1,-}(x,\lambda) = 1 + \sum_{n=1}^{\infty} c_n(\lambda,k)(x+1)^n \tag{3.11}$$

$$y_{2,-}(x,\lambda) = y_{1,-}(x,\lambda) = y_{1,-}(x,\lambda)\ln(1+x) + \sum_{n=0}^{\infty} d_n(\lambda,k)(x+1)^n. \tag{3.12}$$

All of these solutions are real-valued on $(-1,1)$ when $\lambda \in \mathbb{R}$. The pair $\{y_{1,+}(\cdot,\lambda), y_{2,+}(\cdot,\lambda)\}$ form a basis of solutions of equation (2.2) for all $\lambda \in \mathbb{C}$; similarly for the pair of solutions $\{y_{1,-}(\cdot,\lambda), y_{2,-}(\cdot,\lambda)\}$.

The following properties of these solutions are seen to hold, for all $\lambda \in \mathbb{C}$, with the one-sided derivatives defined at ± 1 :

$$y_{1,+}(1,\lambda) = 1 \qquad y'_{1,+}(1,\lambda) = a_1(\lambda,k) \tag{3.13}$$

$$y_{1,-}(-1,\lambda) = 1 \qquad y'_{1,-}(-1,\lambda) = c_1(\lambda,k) \tag{3.14}$$

$$y_{2,+}(x,\lambda) = \ln(1-x) + O(1) \quad (x \to 1^-) \tag{3.15}$$

$$y_{2,-}(x,\lambda) = \ln(1+x) + O(1) \quad (x \to -1^+), \tag{3.16}$$

where the $O(1)$ terms are uniformly bounded on $[-1,1]$ for each $\lambda \in \mathbb{R}$,

$$y'_{2,+}(x,\lambda) = \frac{-1}{1-x} + O(|\ln(1-x)|) \quad (x \to 1^-) \tag{3.17}$$

$$y'_{2,-}(x,\lambda) = \frac{1}{1+x} + O(|\ln(1+x)|) \quad (x \to -1^+), \tag{3.18}$$

where the finite constants implied in the $O(\cdot)$ terms depend on $\lambda \in \mathbb{C}$.

From classical results, see for example [17, Chapter IV], it is known that the differential equation (2.2) has polynomial solutions, called the Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$, when the spectral parameter λ takes the corresponding values λ_n, where λ_n is defined in (1.2). The Legendre polynomial $P_n(x)$ is of degree exactly n for each $n \in \mathbb{N}_0$; it has the explicit representation

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)! x^{n-2k}}{k!(n-k)!(n-2k)!} \quad (n \in \mathbb{N}_0). \tag{3.19}$$

The Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are orthogonal in $L^2(-1,1)$; specifically,

$$(P_n, P_m) = \frac{2}{2n+1}\delta_{n,m} \quad (n, m \in \mathbb{N}_0),$$

where $\delta_{n,m}$ is the Kronecker delta symbol.

We note from properties (3.13) through (3.18) that there exist real numbers $\{K_{n,+}\}_{n=0}^{\infty}$ and $\{K_{n,-}\}_{n=0}^{\infty}$ such that, for all $n \in \mathbb{N}_0$,

$$P_n(x) = K_{n,+}y_{1,+}(x, n(n+1)+k) \quad (x \in (-1,1)), \tag{3.20}$$

and

$$P_n(x) = K_{n,-}y_{1,-}(x, n(n+1)+k) \quad (x \in (-1,1)). \tag{3.21}$$

4. Hilbert function spaces

The two Hilbert function spaces involved in this study of the Legendre differential expression are

(i) the right-definite space $L^2(-1,1)$, and

(ii) the left-definite space $H_1^2(-1,1)$.

The space $L^2(-1,1)$ is the classic integrable-square space of equivalence classes of Lebesgue measurable functions $f : (-1,1) \to \mathbb{C}$ such that $\int_{-1}^{1} |f(x)|^2 \, dx < \infty$ with inner product

$$(f, g) := \int_{-1}^{1} f(x)\overline{g}(x)dx \quad (f, g \in L^2(-1,1)). \tag{4.1}$$

The space $H_1^2(-1,1)$ is defined by

$$H_1^2(-1,1) := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}(-1,1); \; f, (1-x^2)^{1/2}f' \in L^2(-1,1)\} \tag{4.2}$$

with inner product

$$(f, g)_1 := \int_{-1}^{1} \left\{(1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\right\} dx, \tag{4.3}$$

where $k > 0$ is the constant given in the definition of $\ell[\cdot]$ in (2.2). We note that the definition of $H_1^2(-1,1)$ may be simplified to read

$$H_1^2(-1,1) = \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}(-1,1); (1-x^2)^{1/2}f' \in L^2(-1,1)\}.$$

Indeed, if $f \in AC_{loc}(-1,1)$ and $(1-x^2)^{1/2}f' = g \in L^2(-1,1)$, then

$$f(x) = f(0) + \int_0^x \frac{g(t)}{(1-t^2)^{1/2}}dt \quad (x \in [0,1));$$

an application of Hölder's inequality now gives

$$|f(x)|^2 \leq K \left|\ln(1-x)\right| \quad \text{as } x \to 1^-$$

for some $K > 0$; hence $f \in L^2(0,1)$. Similarly $f \in L^2(-1,0)$ and so $f \in L^2(-1,1)$.

The space $H_1^2(-1,1)$ is actually a Hilbert space of functions rather than a space of equivalence classes as in $L^2(-1,1)$; the null element of $H_1^2(-1,1)$ is the zero function on $(-1,1)$. The proof that the vector space $H_1^2(-1,1)$ is complete in the norm derived from the inner product (4.3) is given in [12]; see also [2].

It is well known (see [17, Theorem 3.1.5]) that the set of Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ is a complete, orthogonal set in $L^2(-1,1)$. In fact, the Legendre polynomials also form a complete orthogonal set in $H_1^2(-1,1)$; we prove this fact below in Section 11 (see also [2] and [12]). It can be seen that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are orthogonal in $H_1^2(-1,1)$ through well-known properties of the first derivative of the Gegenbauer polynomials $\{P_n^{(1,1)}\}_{n=0}^{\infty}$; see [17, Chapter 4, Section 21]. In fact, as we will verify in Section 6 below, we find that

$$(P_n, P_m)_1 = \frac{2}{2n+1}(n(n+1)+k)\delta_{n,m} \quad (n, m \in \mathbb{N}_0). \tag{4.4}$$

Lemma 4.1. *For all $\lambda \in \mathbb{C}$, the following properties hold for the solution base*

$$\{y_{1,+}(\cdot,\lambda), y_{2,+}(\cdot,\lambda)\}$$

of the Legendre differential equation (2.2) :

$$\begin{array}{ll} y_{1,+}(\cdot,\lambda) \in L^2(0,1) & y_{2,+}(\cdot,\lambda) \in L^2(0,1) \\ y_{1,+}(\cdot,\lambda) \in H_1^2(0,1) & y_{2,+}(\cdot,\lambda) \notin H_1^2(0,1). \end{array} \tag{4.5}$$

There are corresponding results for the solution base $\{y_{1,-}(\cdot,\lambda), y_{2,-}(\cdot,\lambda)\}$ and the spaces $L^2(-1,0)$ and $H_1^2(-1,0)$.

Proof. These results follow from the properties of the solutions, as $x \to 1^-$, given in equations (3.13) through (3.18) in Section 3. □

5. An integral operator property

In [4], the authors prove the following special case of a theorem regarding certain types of integral operators defined on the Hilbert space $L^2((a,b);w)$, where $w > 0$ (a.e. $x \in (a,b)$) is Lebesgue measurable.

