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The fourth-order Bessel equation: eigenpackets and a generalised Hankel transform

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This paper is dedicated to
the achievements and memory of
Professor Günter Hellwig

In connection with the fourth-order Bessel-type differential equation

$$(L_M y)(x) := (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \ (x > 0)$$

two expansion theorems are established, the convergence being pointwise or in a L^2-setting. If the positive parameter M tends to zero, these two expansion theorems reduce to the classical Hankel transform of order zero. In a previous paper the authors have proved that in one of the introduced Lebesgue-Stieltjes Hilbert function spaces, the differential expression $x^{-1}L_M$ gives rise to exactly one self-adjoint operator S_M. In this paper it is proved, together with the corresponding expansion theorems, that S_M has a complete eigenpacket. The orthogonality property of this eigenpacket is reflected in a distributional orthogonality on which the expansion theorems are based.

1 Introduction

Let H be a Hilbert space. For every self-adjoint operator A in H von Neumann's theorem guarantees the existence of a family $(E(\lambda))_{\lambda\in\mathbb{R}}$ of projection operators (unique to some normalisations) such that

$$Au = \int_{-\infty}^{\infty} \lambda \ dE(\lambda)u \ (u \in D(A)).$$

In the hope of finding a substitute that is in specific situations more amenable to computations, Rellich, in his New York lectures of 1950/51, generalised Hellinger's concept of "eigendifferentials" as follows [8, p. 162].

Let A be a symmetric operator in H. A family $(\phi_\lambda)_{\lambda\in\mathbb{R}}$ of elements in $D(A)$ is called an **eigenpacket** of A if it has the following properties:

(i) The mapping $\lambda \longmapsto \phi_\lambda$ from $\mathbb{R}$ into H is continuous.
(ii) There is some $\lambda_0 \in \mathbb{R}$ with $\phi_{\lambda_0} = 0$, and for every $\lambda \in \mathbb{R}$

$$\int_{\lambda_0}^{\lambda} \mu \ d\phi_\mu = A\phi_\lambda$$

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where the integral term has to exist as a Riemann-Stieltjes integral. (In his later Göttingen lectures Rellich strengthened the second requirement [9, p. 69].)

If A is self-adjoint, $(E(\lambda))_{\lambda\in\mathbb{R}}$ as above, $\lambda_0 \in \mathbb{R}$ and $v \in H$, then

$$\phi_\lambda := \left\{ E(\lambda) - E(\lambda_0) - \sum_{\lambda_0<\lambda_i\le\lambda} [E(\lambda_i) - E(\lambda_i - 0)] \right\} v \ (\lambda \in \mathbb{R})$$

is an eigenpacket of A [8, p.166]. Note, however, that $\phi_\lambda = 0$ $(\lambda \in \mathbb{R})$ if $\sigma_c(A)$, the continuous spectrum of A, is empty.

In the simple case

$$Au = -u'' \text{ with } D(A) = C^2(\mathbb{R}) \subset H = L^2(\mathbb{R})$$

the expansion theorem is given by the Fourier transform. There are two distinct (non-trivial) eigenpackets, generated by sine and cosine, [8, p. 246 f], reflecting the fact that $\sigma(\bar{A}) = \sigma_c(\bar{A}) = [0, \infty)$ has multiplicity two [1, Chapter 6, Sections 85, 86, 89D].

In this paper we are concerned with Bessel's equation

$$(\tau_\nu u)(x) := -u''(x) + \frac{\nu^2 - 1/4}{x^2} u(x) = \lambda u(x) \ (x > 0), \tag{1}$$

and its fourth-order analogue, equation (4) below. For $\nu \ge 0$ the l.h.s. of (1) gives rise to a distinguished self-adjoint operator in $L^2(0, \infty)$, the Friedrichs extension F_ν. It is positive and, according to a result of Rellich [11, Sections 3, 5], the functions in its domain behave at 0 like the principal solution of (1), *i.e.*

$$\begin{aligned} D(F_\nu) = \{f \in C^1((0,\infty)), f' \in AC_{\text{loc}}((0,\infty))\big| \\ f, \tau_\nu f \in L^2(0,\infty) \text{ and } \lim_{x\to 0} f(x)/(x^{1/2-\nu}) = 0\} \end{aligned} \tag{2}$$

if $\nu > 0$, and by

$$\begin{aligned} D(F_0) = \{f \in C^1((0,\infty)), f' \in AC_{\text{loc}}((0,\infty))\big| \\ f, \tau_0 f \in L^2(0,\infty) \text{ and } \lim_{x\to 0} f(x)/(x^{1/2}\ln(x)) = 0\} \end{aligned} \tag{3}$$

if $\nu = 0$. For any $\nu \ge 0$

$$\sigma(F_\nu) = \sigma_c(F_\nu) = [0, \infty)$$

(cf. [5]). The expansion theorem associated with these operators is Hankel's theorem, which can be formulated as follows [13, p. 240].

Theorem 1.1 *Let $\alpha \ge -1/2$ and $\mu > 0$. If $f \in L^1(0,\infty)$ is of bounded variation in a neighbourhood of μ, then*

$$\frac{1}{2}[f(\mu+0) + f(\mu-0)] = \int_0^\infty J_\alpha(\mu x)\sqrt{\mu x} \left(\int_0^\infty J_\alpha(\lambda x)\sqrt{\lambda x} f(\lambda)\, d\lambda \right) dx.$$

More generally, one has the following result which is a special case of a result of Watson, [13, p. 221f] and [12, p. 291ff].

Theorem 1.2 *Let $\alpha \ge -1/2$. The operator $H_\alpha : L^2(0,\infty) \to L^2(0,\infty)$ defined by*

$$f \longmapsto g(\mu) := \frac{d}{d\mu} \int_0^\infty f(x) \left(\int_0^\mu \sqrt{tx} J_\alpha(tx)\, dt \right) dx \text{ for almost all } \mu > 0,$$

is unitary with $H_\alpha^{-1} = H_\alpha$.

$H_{1/2}$ is the Fourier sine, $H_{-1/2}$ the Fourier cosine transform. For a vector space analogous to the Schwartz test function space, where the Hankel transform also acts as an automorphism, see [16, Sections 5.2 to 5.4].

In [6] a hierarchy of higher-order differential equations is introduced which have solutions with orthogonality properties similar to the classical Bessel functions. The simplest of these equations is

$$(L_M y)(x) := (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \ (x > 0) \tag{4}$$

where $M > 0$,

$$\Lambda := \lambda^2(\lambda^2 + 8M^{-1})$$

and $\lambda \in \mathbb{C}$, one solution being

$$J_\lambda^M(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x). \tag{5}$$

The r.h.s. of (5) is defined for $x \geq 0$. Note that if we multiply (4) by $M/8$, let M tend to zero and replace λ^2 by λ, we obtain

$$-(xy'(x))' = \lambda xy(x) \ (x > 0),$$

which is equivalent to (1) with $\nu = 0$ *via* $u = x^{1/2}y$.

The monotonic increasing function

$$m(x) := \begin{cases} -M/2 \text{ if } x = 0 \\ \frac{1}{2}x^2 \quad \text{if } x > 0 \end{cases} \tag{6}$$

gives rise to a Borel measure on $[0, \infty)$, again described by m. We write $L^2([0, \infty); m)$ for the Hilbert space (of equivalent classes) of functions $f : [0, \infty) \to \mathbb{C}$ with

$$\int_0^\infty |f(x)|^2 \, x \, dx < \infty.$$

The scalar product is defined by

$$(f, g)_m := \frac{1}{2}Mf(0)\overline{g}(0) + \int_0^\infty f(x)\overline{g}(x)x \, dx,$$

the norm being $\|f\|_m := [(f, f)_m]^{1/2}$. (We note in passing that $L^2([0, \infty); m)$ can also be viewed as the direct sum of the Hilbert spaces $\frac{1}{2}M\mathbb{C}$ and $L^2((0, \infty); x)$.) It is in this Hilbert space $L^2([0, \infty); m)$ that the l.h.s. of (4) generates exactly one self-adjoint operator S_M. The domain $D(S_M)$ is given by

$$D(S_M) = \{f \in C^3((0, \infty)), f^{(3)} \in AC_{\text{loc}}((0, \infty)) \big| \\ f, x^{-1}L_M f \in L^2((0, \infty); x)\}.$$

It may be shown that for any $f \in D(S_M)$ the following properties hold:

(*i*) For $r = 0, 1, 2$ the derivative $f^{(r)}$ has a continuous extension to 0.
(*ii*) At the endpoint 0 the function f has the boundary values $f'(0) = 0$ and $\lim_{x \to 0}(xf^{(3)}(x)) = 0$.

The operator S_M is positive and given by

$$(S_M f)(x) = \begin{cases} -16M^{-1}f''(0) & \text{if } x = 0 \\ x^{-1}(L_M f)(x) & \text{if } x > 0. \end{cases}$$

Moreover, $\sigma(S_M) = \sigma_c(S_M) = [0, \infty)$.

For the proof of these properties of the operator S_M see [3] and [7].

In this present paper we derive, in Section 2, an orthogonality result (cf. also [6, (4.1)]) for the Bessel-type functions (5) and show that it can be interpreted as an orthogonality relation for an eigenpacket of S_M. Associated with S_M is a generalised Hankel transform which for $M = 0$ reduces to $R^{-1}H_0R$ where R is the operator of multiplication by the square root. Pointwise convergence of this generalised Hankel transform is studied in Section 3, while a Hilbert space setting is considered in Section 4. These results were announced in [4].

2 Orthogonality results

We start by interpreting Theorem 1.1 as a distributional orthogonality, an observation which, in one way or another, can be found in the literature (e.g. [2, pp. 15, 86], [10, p. 367]).