Theorem 5.1. *Let (a,b) be an open interval of the real line (bounded or unbounded). Suppose $\varphi, \psi : (a,b) \to \mathbb{C}$ satisfy the conditions*
(i) $\varphi, \psi \in L^2_{loc}((a,b);w)$;
(ii) there exists $c \in (a,b)$ such that $\varphi \in L^2((a,c];w)$ and $\psi \in L^2([c,b);w)$;
(iii) for all $[\alpha,\beta] \subset (a,b)$

$$\int_\alpha^\beta |\varphi(x)|^2 w(x)dx > 0 \text{ and } \int_\alpha^\beta |\psi(x)|^2 w(x)dx > 0.$$

Define the linear operators A, $B : L^2((a,b);w) \to L^2_{loc}((a,b);w)$ by

$$(Af)(x) = \varphi(x)\int_x^b \psi(x)f(x)w(x)dx \quad (x \in (a,b); f \in L^2((a,b);w)),$$

and

$$(Bf)(x) = \psi(x)\int_a^x \varphi(x)f(x)w(x)dx \quad (x \in (a,b); f \in L^2((a,b);w)).$$

Moreover, define $K : (a,b) \to (0,\infty)$ by

$$K(x) = \left(\int_a^x |\varphi(x)|^2 w(x)dx\right)^{1/2}\left(\int_x^b |\psi(x)|^2 w(x)dx\right)^{1/2} \quad (x \in (a,b)), \tag{5.1}$$

and let $K \in [0,\infty]$ be defined by

$$K := \sup\{K(x) \mid x \in (a,b)\}. \tag{5.2}$$

Then a necessary and sufficient condition that A and B are both bounded operators from $L^2((a,b);w)$ into $L^2((a,b);w)$ is that

$$0 < K < \infty.$$

As a consequence of the proof of Theorem 5.1, we have the following important corollary which is useful in our discussion of the Legendre equation and other applications; we make repeated use of this result throughout the remainder of this paper.

Corollary 5.1. *Suppose the functions φ and ψ are as in Theorem 5.1. Let $g \in L^2((a,b);w)$. Define*

$$g_1(x) = \varphi(x)\int_x^b \psi(x)g(x)w(x)dx \quad (x \in (a,b)), \tag{5.3}$$

$$g_2(x) = \psi(x)\int_a^x \varphi(x)g(x)w(x)dx \quad (x \in (a,b)). \tag{5.4}$$

If $K < \infty$, where K is defined in (5.2), then $g_r \in L^2((a,b);w)$ for $r = 1,2$.

6. The Legendre differential operator T in $L^2(-1,1)$

This section is based on the general GKN (Glazman, Krein, Naimark) theory of self-adjoint differential operators generated by real, Lagrange symmetric (formally self-adjoint) differential expressions in Hilbert spaces; see [13, Chapters IV and V]. Applications of this theory to the classical second-order differential equations having orthogonal polynomial solutions can be found, for example, in the theses of Loveland [12] and Onyango-Otieno [14] .

The maximal operator $T_1 : \mathcal{D}(T_1) \subset L^2(-1,1) \to L^2(-1,1)$ generated by the differential expression $\ell[\cdot]$, given in (2.2), is defined by

$$\mathcal{D}(T_1) := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1);\ f, \ell[f] \in L^2(-1,1)\} \tag{6.1}$$

and

$$T_1 f = \ell[f] \quad (f \in \mathcal{D}(T_1)). \tag{6.2}$$

The Green's formula (2.3) shows that the limits

$$[f,g](-1) := \lim_{x\to -1^+}[f,g](x) \text{ and } [f,g](1) := \lim_{x\to 1^-}[f,g](x) \tag{6.3}$$

both exist and are finite for all $f, g \in \mathcal{D}(T_1)$.

The minimal operator $T_0 : \mathcal{D}(T_0) \subset L^2(-1,1) \to L^2(-1,1)$ is then defined by

$$\mathcal{D}(T_0) := \{f \in \mathcal{D}(T_1) \mid [f,g](1) - [f,g](-1) = 0 \text{ for all } g \in \mathcal{D}(T_1)\} \tag{6.4}$$

and

$$T_0 f = \ell[f] \quad (f \in \mathcal{D}(T_0)). \tag{6.5}$$

From [13, Chapter V], it is known that these linear differential operators have the properties:

(6.6)
(i) T_0 is closed and symmetric in $L^2(-1,1)$;
(ii) $T_0^* = T_1$ and $T_1^* = T_0$ so that T_1 is closed in $L^2(-1,1)$;
(iii) the deficiency indices $\{d^-(T_0), d^+(T_0)\}$ of T_0 are given by
$$d^-(T_0) = d^+(T_0) = 2.$$

Properties (i) and (ii) follow from the general theory; property (iii) follows from the result that the differential expression $\ell[\cdot]$ is in the limit-circle condition in $L^2(-1,1)$ at both endpoints ± 1 of the interval $(-1,1)$; in turn this result follows from the properties in $L^2(-1,1)$ of the solutions $\{y_{r,+}(\cdot,\lambda), y_{r,-}(\cdot,\lambda)\}_{r=1}^2$ of the differential equation $\ell[y] = \lambda y$ on $(-1,1)$, as given in Section 3.

Any self-adjoint operator A in $L^2(-1,1)$, generated by $\ell[\cdot]$ is, from the GKN theory an extension of T_0 and a restriction of T_1; that is

$$T_0 \subset A = A^* \subset T_1. \tag{6.7}$$

The domain $\mathcal{D}(A)$ of such an operator A is determined from the GKN boundary conditions involving the symplectic form $[\cdot,\cdot](\cdot)$, defined in (2.5), and the maximal domain $\mathcal{D}(T_1)$; see [13, Section 18.1, Theorem 4].

Here we are concerned only with the Legendre differential operator, say T, given by

$$Tf = \ell[f] \quad (f \in \mathcal{D}(T)), \tag{6.8}$$

where the domain $\mathcal{D}(T)$ is defined by the GKN separated boundary conditions

$$\mathcal{D}(T) := \{f \in \mathcal{D}(T_1) \mid \lim_{x\to -1^+} [f,1](x) = \lim_{x\to 1^-} [f,1](x) = 0\}; \tag{6.9}$$

equivalently, the boundary conditions take the explicit form

$$\mathcal{D}(T) := \{f \in \mathcal{D}(T_1) \mid \lim_{x\to -1^+} (1-x^2)f'(x) = \lim_{x\to 1^-} (1-x^2)f'(x) = 0\}. \tag{6.10}$$

The spectral properties of the self-adjoint operator T are known and are quoted as

Lemma 6.1. *For the operator T, we have the following properties:*

(i) the spectrum $\sigma(T)$ of T is discrete and simple and is given by $\sigma(T) = \{\lambda_n \mid n \in \mathbb{N}_0\}$, where λ_n is defined in (1.2).

(ii) the operator T is bounded below by kI, where I is the identity operator in $L^2(-1,1)$.

(iii) the eigenvectors of T are the eigenfunctions $\{P_n\}_{n=0}^\infty$, the Legendre polynomials.

(iv) $\{P_n\}_{n=0}^\infty$ is a complete orthogonal set in $L^2(-1,1)$.

Proof. See [1, Volume I, Appendix 1] and [5] for the proofs of (i), (ii), and (iii) and [17, Theorem 3.1.5] for the proof of (iv). □

We now prove some additional properties of the domain $\mathcal{D}(T)$.

Theorem 6.1. *Let $\mathcal{D}(T) \subset L^2(-1,1)$ be defined as in* (6.10) *above. Then for all $f, g \in \mathcal{D}(T)$*

(i) $(1-x^2)^{1/2} f' \in L^2(-1,1)$ *and hence* $\mathcal{D}(T) \subset H_1^2(-1,1)$, *the vector space of functions defined in* (4.2);

(ii) $\lim_{x \to \pm 1} (1-x^2) f'(x) \overline{g}(x) = 0$;

(iii)

$$(Tf, g) = \int_{-1}^{1} \left((1-x^2) f'(x) \overline{g}'(x) + k f(x) \overline{g}(x) \right) dx = (f, g)_1, \tag{6.11}$$

where $(\cdot, \cdot)_1$ *is the inner product defined in* (4.3).

Proof. We note that it is sufficient to prove these results for real-valued $f, g \in \mathcal{D}(T)$.
(i) Let $f \in \mathcal{D}(T)$; then from the Dirichlet formula (2.4), we have for $x \in (0,1)$,

$$\int_0^x \left((1-t^2)(f'(t))^2 + k(f(t))^2 \right) dt = (1-t^2) f'(t) f(t) \mid_0^x + \int_0^x \ell[f](t) f(t) dt. \tag{6.12}$$

If (i) in the theorem does not hold then, for the endpoint 1,

$$\lim_{x \to 1^-} (1-x^2) f'(x) f(x) = \infty;$$

hence for some $K > 0$ and x near 1^- we have

$$(1-x^2) f'(x) f(x) > K.$$

We can suppose then that $(1-x^2) f'(x) > 0$ and $f(x) > 0$ near 1^- and so, for some $\xi \in (0,1)$,

$$\left| ((1-x^2) f'(x))' \right| f(x) > K \frac{|((1-x^2) f'(x))'|}{(1-x^2) f'(x)} \quad (x \in (\xi, 1)).$$

Integration gives, for all $x \in [\xi, 1)$,

$$\begin{aligned} \int_\xi^x \left| ((1-t^2) f'(t))' \right| f(t) dt &\geq K \int_\xi^x \frac{|((1-t^2) f'(t))'|}{(1-t^2) f'(t)} dt \\ &\geq K \left| \int_\xi^x \frac{((1-t^2) f'(t))'}{(1-t^2) f'(t)} dt \right| \\ &= K \left| \ln((1-t^2) f'(t)) \right|_\xi^x \\ &\geq K' \left| \ln((1-x^2) f'(x) \right| \end{aligned}$$

for some K' belonging to $(0, K)$. The right-hand side of this last result tends to ∞ as $x \to 1^-$ from the boundary condition in (6.10). On the other hand, since f and $\ell[f]$ belong to $L^2(-1,1)$, the left-hand side converges to a finite limit as $x \to 1^-$. There is a similar argument for the endpoint -1. This contradiction establishes property (i) of $\mathcal{D}(T)$.

(ii) Let $f, g \in \mathcal{D}(T)$. From part (i) above and (6.12), we see that there exists $L \in \mathbb{R}$ such that

$$\lim_{x \to 1^-} (1-x^2) f'(x) g(x) = L.$$

If $L \neq 0$, then we can assume that $(1-x^2)f'(x) > 0$ near 1^- and then, repeating the argument used in (i) above, we obtain, for some $\xi \in (0,1)$ and some $K > 0$

$$\int_{\xi}^{x} \left| \left((1-t^2)f'(t)\right)' g(t) \right| dt \geq K \left|\ln((1-x^2)f'(x)\right| \quad (x \in (\xi, 1))$$

and this again gives a contradiction unless $L = 0$. There is a similar argument for the endpoint -1.

Finally, the identity in (iii) follows immediately from (ii) and the Dirichlet formula in (2.4).

This completes the proof of the theorem. □

Corollary 6.1. *The result in Theorem 6.1 (ii) extends to give*

$$\lim_{x \to \pm 1} (1-x^2)f'(x)\overline{g}(x) = 0 \tag{6.13}$$

for all $f \in \mathcal{D}(T)$ and for all $g \in H_1^2(-1,1)$. Consequently, we obtain the extended Dirichlet identities

$$(Tf, g) = (f, g)_1 \quad (f \in \mathcal{D}(T), g \in H_1^2(-1,1)) \tag{6.14}$$

$$(f, Tg) = (f, g)_1 \quad (f \in H_1^2(-1,1), g \in \mathcal{D}(T)). \tag{6.15}$$

Proof. The proofs of (6.13)-(6.15) are identical to the proofs of (ii) and (iii) in Theorem 6.1. □

We can now use the above results to prove the essential properties of the sequence $\{P_n\}_{n=0}^{\infty}$ of Legendre polynomials, but now regarded as vectors in the Hilbert function space $H_1^2(-1,1)$.

Theorem 6.2. *Let the Hilbert function space $H_1^2(-1,1)$ be defined as in Section 4 above. Then the sequence $\{P_n\}_{n=0}^{\infty}$ of Legendre polynomials forms a complete orthogonal set in $H_1^2(-1,1)$.*

Proof. We recall from Section 4 that the inner product $(\cdot,\cdot)_1$ for $H_1^2(-1,1)$ is defined by

$$(f,g)_1 = \int_{-1}^{1} \left((1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\right) dx \quad (f, g \in H_1^2(-1,1));$$

observe from (iii) of Theorem 6.1 that

$$(Tf, g) = (f, g)_1 \quad (f, g \in \mathcal{D}(T)).$$

In particular, since $\{P_n\}_{n=0}^{\infty} \subset H_1^2(-1,1) \cap \mathcal{D}(T)$, we see that

$$(P_n, P_m)_1 = (TP_n, P_m) = \lambda_n(P_n, P_m) = \frac{2}{2n+1}(n(n+1)+k)\delta_{n,m} \quad (n, m \in \mathbb{N}_0), \tag{6.16}$$

which confirms (4.4) in Section 4. Suppose $g \in H_1^2(-1,1)$; from (6.14), we see that

$$(P_n, g)_1 = (TP_n, g) = \lambda_n(P_n, g) \quad (n \in \mathbb{N}_0).$$

Hence, if $(P_n, g)_1 = 0$ for all $n \in \mathbb{N}_0$, we see that $(P_n, g) = 0$ for all $n \in \mathbb{N}_0$ (since $\lambda_n \neq 0$). Since the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are complete in $L^2(-1,1)$, we see that g is the null function in $L^2(-1,1)$; that is, $g(x) = 0$ for almost all $x \in (-1,1)$. Since $g \in H_1^2(-1,1)$, this last result implies that $g(x) = 0$ for all $x \in (-1,1)$ and hence g is the null vector in $H_1^2(-1,1)$. From standard results in Hilbert space theory it follows that the orthogonal sequence $\{P_n\}_{n=0}^{\infty}$ is complete in $H_1^2(-1,1)$. □

7. Equivalent properties of the domain $\mathcal{D}(T)$

In this section we prove the following theorem:

Theorem 7.1. *Let the Legendre operator domain $\mathcal{D}(T)$ be defined as in Section six above. Then*
(i) $f \in \mathcal{D}(T)$ if and only if $f \in \mathcal{D}(T_1)$ and $f' \in L^2(-1,1)$. Furthermore, this result is best possible in that there exists $g \in \mathcal{D}(T)$ such that $g' \notin L^p(-1,1)$ for any $p > 2$ and where g is independent of p.
(ii) $f \in \mathcal{D}(T)$ if and only if $f \in \mathcal{D}(T_1)$ and f is bounded on $(-1,1)$.
(iii) $f \in \mathcal{D}(T)$ if and only if $f \in \mathcal{D}(T_1)$ and $f \in AC[-1,1]$.
(iv) $f \in \mathcal{D}(T)$ if and only if $f \in \mathcal{D}(T_1)$ and $(1-x^2)^{1/2}f' \in L^2(-1,1)$.

Proof. We begin by noting that it suffices to prove the results in (i), (ii), (iii), and (iv) for real-valued functions $f \in \mathcal{D}(T_1)$. We prove these results in eight steps.

Step 1 Let $f \in \mathcal{D}(T)$; we show that

$$f \in \mathcal{D}(T_1) \text{ and } f' \in L^2(-1,1). \tag{7.1}$$

Clearly

$$f \in \mathcal{D}(T_1) \tag{7.2}$$

follows from the definition of $\mathcal{D}(T)$. To prove that $f' \in L^2(-1,1)$, we note that $f \in \mathcal{D}(T)$ implies that $\lim_{x\to 1^-}(1-x^2)f'(x) = 0$. From the definition of $\mathcal{D}(T_1)$ we obtain $((1-x^2)f')' \in L^2(-1,1) \subset L^1(-1,1)$; thus for all $x \in [0,1)$

$$\begin{aligned} \int_x^1 \left((1-t^2)f'(t)\right)' dt &= \lim_{\xi\to 1^-} \int_x^{\xi} \left((1-t^2)f'(t)\right)' dt \\ &= \lim_{\xi\to 1^-} (1-\xi^2)f'(\xi) - (1-x^2)f'(x) \\ &= -(1-x^2)f'(x). \end{aligned} \tag{7.3}$$

Hence, for all $x \in [0,1)$

$$f'(x) = \frac{-1}{1-x^2}\int_x^1 \left((1-t^2)f'(t)\right)' dt. \tag{7.4}$$

The integrand on the right-hand side is in $L^2(0,1)$ and so we can apply (5.3), with

$$\varphi(x) = 1/(1-x^2), \ \psi(x) = 1 \quad (x \in [0,1))$$

to obtain $f' \in L^2(0,1)$; an entirely similar argument puts $f' \in L^2(-1,0)$ so

$$f' \in L^2(-1,1). \tag{7.5}$$

Combining (7.2) and (7.5), we obtain (7.1).