PROPOSITION 2.1 *Let $\alpha \geq -1/2, \mu > 0$. Then*

$$\lambda \int_0^\infty J_\alpha(\lambda x) J_\alpha(\mu x) x \; dx = \delta(\lambda - \mu) \; (\lambda > 0)$$

in the sense of distributions on $C_0^\infty((0,\infty))$, i.e. with

$$f_b(\alpha, \lambda, \mu) := \lambda \int_0^b J_\alpha(\lambda x) J_\alpha(\mu x) x \; dx \; (b > 0) \tag{7}$$

we have

$$\lim_{b\to\infty} \int_0^\infty f_b(\alpha, \lambda, \mu)\varphi(\lambda) \; d\lambda = \varphi(\mu) \; (\varphi \in C_0^\infty((0,\infty))). \tag{8}$$

Proof Changing the order of integration on the l.h.s. of (8), we have

$$\int_0^\infty f_b(\alpha, \lambda, \mu)\varphi(\lambda) \; d\lambda = \frac{1}{\sqrt{\mu}} \int_0^b J_\alpha(\mu x)\sqrt{\mu x} \left(\int_0^\infty J_\alpha(\lambda x)\sqrt{\lambda x}\sqrt{\lambda}\varphi(\lambda) \; d\lambda \right) dx.$$

By Theorem 1.1, the r.h.s. tends to $\varphi(\mu)$ as $b \to \infty$. □

We frequently take advantage of the well-known asymptotic behaviour of the Bessel functions.

Remark 1 For $\nu \geq 0$ there exists a continuous function R_ν, defined on $[1, \infty)$ say, with the property that $sR_\nu(s)$ is bounded for $s \geq 1$ and

$$J_\nu(s) = \left(\frac{2}{\pi s}\right)^{1/2} [\cos(s - (\nu\pi)/2 - \pi/4) + R_\nu(s)] \; (s \geq 1)$$

[15, Section 7.3].

We also note the following estimate

$$|J_\nu(s)| \leq \frac{(s/2)^\nu}{\Gamma(\nu + 1)} \; (\nu, s \geq 0) \tag{9}$$

[15, p. 48 (6)].

In the sequel M denotes a non-negative real number.

LEMMA 2.2 *For $\nu, a, s \geq 0$ we define*

$$\psi_1(s) := \int_0^a \sqrt{st} J_\nu(st)\ dt,$$

$$\psi_2(s) := \frac{\sqrt{s}}{1+M(s/2)^2} \int_0^a J_s^M(t)\sqrt{t}\ dt,$$

$$\psi_3(s) := \int_0^a J_t^M(s) \frac{\sqrt{t}}{1+M(t/2)^2}\ dt.$$

Let $0 < \alpha < \beta < \infty$. For any $\varepsilon \in (0,1)$ there is a $C > 0$ such that

$$|\psi_i(s)| \leq \frac{C}{s^{1-\varepsilon}}\ (i \in \{1,2\}) \text{ and } |\psi_3(s)| \leq \frac{C}{s^{3/2-\varepsilon}}$$

for $a \in [\alpha, \beta]$ and $s \geq 1/\alpha$.

Proof In view of Remark *1* the assertion concerning ψ_1 is clear from

$$\psi_1(s) = \frac{1}{s}\left\{\int_0^{a/\alpha} \sqrt{x} J_\nu(x)\ dx + \left(\frac{2}{\pi}\right)^{1/2} \int_{a/\alpha}^{as} [\cos(x - (\nu\pi)/2 - \pi/4) + R_\nu(x)]\ dx\right\}.$$

As for ψ_2, we can write

$$\psi_2(s) = \int_0^a \sqrt{st} J_0(st)\ dt - 2M\frac{(s/2)^2}{1+M(s/2)^2} \int_0^a \frac{J_1(st)}{\sqrt{st}}\ dt.$$

The second integral can be taken care of by using the estimate (9) on a compact interval and Remark *1* outside. The same argument applies to

$$\psi_3(s) = \frac{1}{s^{3/2}}\left\{\int_0^{as} \sqrt{x} J_0(x)\ dx - 2M\int_0^{as} \frac{(x/2)^2}{s^2+(x/2)^2} \frac{J_1(x)}{\sqrt{x}}\ dx\right\}.$$

□

THEOREM 2.3 *Let $\nu \geq 0$. The functions defined by, for all $x \geq 0$,*

$$\phi_\lambda(x) := \begin{cases} \displaystyle\int_0^{\sqrt{\lambda}} \sqrt{xt} J_\nu(xt)\ dt & \text{if } \lambda \geq 0 \\ 0 & \text{if } \lambda < 0 \end{cases}$$

form an eigenpacket of the Friedrichs extension F_ν of $-d^2/dx^2 + (\nu^2 - 1/4)x^{-2}$ in $L^2(0,\infty)$. If $f \in L^2(0,\infty)$ is a function with

$$(f, \phi_\lambda) = 0\ (\lambda \in \mathbb{R}), \tag{10}$$

then $f = 0$ *a.e on* $(0,\infty)$, *i.e. the eigenpacket is complete in the sense of [8, p. 170]. Moreover, if* $0 < \lambda_1 < \lambda_2 < \infty, 0 < \mu_1 < \mu_2 < \infty$ *and the intervals* $I_1 := [\sqrt{\lambda_1}, \sqrt{\lambda_2}], I_2 := [\sqrt{\mu_1}, \sqrt{\mu_2}]$ *have at most one point in common, then*

$$(\phi_{\lambda_2} - \phi_{\lambda_1}, \phi_{\mu_2} - \phi_{\mu_1}) = 0. \tag{11}$$

Proof Let $\lambda > 0$. On account of Lemma 2.2 we have $\phi_\lambda \in L^2(0,\infty)$. Furthermore,

$$\lim_{\mu\to\lambda} \|\phi_\mu - \phi_\lambda\| = 0$$

by Lebesgue's theorem, since there is a neighbourhood I of λ where ϕ_μ can be dominated by an L^2-function independent of $\mu \in I$.