Step 2 Let $f \in \mathcal{D}(T)$; we show that

$$f \in \mathcal{D}(T_1) \text{ and } f \text{ is bounded on } (-1,1). \tag{7.6}$$

Again, it is clear that $f \in \mathcal{D}(T_1)$. From the result in (7.1), we have $f' \in L^2(-1,1) \subset L^1(-1,1)$. With appropriate definition of f at the endpoints ± 1, if necessary, it follows that $f \in AC[-1,1]$ and so f is bounded. This proves (7.6).

Step 3 Suppose $f \in \mathcal{D}(T_1)$ and f is bounded on $(-1,1)$. We now show that

$$\lim_{x \to 1^-} (1-x^2) f'(x) = 0; \tag{7.7}$$

a similar argument will show that

$$\lim_{x \to -1^+} (1-x^2) f'(x) = 0,$$

and hence that $f \in \mathcal{D}(T)$. We begin by showing that $(1-x^2)^{1/2} f' \in L^2(0,1)$; for if not, then the Dirichlet formula (2.4) shows that there exists a positive number K_1 such that for all x near to 1^- we have $(1-x^2)f'(x)f(x) > K_1$. Hence, for some positive number K_2 it follows that $f'(x)f(x) > K_2(1-x^2)^{-1}$ near to 1^- and integrating this result gives a positive number K_3 such that $(f(x))^2 \geq K_3 \left|\ln(1-x)\right|$ for x near to 1^-. However, this contradicts our assumption that f is bounded on $(-1,1)$ and so we have

$$(1-x^2)^{1/2} f' \in L^2(0,1). \tag{7.8}$$

From the definition of $\mathcal{D}(T_1)$ we have $((1-x^2)f')' \in L^2(0,1) \subset L^1(0,1)$; hence

$$\int_0^x \left((1-t^2)f'(t)\right)' dt = (1-t^2)f'(t) \mid_0^x = (1-x^2)f'(x) - f'(0)$$

and thus there exists a real number K_4 such that

$$\lim_{x \to 1^-} (1-x^2) f'(x) = K_4.$$

If $K_4 \neq 0$ then for all x near to 1^-

$$(1-x^2)^{1/2} \left|f'(x)\right| \geq \frac{1}{2} \left|K_4\right| (1-x^2)^{-1/2};$$

however, this contradicts (7.8). Thus we conclude that (7.7) holds and $f \in \mathcal{D}(T)$.

Step 4 Suppose $f \in \mathcal{D}(T_1)$ and $f' \in L^2(-1,1)$. Then $f' \in L^1(-1,1)$; repeating the argument in Step 2, we see that $f \in AC[-1,1]$ and is therefore bounded on $(-1,1)$. By Step 3, $f \in \mathcal{D}(T)$.

Following the methods used in [6, Theorem 1.1] and [7, Theorem 1.1], define the function $g : (-1,1) \to \mathbb{R}$ by

$$g(x) = \int_0^x g'(t)dt \quad (x \in (-1,1)),$$

where the derivative is given by

$$g'(x) = ((1-x)^{1/2} \ln(1/(1-x)))^{-1} \quad (x \in [\tfrac{1}{2}, 1))$$

and then completing the definition of $g'(x)$ on $(-1, \frac{1}{2})$ by polynomial extension so that $g' \in C^1(-1,1)$. A calculation, details omitted, shows that g possesses the properties (i) $g \in$

$L^2(-1,1)$, (ii) $g, g' \in AC_{loc}(-1,1)$, (iii) $\ell[g] \in L^2(-1,1)$, and (iv) $\lim_{x\to\pm 1}(1-x^2)g'(x) = 0$. A further calculation shows that $g' \in L^2(-1,1)$ but $g' \notin L^p(-1,1)$ for any $p > 2$.

Step 5 Suppose $f \in \mathcal{D}(T)$; by definition $f \in \mathcal{D}(T_1)$. From Step 1, $f' \in L^2(-1,1)$. Hence, as in Steps 2 and 4, we obtain $f \in AC[-1,1]$.

Step 6 Suppose $f \in \mathcal{D}(T_1)$ and $f \in AC[-1,1]$. In particular, f is bounded on $(-1,1)$ so by Step 3, $f \in \mathcal{D}(T)$.

Step 7 Suppose $f \in \mathcal{D}(T)$ so $f \in \mathcal{D}(T_1)$; from Step 1, we see that $f' \in L^2(-1,1)$ and hence $(1-x^2)^{1/2}f' \in L^2(-1,1)$.

Step 8 Suppose $f \in \mathcal{D}(T_1)$ and $(1-x^2)^{1/2}f' \in L^2(-1,1)$. We show that

$$\lim_{x\to 1^-}(1-x^2)f'(x) = 0;$$

a similar result will hold as $x \to -1^+$ and this shows $f \in \mathcal{D}(T)$. Let $g \in \mathcal{D}(T_1)$ and suppose $(1-x^2)^{1/2}g' \in L^2(-1,1)$. Using integration by parts, we see that, for $0 < x < 1$,

$$\begin{aligned} &\int_0^x \ell[f](t)\overline{g}(t)dt + (1-x^2)f'(x)\overline{g}(x) \\ &= f'(0)\overline{g}(0) + \int_0^x \left((1-t^2)f'(t)\overline{g}'(t) + kf(t)\overline{g}(t)\right) dt. \end{aligned} \tag{7.9}$$

Since both of the integral terms in (7.9) are convergent as $x \to 1^-$, we see that

$$\lim_{x\to 1^-}(1-x^2)f'(x)\overline{g}(x)$$

exists and is finite. In particular, by taking $g = 1$, we see that

$$\lim_{x\to 1^-}(1-x^2)f'(x) = L$$

for some $L \in \mathbb{R}$. If $L \neq 0$, we may assume that $L > 0$. Consequently, we see that there exists $\eta \in (0,1)$ such that

$$(1-x^2)f'(x) > \frac{L}{2} \quad (x \in [\eta, 1)).$$

Hence,

$$(1-x^2)^{1/2}f'(x) > \frac{L}{2}(1-x^2)^{-1/2} \quad (x \in [\eta, 1)).$$

However, this inequality implies that

$$(1-x^2)^{1/2}f'(x) \notin L^2(-1,1),$$

contradicting our assumption on f.

This completes the proof of the theorem. □

We remark here that some of these characterizations of the domain $\mathcal{D}(T)$ have been proved elsewhere in the literature; we refer the reader to [1, Volume II, Appendix II], the paper by Everitt [5, Pages 97-99], and the paper by Kaper, Kwong, and Zettl [9, Lemma 2]. However, these references do not include the critical result that $f \in \mathcal{D}(T)$ if and only if $f \in \mathcal{D}(T_1)$ and $f' \in L^2(-1,1)$; the only known proofs of this best possible result are the proof given above using the results in [4], and the proof given in [2] using some remarkable properties of the Legendre polynomials.

8. Another property of the domain $\mathcal{D}(T)$

We now give a new proof of the following characterization of $\mathcal{D}(T)$; this theorem was reported on in [2, Corollary 7.1], based on a general left-definite theory of Littlejohn and Wellman [11].

Theorem 8.1. *Given the domain* $\mathcal{D}(T)$ *in* (6.10), *we have the following characterization*

$$\mathcal{D}(T) = \mathcal{D}_1 := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); (1-x^2)f'' \in L^2(-1,1)\}.$$

Furthermore, this result is best possible in the sense that there exists $g \in \mathcal{D}(T)$ *such that* $(1-x^2)g'' \notin L^p(-1,1)$ *for any* $p > 2$, *and where* g *is independent of* p.

Proof. We first prove the inclusion $\mathcal{D}(T) \subset \mathcal{D}_1$. Let $f \in \mathcal{D}(T)$ so that

$$f, f' \in AC_{loc}(-1,1). \tag{8.1}$$

By Theorem 7.1, $f' \in L^2(-1,1)$, and since $\ell[f] \in L^2(-1,1)$, we see that

$$(1-x^2)f'' = -\ell[f] - 2xf' + kf \in L^2(-1,1); \tag{8.2}$$

(8.1) and (8.2) together give the inclusion $\mathcal{D}(T) \subset \mathcal{D}_1$.

To show $\mathcal{D}_1 \subset \mathcal{D}(T)$, let $f \in \mathcal{D}_1$ and set

$$g(x) = (1-x^2)f''(x) \quad (x \in (-1,1))$$

so that $g \in L^2(-1,1)$. Integration on $[0,x] \subset [0,1)$ yields

$$f'(x) = f'(0) + \int_0^x f''(t)dt = f'(0) + \int_0^x \frac{g(t)}{1-t^2}dt. \tag{8.3}$$

We now apply (5.4) in Corollary 5.1 with

$$\varphi(x) = 1/(1-x^2), \psi(x) = 1 \quad (x \in (-1,1))$$

to obtain $f' \in L^2(0,1)$. A similar argument gives $f' \in L^2(-1,0)$; thus

$$f' \in L^2(-1,1). \tag{8.4}$$

Consequently, $f \in AC[-1,1]$ and so $f \in L^2(-1,1)$; hence

$$\ell[f] = -(1-x^2)f'' + 2xf' + kf \in L^2(-1,1).$$

Thus $f \in \mathcal{D}(T_1)$; moreover, from (8.4) and Theorem 7.1 (i), we see that $f \in \mathcal{D}(T)$. Hence $\mathcal{D}_1 \subset \mathcal{D}(T)$.

For the best possible function g, we can use the same g as in Theorem 7.1 (i).

This completes the proof of the theorem. □

9. Other self-adjoint operators in $L^2(-1,1)$

Recalling our discussion in Section 6 of the GKN description of all self-adjoint extensions in $L^2(-1,1)$ of the minimal operator T_0, each of these self-adjoint extensions has the property that it is a restriction of the maximal operator T_1; in particular, if A is such an extension, $f \in \mathcal{D}(T_1)$ for all $f \in \mathcal{D}(A)$. We now prove the following theorem.

Theorem 9.1. *Suppose that $A \neq T$ is a self-adjoint extension in $L^2(-1,1)$ of the Legendre minimal operator T_0; here T is the Legendre differential operator defined in* (6.8) *and* (6.10). *Then there exists $f \in \mathcal{D}(A)$ such that*

$$(1-x^2)f'' \notin L^2(-1,1) \text{ and } f' \notin L^2(-1,1).$$

Proof. Let A be any self-adjoint extension of the Legendre minimal operator T_0 in $L^2(-1,1)$ defined in (6.4) and (6.5). Suppose, for all $f \in \mathcal{D}(A)$, we have

$$(1-x^2)f'' \in L^2(-1,1). \tag{9.1}$$

Since

$$f \in \mathcal{D}(T_1), \tag{9.2}$$

we see that

$$\ell[f] \in L^2(-1,1). \tag{9.3}$$

Combining (9.1) and (9.3), we obtain

$$2xf' \in L^2(-1,1). \tag{9.4}$$

However, since $f, f' \in AC_{loc}(-1,1)$, we have $f' \in AC[-\frac{1}{2},\frac{1}{2}]$ and hence

$$f' \in L^2(-\tfrac{1}{2},\tfrac{1}{2}). \tag{9.5}$$

Since the function $g(x) = x$ is bounded on $[-1,-\frac{1}{2}]$ and $[\frac{1}{2},1]$, we see from (9.4) that

$$f' \in L^2(-1,-\tfrac{1}{2}) \text{ and } f' \in L^2(\tfrac{1}{2},1); \tag{9.6}$$

combining (9.5) and (9.6), we obtain

$$f' \in L^2(-1,1). \tag{9.7}$$

From (9.2), (9.7), and Theorem 7.1 (i), we see that $f \in \mathcal{D}(T)$ and hence that $A \subset T$. However, since A and T are both self-adjoint, it follows that $A = T$. Consequently, if $A \neq T$, we see that there exists $f \in \mathcal{D}(A)$ such that

$$(1-x^2)f'' \notin L^2(-1,1).$$

Moreover, since $\ell[f] = -(1-x^2)f'' + 2xf' - kf$ and f are necessarily in $L^2(-1,1)$, we see that this f also satisfies

$$f' \notin L^2(-1,1);$$

this completes the proof of this theorem. □

10. A property of the maximal domain $\mathcal{D}(T_1)$

In this theorem, we prove the following surprising and remarkable theorem:

Theorem 10.1. *Let $f \in \mathcal{D}(T_1)$, the maximal domain defined in* (6.1). *Then*

$$f' \in L^2(-1,1) \text{ if and only if } f' \in L^1(-1,1).$$

Proof. Let $f \in \mathcal{D}(T_1)$. First, suppose $f' \in L^2(-1,1)$; since $L^2(-1,1) \subset L^1(-1,1)$, we see that $f' \in L^1(-1,1)$. Conversely, if $f' \in L^1(-1,1)$, then $f \in AC[-1,1]$; in particular, f is bounded on $(-1,1)$. Consequently, from Theorem 7.1 (ii), we have that $f \in \mathcal{D}(T)$. An application of Theorem 7.1 (i) yields $f' \in L^2(-1,1)$; this completes the proof of the theorem. □

We note that the authors in [9] show, among other results, that for $f \in \mathcal{D}(T_1)$, then $f \in \mathcal{D}(T)$ if and only if $f' \in L^1(-1,1)$.

11. The Legendre differential operator S in $H_1^2(-1,1)$

We now define the self-adjoint differential operator S, the so-called left-definite operator, generated by the differential expression $\ell[\cdot]$ in the left-definite Hilbert-Sobolev function space $H_1^2(-1,1)$.

Definition 11.1. *The linear operator $S : \mathcal{D}(S) \subset H_1^2(-1,1) \to H_1^2(-1,1)$ is given by*

$$\mathcal{D}(S) := \{f \in \mathcal{D}(T) \mid Tf \in H_1^2(-1,1)\} \tag{11.1}$$
$$Sf = Tf = \ell[f] \quad (f \in \mathcal{D}(S));$$

here, T is the Legendre differential operator defined in Section 6. We call S the left-definite operator associated with the pair $(L^2(-1,1), T)$.

A few historical remarks on this left-definite operator are in order. The first definition of the left-definite Legendre differential operator in $H_1^2(-1,1)$ is due to Pleijel in [15] and [16]. As remarked earlier, Pleijel first observed that the Legendre differential expression (2.2) is limit-point at $x = \pm 1$ in this setting; see also [3] for a generalization of the limit-point/limit-circle theory in a left-definite context. Pleijel's work on this subject was followed by Everitt [5] in 1980 who used a different approach to study the left-definite operator in $H_1^2(-1,1)$; at the time, however, it was not clear that Everitt's left-definite operator was a differential operator generated by the Legendre differential expression. In Theorem 11.4 below, we show that the Everitt left-definite operator $\widetilde{S}$, say, in fact, is identical to our S. Our definition above of S was first recorded in the 1988 unpublished manuscript [8] of Everitt and Marić. In 1993, Krall and Littlejohn [10] independently used this same definition and gave the first known proof of self-adjointness of S in $H_1^2(-1,1)$; this proof differs from the proof in this current paper (see Theorem 11.2). In 2000, Vonhoff [19], presented yet another new approach to the left-definite theory of the Legendre expression (2.2); we show below in Theorem 11.4 that his left-definite operator $S_{\max}$ is also identical to our S. Also in 2000, Arvesú, Littlejohn, and Marcellán [2] defined the left-definite Legendre operator in still a different way using a general left-definite theory of self-adjoint, bounded below operators A in a Hilbert space H, developed earlier by Littlejohn and Wellman in [11]. More specifically, in [11], they construct, with the aid of the Hilbert space spectral theorem, a continuum of left-definite

Hilbert spaces $\{(V_r, (\cdot,\cdot)_r)\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$ associated with (H, A). In particular, in [2], the authors show that their left-definite operator A_1 has domain V_3, which they explicitly construct. We show in Section 12 below that A_1 is identical to the operator S defined above.