For $x > 0$ we find

$$-\phi_\lambda''(x) + \frac{\nu^2 - 1/4}{x^2}\phi_\lambda(x) = \int_0^{\sqrt{\lambda}} t^2 \sqrt{xt} J_\nu(xt)\, dt, \tag{12}$$

using Bessel's equation. Changing the order of integration in

$$\int_0^\lambda \left(\int_0^{\sqrt{\mu}} \sqrt{xt} J_\nu(xt)\, dt \right) d\mu,$$

we see that the r.h.s. of (12) can be written as

$$\lambda\phi_\lambda(x) - \int_0^\lambda \phi_\mu(x)\, d\mu = \int_0^\lambda \mu\, d\phi_\mu(x). \tag{13}$$

Owing to Lemma 2.2, the l.h.s. of (13) is in $L^2(0,\infty)$.

To complete the proof that $\phi_\lambda \in D(F_\nu)$, we have to check that the limits in (2) and (3) are indeed zero, but this follows immediately from the estimate (9).

In the case when (10) holds, we have $g = H_\nu f = 0$ by Theorem 1.2. Hence $f = 0$.

By virtue of an abstract result [8, p. 164f.], any eigenpacket enjoys the orthogonality property (11), but it is instructive to view (11) as a consequence of Proposition 2.1. Take a sequence (φ_i) of non-negative functions in $C_0^\infty(\mathbb{R})$ with

$$\lim_{i\to\infty} \varphi_i(x) = \begin{cases} 1 \text{ if } x \in \text{int}(I_1) \\ 0 \text{ if } x \in \mathbb{R}\setminus I_1 \end{cases}$$

and a similar sequence (ψ_k) for the interval I_2. In the notation of Proposition 2.1 we have

$$\begin{aligned} \lim\nolimits_{b\to\infty} \int_0^b \left(\int_0^\infty \sqrt{xt} J_\nu(xt) \varphi_i(t)\, dt \right) \left(\int_0^\infty \sqrt{xs} J_\nu(xs) \psi_k(s)\, ds \right) dx \\ = \lim\nolimits_{b\to\infty} \int_0^\infty t^{1/2} \varphi_i(t) \left(\int_0^\infty s^{-1/2} \psi_k(s) f_b(\nu, s, t)\, ds \right) dt. \end{aligned}$$

The right hand side equals, by Proposition 2.1,

$$\int_0^\infty \varphi_i(t)\psi_k(t)\, dt.$$

Then in the limit $i, k \to \infty$ we therefore find

$$(\phi_{\lambda_2} - \phi_{\lambda_1}, \phi_{\mu_2} - \phi_{\mu_1}) = (\chi_{I_1}, \chi_{I_2})$$

where χ_{I_1}, χ_{I_2} are the characteristic functions of the intervals I_1, I_2. □

The two orthogonality results in Proposition 2.5 below which use the Bessel-type function (5) reduce to the single relation of Proposition 2.1 with $\alpha = 0$ when $M = 0$. We first present the identities that are at the basis of these orthogonality results.

LEMMA 2.4 *For $b > 0$ we have, for $\lambda, \mu > 0$,*

$$\begin{cases} \lambda[1+M(\lambda/2)^2]^{-2}\left\{\int_0^b J_\lambda^M(z)J_\mu^M(z)z\ dz + \frac{1}{2}M\right\} \\ \qquad = [1+M(\mu/2)^2]\lambda[1+M(\lambda/2)^2]^{-1}\int_0^b J_0(\lambda z)J_0(\mu z)z\ dz \\ \qquad\quad +\frac{1}{2}M\lambda[1+M(\lambda/2)^2]^{-2}J_0(\lambda b)J_0(\mu b) \\ \qquad\quad -\frac{1}{2}\mu(M/2)^2\lambda^2[1+M(\lambda/2)^2]^{-2}J_1(\lambda b)J_1(\mu b), \end{cases} \tag{14}$$

and for $x, y > 0$

$$\begin{cases} \int_0^b J_\lambda^M(x)J_\lambda^M(y)y\lambda[1+M(\lambda/2)^2]^{-2}\ d\lambda \\ \qquad = y\int_0^b J_0(x\lambda)J_0(y\lambda)\lambda\ d\lambda - \frac{2M}{x}\frac{(b/2)^2}{1+M(b/2)^2}J_1(bx)J_1(by). \end{cases} \tag{15}$$