Lemma 11.1. *The operator S, given in Definition 11.1, is closed in $H_1^2(-1,1)$.*

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{D}(S)$ be a sequence in $H_1^2(-1,1)$ such that

$$\begin{array}{ll} (i) & \{f_n\} \to f \text{ in } H_1^2(-1,1) \\ (ii) & \{Sf_n\} \to g \text{ in } H_1^2(-1,1). \end{array}$$

From (ii) and the completeness of $H_1^2(-1,1)$ (see Theorem 6.2), we see that $g \in H_1^2(-1,1)$ and since $\|h\|_1^2 \geq k\|h\|^2$ for all $h \in H_1^2(-1,1)$, it follows that (i) and (ii) imply that $\{f_n\} \to f$ in $L^2(-1,1)$ and $\{Sf_n\} = \{Tf_n\} \to g$ in $L^2(-1,1)$. Since T is closed (being self-adjoint) in $L^2(-1,1)$, we have

$$f \in \mathcal{D}(T) \tag{11.2}$$

and

$$Tf = g \in H_1^2(-1,1). \tag{11.3}$$

Together, (11.2) and (11.3) show that $f \in \mathcal{D}(S)$ and $Sf = Tf = \ell[f] \in H_1^2(-1,1)$; hence S is closed. □

Theorem 11.1. *The operator S is symmetric in $H_1^2(-1,1)$.*

Proof. Since the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ satisfy

$$\begin{array}{ll} (i) & P_n \in \mathcal{D}(S) \quad (n \in \mathbb{N}_0) \\ (ii) & \{P_n\}_{n=0}^{\infty} \text{ is a complete orthogonal set in } H_1^2(-1,1) \end{array}$$

where in (ii), we use Theorem 6.2, it follows that the domain $\mathcal{D}(S)$ is dense in $H_1^2(-1,1)$. Let $f, g \in \mathcal{D}(S)$; then

$$\begin{aligned} (Sf,g)_1 &= \int_{-1}^{1} \left((1-x^2)(Sf(x))'\overline{g}'(x) + kSf(x)\overline{g}(x)\right) dx \\ &= (1-x^2)Sf(x)\overline{g}'(x) \mid_{-1}^{1} + \int_{-1}^{1} Sf(x)\left(-((1-x^2)\overline{g}'(x))' + k\overline{g}(x)\right) dx \\ &= \int_{-1}^{1} Sf(x)\overline{Sg}(x)dx \end{aligned} \tag{11.4}$$

since with $g \in \mathcal{D}(S) \subset \mathcal{D}(T)$ and $Sf \in H_1^2(-1,1)$ we apply (6.13) in Corollary 6.1 to give

$$\lim_{x\to\pm1}(1-x^2)\overline{g}'(x)Sf(x) = 0.$$

A similar calculation also shows that

$$(f, Sg)_1 = \int_{-1}^{1} Sf(x)\overline{Sg}(x)dx \quad (f, g \in \mathcal{D}(S))$$

and this completes the proof of the symmetry of S in $H_1^2(-1,1)$. □

Theorem 11.2. *The operator S is self-adjoint in $H_1^2(-1,1)$.*

Proof. Let

$$\varphi_n = \sqrt{\frac{2n+1}{2n^2+2n+2k}}P_n \quad (n \in \mathbb{N}_0),$$

where P_n is the n^{th} Legendre polynomial defined in (3.19). From (4.4) or (6.16), we see that the normalized Legendre polynomials $\{\varphi_n\}_{n=0}^{\infty}$ are orthonormal:

$$(\varphi_n, \varphi_m)_1 = \delta_{n,m} \quad (n, m \in \mathbb{N}_0).$$

From Theorem 6.2 in Section 6, $\{\varphi_n\}_{n=0}^{\infty}$ is, in fact, a complete orthonormal set in $H_1^2(-1,1)$ and

$$S\varphi_n = \lambda_n \varphi_n \quad (n \in \mathbb{N}_0),$$

where λ_n is defined in (1.2). Let $g \in H_1^2(-1,1)$; we consider the solution of the equation $Sf = g$. Write

$$g = \sum_{n=0}^{\infty} c_n \varphi_n,$$

where

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

Define the sequence $\{f_N\}_{N=0}^{\infty}$ in $H_1^2(-1,1)$ by

$$f_N = \sum_{n=0}^{N} \frac{c_n}{\lambda_n} \varphi_n.$$

Since

$$\sum_{n=0}^{\infty} \left| \frac{c_n}{\lambda_n} \right|^2 \leq \frac{1}{k^2} \sum_{n=0}^{\infty} |c_n|^2 < \infty,$$

we see that $\{f_N\}_{N=0}^{\infty}$ converges in $H_1^2(-1,1)$ to a vector f (say) in $H_1^2(-1,1)$. Now

$$Sf_N = \sum_{n=0}^{N} \frac{c_n}{\lambda_n} S\varphi_n = \sum_{n=0}^{N} c_n \varphi_n \to g \text{ in } H_1^2(-1,1).$$

Since S is closed by Lemma 11.1, we have that $f \in \mathcal{D}(S)$ and $Sf = g$. Thus the range of the symmetric operator S is all of $H_1^2(-1,1)$; consequently by a well-known result (see [1, Volume I, Section 41, Theorem 1]), S is self-adjoint. This completes the proof of the theorem. □

Theorem 11.3. *The self-adjoint operator S in $H_1^2(-1,1)$ is unique in the following sense: if S' is another self-adjoint operator in $H_1^2(-1,1)$ with*

(i) $\mathcal{D}(S') \subset \mathcal{D}(T)$, where T is the Legendre differential operator defined in Section 6,
(ii) $S'f = Tf$ for all $f \in \mathcal{D}(S')$,

then $S = S'$.

Proof. It suffices to show

$$\mathcal{D}(S') \subset \mathcal{D}(S); \tag{11.5}$$

indeed, from (ii) above, we then have $S' \subset S$. Since both S and S' are self-adjoint in $H_1^2(-1,1)$, we obtain $S = S'$. To show (11.5), let $f \in \mathcal{D}(S')$; in particular, from (i), we see that

$$f \in \mathcal{D}(T). \tag{11.6}$$

Moreover, since S' is an operator in $H_1^2(-1,1)$, we see from (ii) that

$$S'f = Tf \in H_1^2(-1,1). \tag{11.7}$$

From (11.6), (11.7), and the definition of $\mathcal{D}(S)$ in Definition 11.1, we obtain (11.5); this completes the proof of this theorem. □

This uniqueness theorem has an interesting and important consequence pertaining to the left-definite operator approach discussed in [5]. Indeed, in [5], Everitt defines the left-definite operator - which we denote here by $\widetilde{S}$ - in $H_1^2(-1,1)$ in a different, but equivalent, way. To begin this discussion, observe from Lemma 6.1 (i) that $0 \in \rho(T)$, the resolvent set of T, and hence T^{-1} exists as a bounded operator from $\mathcal{D}(T^{-1}) = L^2(-1,1)$ onto $\mathcal{D}(T)$. From the set inclusions (see Theorem 6.1 (i))

$$\mathcal{D}(T) \subset H_1^2(-1,1) \subset L^2(-1,1),$$

we see that the linear operator B, defined by

$$\begin{aligned} \mathcal{D}(B) &:= H_1^2(-1,1) \\ Bf &= T^{-1}f \quad (f \in \mathcal{D}(B)) \end{aligned}$$

maps $H_1^2(-1,1)$ into $\mathcal{D}(T) \subset H_1^2(-1,1)$; that is, B is a linear operator in $H_1^2(-1,1)$. Using the extended Dirichlet identities (6.14) and (6.15), Everitt shows, in fact, that B is a one-to-one, bounded, self-adjoint operator in $H_1^2(-1,1)$. Consequently, the operator $\widetilde{S} : \mathcal{D}(\widetilde{S}) \subset H_1^2(-1,1) \to H_1^2(-1,1)$ defined by

$$\mathcal{D}(\widetilde{S}) := B(H_1^2(-1,1)) \tag{11.8}$$

$$\widetilde{S}f = B^{-1}f \quad (f \in \mathcal{D}(\widetilde{S})) \tag{11.9}$$

is also self-adjoint in $H_1^2(-1,1)$. In Theorem 11.4 below, we show that $S = \widetilde{S}$.

In [19], Vonhoff defines the left-definite Legendre operator $S_{\max}$ (his notation) as

$$\begin{aligned} \mathcal{D}(S_{\max}) = \{f : (-1,1) \to \mathbb{C} \mid & f, f', \ell[f] \in AC_{loc}(-1,1); \\ & f, (1-x^2)^{1/2}f', (1-x^2)^{1/2}(\ell[f])', \ell[f] \in L^2(-1,1)\} \end{aligned} \tag{11.10}$$

$$S_{\max}f = Tf \quad (f \in \mathcal{D}(S_{\max})), \tag{11.11}$$

and then proves that $S_{\max}$ is self-adjoint in $H_1^2(-1,1)$. Observe that, since $H_1^2(-1,1) \subset L^2(-1,1)$, we see that

$$\mathcal{D}(S_{\max}) = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); f, \ell[f] \in H_1^2(-1,1)\}.$$

Comparing this formulation of $\mathcal{D}(S_{\max})$ with the definition of the domain of the maximal operator $\mathcal{D}(T_1)$ in (6.1), it is indeed appropriate for Vonhoff to call his operator $S_{\max}$ the maximal operator in $H_1^2(-1,1)$ generated by the Legendre expression $\ell[\cdot]$.

We now prove

Theorem 11.4. $S = \widetilde{S} = S_{\max}$, *where* $\widetilde{S}$ *and* $S_{\max}$ *are defined in* (11.8), (11.9) *and* (11.10), (11.11), *respectively.*

Proof. To show that $S = \widetilde{S}$, it suffices to show, from Theorem 11.3, that

$$\begin{aligned} &\text{(i)} \quad \mathcal{D}(\widetilde{S}) \subset \mathcal{D}(T), \\ &\text{(ii)} \quad \widetilde{S}f = Tf \quad (f \in \mathcal{D}(\widetilde{S})). \end{aligned}$$

To prove (i), let $f \in \mathcal{D}(\widetilde{S})$; since $B(H_1^2(-1,1)) \subset \mathcal{D}(T)$, we have

$$f \in \mathcal{D}(T), \tag{11.12}$$

establishing (i).

To demonstrate (ii), take $f \in \mathcal{D}(\widetilde{S})$; as above, $f \in \mathcal{D}(T)$; moreover, by definition of $\mathcal{D}(\widetilde{S})$, there exists $g \in H_1^2(-1,1)$ such that $Bg = T^{-1}g = f$. Hence, since $\widetilde{S}$ and B are inverses of each other,

$$\widetilde{S}f = \widetilde{S}(Bg) = g. \tag{11.13}$$

On the other hand,

$$Tf = T(T^{-1}g) = g. \tag{11.14}$$

The identities in (11.13) and (11.14) yield $\widetilde{S}f = Tf$; this completes the proof that $S = \widetilde{S}$.

We now proceed to show $S = S_{\max}$. Let $f \in \mathcal{D}(S_{\max})$; in particular, we see that $f \in \mathcal{D}(T_1)$ and $(1-x^2)^{1/2}f' \in L^2(-1,1)$. From Theorem 7.1 (iv), we have $f \in \mathcal{D}(T)$; that is, $\mathcal{D}(S_{\max}) \subset \mathcal{D}(T)$. Since, from (11.11), $S_{\max}f = Tf$, another application of Theorem 11.3 yields $S = S_{\max}$.

This completes the proof of the theorem. ☐

Theorem 11.5. *The self-adjoint operator* S *in* $H_1^2(-1,1)$ *has a discrete, simple spectrum* $\sigma(S)$ *given by*

$$\sigma(S) = \{n(n+1) + k \mid n \in \mathbb{N}_0\}$$

with the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ *as eigenfunctions; that is, the self-adjoint operators* S *and* T *have the same spectrum and the same eigenfunctions.*

Proof. Note that the n^{th} Legendre polynomial $P_n \in \mathcal{D}(S)$ for all $n \in \mathbb{N}_0$. Moreover,

$$SP_n = TP_n = \ell[P_n] = (n(n+1)+k)P_n;$$

that is P_n is an eigenfunction of S with corresponding eigenvalue $\lambda_n = n(n+1)+k$. From Section 6 we see that the set of Legendre polynomials is a complete, orthogonal set in $H_1^2(-1,1)$. Since the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ have no finite limit point (as $n \to \infty$) in $\mathbb{C}$ it follows that the spectrum of S consists only of these eigenvalues. From the completeness of $\{P_n\}_{n=0}^{\infty}$ in $H_1^2(-1,1)$, it follows that each eigenvalue λ_n is simple. ☐

Theorem 11.6. *The deficiency indices of the self-adjoint operator S are $d^{\pm}(S) = 0$.*

Proof. We note that since S is self-adjoint, we have $d^+(S) = d^-(S) = 0$. We confirm this result, which is also proved in [16] and [19, Lemma 3.3], using properties of the solutions of the Legendre differential equation, as discussed in Sections 3 and 4. By definition,

$$\begin{aligned} d^+(S) &= \dim\{f \in \mathcal{D}(S^*) \mid S^* f = if\} = \dim\{f \in \mathcal{D}(S) \mid Sf = if\} \\ &= \dim\{y \in \mathcal{D}(\ell) \mid \ell[y] = iy \text{ on } (-1,1) \text{ and } y \in H_1^2(-1,1)\}. \end{aligned} \tag{11.15}$$

From the general properties of the Legendre differential equation (2.1) and the special properties of the solutions $\{y_{1,+}, y_{2,+}, y_{1,-}, y_{2,-}\}$ given in Lemma 4.1, it follows that if a non-null solution $y(\cdot)$ of $\ell[y] = iy$ on $[0,1)$ is to satisfy $y(\cdot) \in H_1^2(0,1)$, then for some $\alpha_+ \in \mathbb{C}$, $\alpha_+ \neq 0$,

$$y(x) = \alpha_+ y_{1,+}(x,i) \quad (x \in [0,1)). \tag{11.16}$$

This solution $y(\cdot)$ has a continuation into $(-1,0)$ and so can be represented in the form

$$y(x) = \alpha_- y_{1,-}(x,i) + \beta_- y_{2,-}(x,i) \quad (x \in (-1,0)). \tag{11.17}$$

If this $y(\cdot)$, now defined on $(-1,1)$, is in $H_1^2(-1,1)$, then $y(\cdot) \in H_1^2(-1,0)$. This being so, and returning to the properties given in Lemma 4.1, it follows from (11.17) that we have to take $\beta_- = 0$ and $\alpha_- \neq 0$. Thus we have

$$y(x) = \alpha_+ y_{1,+}(x,i) = \alpha_- y_{1,-}(x,i) \quad (x \in (-1,1)). \tag{11.18}$$

This last result implies, since $y_{1,\pm}(x,i) \in \mathcal{D}(T)$,

$$\lim_{x \to 1^-} (1-x^2) y_{1,+}'(x,i) = \lim_{x \to -1^+} (1-x^2) y_{1,-}'(x,i) = 0,$$

that $y(\cdot) \in \mathcal{D}(T)$ and $\ell[y] = iy$ on $(-1,1)$. However this gives $Ty = iy$ and so $\lambda = i$ is an eigenvalue of T. This is impossible since T is self-adjoint and, necessarily, the spectrum $\sigma(T)$ of T is real. Returning to (11.15), it follows that $d^+(S) = 0$. A similar argument shows $d^-(S) = 0$ and this completes the proof of the theorem. □

12. A property of the domain $\mathcal{D}(S)$

In this section we prove the following characterization of the domain of the left-definite operator S defined and discussed in the last section; this proof is based on the notes in [8]. This characterization was independently obtained by the authors in [2, Theorem 8.2] using different methods.

Theorem 12.1. *Let*

$$\mathcal{D} := \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{loc}(-1,1);\ (1-x^2)^{3/2} f^{(3)} \in L^2(-1,1)\}. \tag{12.1}$$

Then

$$\mathcal{D}(S) = \mathcal{D},$$

where $\mathcal{D}(S)$, the domain of the left-definite operator S, is defined in Definition 11.1. Furthermore, this result is best possible in the sense that there exists $g \in \mathcal{D}(S)$ such that $(1-x^2)^{3/2} g^{(3)} \notin L^p(-1,1)$ for any $p > 2$ and where g is independent of p.