Proof From (5), the property $J_0' = -J_1$ and

$$J_1'(s) + s^{-1}J_1(s) = J_0(s)\ (s > 0), \tag{16}$$

see [15, p. 45], we derive the identity

$$J_\lambda^M(z)J_\mu^M(z)z = [1+M(\lambda/2)^2][1+M(\mu/2)^2]J_0(\lambda z)J_0(\mu z)z$$
$$+\frac{1}{2}M\frac{d}{dz}[J_0(\lambda z)J_0(\mu z)] - \frac{\lambda\mu}{2}(M/2)^2\frac{d}{dz}[J_1(\lambda z)J_1(\mu z)]$$

integration of which yields

$$\int_0^b J_\lambda^M(z)J_\mu^M(z)z\ dz = [1+M(\lambda/2)^2][1+M(\mu/2)^2]\int_0^b J_0(\lambda z)J_0(\mu z)z\ dz$$
$$+\frac{1}{2}M[J_0(\lambda b)J_0(\mu b) - 1] - \frac{\lambda\mu}{2}(M/2)^2[J_1(\lambda b)J_1(\mu b)],$$

which is (14).

From the definition of J_λ^M together with (16) we conclude

$$\frac{\lambda y J_\lambda^M(x) J_\lambda^M(y)}{[1+M(\lambda/2)^2]^2} = \lambda y J_0(\lambda x) J_0(\lambda y) + \frac{2M}{x}\left\{ -\frac{(\lambda/2)^2}{1+M(\lambda/2)^2}\frac{d}{d\lambda}(J_1(\lambda x)J_1(\lambda y)) \right.$$

$$\left. + J_1(\lambda x) J_1(\lambda y)\left[\frac{M(\lambda/2)^3}{[1+M(\lambda/2)^2]^2} - \frac{(\lambda/2)}{1+M(\lambda/2)^2}\right]\right\},$$

which integrates to (15). □

Proposition 2.5 *Let $\mu, x > 0$. Then, see also [6, Section 4, Corollary 4.3],*

$$\lambda[1+M(\lambda/2)^2]^{-2}\left\{\int_0^\infty J_\lambda^M(z) J_\mu^M(z) z\; dz + \frac{1}{2}M\right\} = \delta(\lambda-\mu)\ (\lambda>0)$$

and

$$y\int_0^\infty \frac{J_\lambda^M(x) J_\lambda^M(y)\lambda}{[1+M(\lambda/2)^2]^2}\; d\lambda = \delta(x-y)\ (y>0)$$

in the sense of distributions on $C_0^\infty((0,\infty))$, i.e. with

$$g_b(M,\lambda,\mu) := \frac{\lambda}{[1+M(\lambda/2)^2]^2}\int_0^b J_\lambda^M(z) J_\mu^M(z) z\; dz,$$

$$h_b(M,x,y) := y\int_0^b \frac{J_\lambda^M(x) J_\lambda^M(y)\lambda}{[1+M(\lambda/2)^2]^2} d\lambda$$

defined for all $b>0$, we then have

$$\lim_{b\to\infty}\int_0^\infty g_b(M,\lambda,\mu)\varphi(\lambda)\; d\lambda + \frac{1}{2}M\int_0^\infty \frac{\varphi(\lambda)\lambda}{[1+M(\lambda/2)^2]^2} d\lambda = \varphi(\mu)$$

and

$$\lim_{b\to\infty}\int_0^\infty h_b(M,x,y)\varphi(y)\; dy = \varphi(x)$$

for all $\varphi \in C_0^\infty((0,\infty))$.

Proof We multiply (14) by $\varphi(\lambda)$ and integrate with respect to λ. Using Remark *1* and the functions defined in (7), we then have, as $b\to\infty$,

$$\int_0^\infty g_b(M,\lambda,\mu)\varphi(\lambda)\; d\lambda + \frac{1}{2}M\int_0^\infty \frac{\varphi(\lambda)\lambda}{[1+M(\lambda/2)^2]^2} d\lambda$$

$$= [1+M(\mu/2)^2]\int_0^\infty f_b(0,\lambda,\mu)\frac{\varphi(\lambda)}{1+M(\lambda/2)^2} d\lambda + \mathcal{O}\left(\frac{1}{b}\right).$$

Owing to Proposition 2.1, the first term tends to $\varphi(\mu)$ as $b\to\infty$.

The second assertion follows immediately from

$$\int_0^\infty h_b(M,x,y)\varphi(y)\ dy = \int_0^\infty f_b(0,x,y)\varphi(y)\ dy + \mathcal{O}\left(\frac{1}{b}\right).$$

□

3 Generalised Hankel transformation: pointwise convergence

Lemma 2.4 can be simplified notationally by introducing the monotonic increasing function

$$n(\lambda) := \int_0^\lambda \frac{\tau}{[1+M(\tau/2)^2]^2} d\tau = \frac{\lambda^2}{2[1+M(\lambda/2)^2]}\ (\lambda \geq 0). \tag{17}$$

It is important to evaluate the following discontinuous integral.