Proof. We first prove the inclusion

$$\mathcal{D}(S) \subset \mathcal{D}. \tag{12.2}$$

Let $f \in \mathcal{D}(S)$ so $f \in \mathcal{D}(T)$; in particular,

$$f, f' \in AC_{loc}(-1,1). \tag{12.3}$$

Since $Tf \in H_1^2(-1,1)$, we see that $Tf \in AC_{loc}(-1,1)$; that is,

$$-((1-x^2)f'(x))' + kf(x) = -(1-x^2)f''(x) + 2xf'(x) + kf(x) \in AC_{loc}(-1,1). \tag{12.4}$$

From (12.2) and the fact that $(1-x^2) \in AC[-1,1]$, we now see from (12.4) that

$$f'' \in AC_{loc}(-1,1). \tag{12.5}$$

Again, since $Tf \in H_1^2(-1,1)$, we obtain

$$(1-x^2)^{1/2}\Big(-\big((1-x^2)f'(x)\big)' + kf(x)\Big)' \in L^2(-1,1). \tag{12.6}$$

Written out, (12.6) simplifies to

$$-(1-x^2)^{3/2}f^{(3)} + 4x(1-x^2)^{1/2}f''(x) + (k+2)(1-x^2)^{1/2}f'(x) \in L^2(-1,1).$$

From the fact that $f \in \mathcal{D}(T)$ we have, from Theorem 7.1 (i), $f' \in L^2(-1,1)$ and so this last result simplifies to

$$g(x) := (1-x^2)^{3/2}f^{(3)}(x) - 4x(1-x^2)^{1/2}f''(x) \in L^2(-1,1). \tag{12.7}$$

We next show that

$$(1-x^2)^{1/2}f'' \in L^2(-1,1); \tag{12.8}$$

this result (12.8), together with (12.7), yields

$$(1-x^2)^{3/2}f^{(3)} \in L^2(-1,1). \tag{12.9}$$

To prove (12.8), it suffices to show that $(1-x^2)^{1/2}f'' \in L^2(0,1)$; a similar proof gives $(1-x^2)^{1/2}f'' \in L^2(-1,0)$. Multiplying (12.7) by $(1-x^2)^{1/2}$ yields

$$(1-x^2)^{1/2}g(x) = \big((1-x^2)^2f''(x)\big)' \quad (x \in [0,1)). \tag{12.10}$$

Integrating (12.10) over $[0,x]$ gives

$$(1-x^2)^2f''(x) = f''(0) + \int_0^x (1-t^2)^{1/2}g(t)dt \quad (x \in [0,1)). \tag{12.11}$$

Since the integrand on the right hand side of (12.11) is in $L^2(0,1) \subset L^1(0,1)$, we see that

$$\lim_{x\to 1^-}(1-x^2)^2f''(x) = L \text{ (say)}$$

exists and is finite. If $L \neq 0$, we may assume, without loss of generality, that $L > 0$. Consequently, for some $K > 0$, $f''(x) > K(1-x^2)^{-2}$ near 1^- (here, K is the Littlewood "K" which represents a positive number but not necessarily the same number in each appearance); integration over $[x_0, x] \subset [0,1)$ gives $f'(x) > K(1-x)$ near 1^- and this result takes f' out of

$L^2(0,1)$, contradicting Theorem 7.1. Thus $L = 0$. Now integrate (12.10) over $[x,\xi] \subset [0,1)$ to give

$$\int_x^\xi (1-t^2)^{1/2} g(t)dt = (1-\xi^2)^2 f''(\xi) - (1-x^2)^2 f''(x).$$

In this last result, let $\xi \to 1^-$ to obtain

$$(1-x^2)^2 f''(x) = -\int_x^1 (1-t^2)^{1/2} g(t)dt \quad (x \in [0,1)).$$

Hence

$$(1-x^2)^{1/2} f''(x) = \frac{-1}{(1-x^2)^{3/2}} \int_x^1 (1-t^2)^{1/2} g(t)dt \quad (x \in [0,1)). \tag{12.12}$$

Since $g \in L^2(0,1)$, we can now apply (5.3) with

$$\varphi(x) = (1-x^2)^{-3/2}, \psi(x) = (1-x^2)^{1/2} \quad (x \in [0,1))$$

to obtain

$$(1-x^2)^{1/2} f'' \in L^2(0,1).$$

We now see that (12.3), (12.5), and (12.9) give us the required inclusion in (12.2). To show that

$$\mathcal{D} \subset \mathcal{D}(S), \tag{12.13}$$

let $f \in \mathcal{D}$. In particular,

$$f, f', f'' \in AC_{loc}(-1,1). \tag{12.14}$$

Set $g(x) = (1-x^2)^{3/2} f^{(3)}$ $(x \in (-1,1))$ so that $g \in L^2(-1,1)$. Integrating over $[0,x] \subset [0,1)$, we find

$$f''(x) = f''(0) + \int_0^x \frac{g(t)}{(1-t^2)^{3/2}} dt \quad (x \in [0,1))$$

and so

$$(1-x^2)^{1/2} f''(x) = (1-x^2)^{1/2} f''(0) + (1-x^2)^{1/2} \int_0^x \frac{g(t)}{(1-t^2)^{3/2}} dt \quad (x \in [0,1)). \tag{12.15}$$

Applying (5.4) to the integral on the right hand side of (12.15) with

$$\varphi(x) = (1-x^2)^{1/2}, \ \psi(x) = 1/(1-x^2)^{1/2} \quad (x \in [0,1)),$$

we find

$$(1-x^2)^{1/2} f'' \in L^2(0,1);$$

a similar computation gives $(1-x^2)^{1/2} f'' \in L^2(-1,0)$ and so

$$h(x) := (1-x^2)^{1/2} f'' \in L^2(-1,1). \tag{12.16}$$

From (12.16) we obtain

$$f'(x) = f'(0) + \int_0^x f''(t)dt = f'(0) + \int_0^x \frac{h(t)}{(1-t^2)^{1/2}} dt \quad (x \in [0,1)); \tag{12.17}$$

another application of (5.4) to the integral on the right side of (12.17), with

$$\varphi(x) = 1/(1-x^2)^{1/2},\ \psi(x) = 1 \quad (x \in [0,1)),$$

implies that $f' \in L^2(0,1)$. It now follows that

$$f' \in L^2(-1,1), \tag{12.18}$$

and hence that

$$f \in L^2(-1,1). \tag{12.19}$$

We can now prove that

$$\ell[f] \in H_1^2(-1,1). \tag{12.20}$$

Indeed, from (12.16), (12.18), and since $(1-x^2)^{3/2}f^{(3)} \in L^2(-1,1)$, we have

(12.21)
$$(1-x^2)^{1/2}(\ell[f])' = -(1-x^2)^{3/2}f^{(3)} + 4x(1-x^2)^{1/2}f'' + k(1-x^2)^{1/2}f' \in L^2(-1,1).$$

From (12.14) it follows that

$$\ell[f] = -(1-x^2)f'' + 2xf' + kf \in AC_{loc}(-1,1). \tag{12.22}$$

Hence, from (12.21) and (12.22), we see that (12.20) holds. Furthermore, from (12.16), we see that

$$(1-x^2)f'' \in L^2(-1,1). \tag{12.23}$$

Hence, from (12.18), (12.19), and (12.23) we see that

$$\ell[f] = -(1-x^2)f'' + 2xf' + kf \in L^2(-1,1). \tag{12.24}$$

Taken together, (12.21), (12.22), and (12.24) imply that

$$\ell[f] \in H_1^2(-1,1). \tag{12.25}$$

Furthermore, from (12.14), (12.19), and (12.24), we have $f \in \mathcal{D}(T_1)$. Also, from (12.18) and Theorem 7.1 (i), we see that $f \in \mathcal{D}(T)$. Hence, from (12.25), $Tf = \ell[f] \in H_1^2(-1,1)$. We now see that these results imply that (12.13) holds.

Finally, to prove that the condition in (12.9) is best possible, define the function $g : [\frac{1}{2},1) \to \mathbb{R}$ by

$$g(x) = \int_{1/2}^{x} \ln(\ln(1/(1-t)))dt \quad (x \in [\tfrac{1}{2},1))$$

and then complete the definition of $g(x)$ on $(-1,\frac{1}{2})$ by polynomial extension so that $g, g', g'' \in AC_{loc}(-1,1)$. A calculation shows that

(12.26)
$$(1-x^2)^{3/2}g^{(3)}(x) = \frac{(1+x)^{3/2}}{(1-x)^{1/2}\ln(1/(1-x))} - \frac{(1+x)^{3/2}}{(1-x)^2\ln^2(1/(1-x))} \quad (x \in [\tfrac{1}{2},1))$$

and it now follows that $(1-x^2)^{3/2}g^{(3)} \in L^2(-1,1)$. Moreover, (12.26) shows that, for $p > 2$, $(1-x^2)^{3/2}g^{(3)} \notin L^p(-1,1)$. This completes the proof of the theorem. □

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