LEMMA 3.1 *For $\dot{M} > 0$ we have*

$$\int_0^\infty J_\lambda^M(y)\ dn(\lambda) = \begin{cases} 2/M & \text{if } y = 0 \\ 0 & \text{if } y > 0. \end{cases}$$

Proof Since $J_\lambda^M(0) = 1$, the first claim is clear from (17).

For $y > 0$ the integral is absolutely convergent. Observing, from [15, p. 38],

$$\lambda y J_0''(\lambda y) + J_0'(\lambda y) + \lambda y J_0(\lambda y) = 0\ (\lambda > 0) \tag{18}$$

we can write

$$\begin{aligned} y\int_0^\infty J_\lambda^M(y)\ dn(\lambda) &= \int_0^\infty \left\{ \frac{-\lambda d(J_0'(\lambda y))/d\lambda - J_0'(\lambda y)}{1+M(\lambda/2)^2} + \frac{2M(\lambda/2)^2}{[1+M(\lambda/2)^2]^2} J_0'(\lambda y) \right\} d\lambda \\ &= \int_0^\infty J_0'(\lambda y) \left\{ \frac{d}{d\lambda}\frac{\lambda}{1+M(\lambda/2)^2} - \frac{1}{1+M(\lambda/2)^2} + \frac{2M(\lambda/2)^2}{[1+M(\lambda/2)^2]^2} \right\} d\lambda, \end{aligned}$$

which proves the second assertion. □

The next result offers two expansion theorems that both reduce to Theorem 1.1 with $\alpha = 0$ when $M = 0$.

THEOREM 3.2 *a) Let $\mu > 0$. If $g : (0,\infty) \to \mathbb{R}$ is a function with the property that*

$$f(\lambda) := \frac{g(\lambda)\sqrt{\lambda}}{1+M(\lambda/2)^2}\ (\lambda > 0)$$

is in $L^1(0,\infty)$ and is of bounded variation in a neighbourhood of μ, then

$$\frac{1}{2}[g(\mu+0) + g(\mu-0)] = \int_0^\infty J_\mu^M(x) \left(\int_0^\infty J_\lambda^M(x) g(\lambda)\ dn(\lambda) \right) x\ dx + \frac{1}{2}M \int_0^\infty g\ dn.$$

b) Let $f : (0,\infty) \to \mathbb{R}$ be a function with the property that

$$h(y) := f(y)\sqrt{y}\ (y > 0)$$

is in $L^1(0,\infty)$. *Then*

$$\int_0^\infty J_\lambda^M(x)\left(\int_0^\infty J_\lambda^M(y)f(y)y\;dy\right)dn(\lambda) \tag{19}$$

is zero if $x=0$, *and equal to* $\frac{1}{2}[f(x+0)+f(x-0)]$ *for* $x>0$, *provided* h *is of bounded variation in a neighbourhood of* x.

Proof a) Let $a,b>0$. From (14) we have

$$\begin{aligned}\int_0^b J_\mu^M(x)&\left(\int_0^a J_\lambda^M(x)g(\lambda)\;dn(\lambda)\right)x\;dx+(M/2)\int_0^a g(\lambda)\;dn(\lambda)\\ &=[1+M(\mu/2)^2]\mu^{-1/2}\int_0^b J_0(\mu x)\sqrt{\mu x}\left(\int_0^a J_0(\lambda x)\sqrt{\lambda x}f(\lambda)\;d\lambda\right)dx\\ &\quad+(M/2)J_0(\mu b)\int_0^a\frac{\sqrt{\lambda}J_0(\lambda b)}{1+M(\lambda/2)^2}f(\lambda)\;d\lambda\\ &\quad-\frac{\mu}{2}(M/2)^2J_1(\mu b)\int_0^a\frac{\lambda^{3/2}J_1(\lambda b)}{1+M(\lambda/2)^2}f(\lambda)\;d\lambda.\end{aligned} \tag{20}$$

There is no problem in letting $a\to\infty$ since $\sqrt{\lambda}J_\lambda^M(x)[1+M(\lambda/2)^2]^{-1}$ is a bounded function of λ and $|J_n(s)|\le 1$ ($s\ge 0, n\in\mathbb{N}_0$; see [15, p. 19]). Next, sending b to infinity, the second and third integrals on the r.h.s. of (20) (with $a=\infty$) tend to zero because of Remark *1*, while the first integral (with $a=\infty$) tends to $\frac{1}{2}[f(\mu+0)+f(\mu-0)]$ by Theorem 1.1.

b) For $x=0$, (19) reduces to the absolutely convergent integral

$$\int_0^\infty\left(\int_0^\infty J_\lambda^M(y)h(y)\sqrt{y}\;dy\right)dn(\lambda).$$

Changing the order of integration, the assertion follows from Lemma 3.1.

Next, let $x>0$. With $a,b>0$ we write (15) as

$$\begin{aligned}\int_0^b J_\lambda^M(x)&\left(\int_0^a J_\lambda^M(y)f(y)y\;dy\right)dn(\lambda)\\ &=x^{-1/2}\int_0^b J_0(xs)\sqrt{xs}\left(\int_0^a J_0(ys)\sqrt{ys}h(y)\;dy\right)ds\\ &\quad-\frac{2M}{x}\frac{(b/2)^2}{1+M(b/2)^2}J_1(bx)\int_0^a\frac{J_1(by)}{\sqrt{y}}h(y)\;dy.\end{aligned}$$

As before, the desired result follows by dint of Theorem 1.1. □

In order to interpret Theorem 3.2 in terms of linear operators, we propose the following result.

Lemma 3.3 a) *Let*

$$D:=\{f:[0,\infty)\to\mathbb{R}\,|\,f\text{ is continuous on }(0,\infty),(1+x)\sqrt{x}f\in L^1(0,\infty)\}.$$

For $f\in D$

$$g(\lambda):=\int_0^\infty J_\lambda^M(x)f(x)x\;dx\;(\lambda>0)$$

is in $C^1((0,\infty))$.

b) For $g \in C_0^3((0,\infty))$

$$f(x) := \int_0^\infty J_\lambda^M(x) g(\lambda)\ dn(\lambda)\ (x \geq 0)$$

is in D.

Proof a) Using $J_1 = -J_0'$ and (18), we see

$$d(J_\lambda^M(x))/d\lambda = -[1 + M(\lambda/2)^2] x J_1(\lambda x)\ (\lambda, x \geq 0).$$

This together with Remark *1* shows that every $\lambda > 0$ has a neighbourhood where $(d(J_\lambda^M(x))/d\lambda) f(x) x$ can be majorised by an integrable function that is independent of λ.

b) It suffices to prove

$$f(x) = \mathcal{O}(x^{-3}) \text{ as } x \to \infty.$$

For $m \in \mathbb{N}_0$ and $\nu \in \mathbb{R}$ we have, see [15, p. 46],

$$\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^m (\lambda^\nu J_\nu(\lambda x)) = x^m \lambda^{\nu - m} J_{\nu - m}(\lambda x)\ (\lambda, x > 0).$$

Hence

$$\begin{aligned} f(x) &= \int_0^\infty J_0(\lambda x) \frac{g(\lambda)\lambda}{1 + M(\lambda/2)^2} d\lambda - \frac{2M}{x} \int_0^\infty \frac{(\lambda/2)^2}{[1 + M(\lambda/2)^2]^2} J_1(\lambda x) g(\lambda)\ d\lambda \\ &= \frac{1}{x^3} \left\{ \int_0^\infty \left[\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^3 (\lambda^3 J_3(\lambda x)) \right] \frac{g(\lambda)\lambda}{1 + M(\lambda/2)^2} d\lambda \right. \\ &\qquad \left. - \frac{M}{2x} \int_0^\infty \left[\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^3 (\lambda^4 J_4(\lambda x)) \right] \frac{g(\lambda)\lambda}{[1 + M(\lambda/2)^2]^2} d\lambda \right\}, \end{aligned}$$

and the assertion follows when integrating by parts. □

Let D be as in Lemma 3.3 and

$$R := \left\{ g \in C^1((0,\infty)) \,\middle|\, \lambda \longmapsto \frac{g(\lambda)\sqrt{\lambda}}{1 + M(\lambda/2)^2} \text{ isin } L^1(0,\infty) \right\}.$$

Then we can define linear operators by

$$T_0 : R \to C^0[0,\infty), g \longmapsto \int_0^\infty J_\lambda^M(x) g(\lambda)\ dn(\lambda)\ (x \geq 0),$$

$$U_0 : D \to R, f \longmapsto \int_0^\infty J_\lambda^M(x) f(x) x\ dx + \frac{1}{2} M f(0)\ (\lambda > 0).$$

Owing to Theorem 3.2 *b*) and Lemma 3.1 we have

$$T_0 U_0 f = f$$

for all $f \in D$, *i.e.* U_0 is injective. For $g \in C_0^3((0,\infty)) \subset R$, $T_0 g \in D$ and

$$U_0 T_0 g = g$$

by Theorem 3.2 *a*), which shows that U_0^{-1} equals T_0 on $C_0^3((0,\infty))$. This is sufficient for our purposes (see Remark *1* below), but we emphasise that characterisations of the range of the classical Hankel transform exist, see [14].

4 Generalised Hankel transformation: L^2-convergence

We extend T_0 to the Hilbert space $L^2((0,\infty);n)$ where n is the measure generated by the function in (17). Scalar product and norm are defined by, respectively,

$$(f,g)_n := \int_0^\infty f(\lambda)\overline{g}(\lambda)\; dn(\lambda) \text{ and } \|f\|_n := [(f,f)_n]^{1/2}.$$

For the definition of the measure m we refer to (6).

THEOREM 4.1 *a) Let* $g \in L^2((0,\infty);n)$. *Then there is exactly one* $f \in L^2([0,\infty);m)$ *such that*

$$\|f\|_m = \|g\|_n, \tag{21}$$

viz., the function defined by

$$(Tg)(x) := f(x) := \begin{cases} \displaystyle\int_0^\infty g\; dn & \text{if } x = 0 \\ \displaystyle\frac{1}{\sqrt{x}}\frac{d}{dx}\int_0^\infty g(\lambda)\left(\int_0^x J_\lambda^M(t)\sqrt{t}\; dt\right) dn(\lambda) & \text{f.a.a. } x > 0. \end{cases} \tag{22}$$

Moreover,

$$\int_0^\Lambda J_\lambda^M(\cdot)g(\lambda)\; dn(\lambda) \in L^2([0,\infty);m) \text{ forall } \Lambda \in (0,\infty)$$

and

$$\lim_{\Lambda\to\infty}\int_0^\infty \left| f(x) - \int_0^\Lambda J_\lambda^M(x)g(\lambda)\; dn(\lambda)\right|^2 dm(x) = 0. \tag{23}$$

b) Let $f \in L^2([0,\infty);m)$. *Then there is exactly one* $g \in L^2((0,\infty);n)$ *with the property* (21). *This function* g *satisfies*

$$\int_0^\infty g\; dn = f(0) \tag{24}$$

and is given by

$$(Uf)(\lambda) := g(\lambda) := \begin{cases} \displaystyle\frac{1+M(\lambda/2)^2}{\sqrt{\lambda}}\frac{d}{d\lambda}\int_0^\infty f(x)\left(\int_0^\lambda J_\mu^M(x)\frac{\sqrt{\mu}}{1+M(\mu/2)^2}d\mu\right) x\; dx \\ \qquad\qquad + \tfrac{1}{2}Mf(0) \text{ f.a.a. } \lambda > 0. \end{cases} \tag{25}$$

Moreover,

$$\int_{[0,X]} J^M_{(\cdot)}(x)f(x)\ dm(x) \in L^2((0,\infty);n) \text{ forall } X \in (0,\infty)$$

and

$$\lim_{X\to\infty}\int_0^\infty \left| g(\lambda) - \int_{[0,X]} J^M_\lambda(x)f(x)\ dm(x)\right|^2 dn(\lambda) = 0. \tag{26}$$

Proof *a*) Let $l \in \{1,2\}$. If $(\gamma_i^{(l)})$ is a sequence in $C_0^\infty((0,\infty))$, we define

$$f_i^{(l)}(x) := \int_0^\infty J^M_\lambda(x)\gamma_i^{(l)}(\lambda)\ dn(\lambda)\ (x \geq 0 \text{ and } i \in \mathbb{N}).$$

For $X > 0$ we then have

$$\int_0^X \left|f_i^{(l)}(x)\right|^2 x\ dx = \int_0^\infty \left[\int_0^X J^M_\mu(x)\left(\int_0^\infty J^M_\lambda(x)\gamma_i^{(l)}(\lambda)\ dn(\lambda)\right) x\ dx\right]\overline{\gamma_i^{(l)}(\mu)}\ dn(\mu).$$

By Theorem 3.2 *a*) (or Proposition 2.5) the term in brackets tends to

$$\gamma_i^{(l)}(\mu) - \frac{1}{2}M\int_0^\infty \gamma_i^{(l)}\ dn\ (\mu > 0)$$

as $X \to \infty$. Hence

$$\left\|f_i^{(1)} - f_k^{(2)}\right\|^2_{L^2((0,\infty);x)} = \left\|\gamma_i^{(1)} - \gamma_k^{(2)}\right\|_n^2 - \frac{1}{2}M\left|\int_0^\infty \left(\gamma_i^{(1)} - \gamma_k^{(2)}\right) dn\right|^2 \tag{27}$$

for $i,k \in \mathbb{N}$. Given $g \in L^2((0,\infty);n)$, there is a sequence (γ_i) in $C_0^\infty((0,\infty))$ with $\|g-\gamma_i\|_n \to 0$ as $i \to \infty$. Using this single sequence (γ_i) in (27) and observing

$$\left(\int_0^\infty |\gamma_i - \gamma_k|\ dn\right)^2 \leq \frac{2}{M}\|\gamma_i - \gamma_k\|_n^2 \tag{28}$$

(note Lemma 3.1), we see that the corresponding sequence (f_i) is a Cauchy sequence in $L^2((0,\infty);x)$ and therefore convergent to an element $f \in L^2((0,\infty);x)$. Now (21) follows when we define $f(0) := \int_0^\infty g\ dn$.

Next let $x > 0$. Then

$$h(\lambda) := \frac{\sqrt{\lambda}}{1+M(\lambda/2)^2}\int_0^x J^M_\lambda(t)\sqrt{t}\ dt\ (\lambda > 0)$$

is in $L^2(0,\infty)$ by Lemma 2.2. As a consequence, we can perform the limit $i \to \infty$ in

$$\int_0^x f_i(t)\sqrt{t}\ dt = \int_0^\infty \frac{\gamma_i(\lambda)\sqrt{\lambda}}{1+M(\lambda/2)^2}h(\lambda)\ d\lambda$$

to obtain

$$\int_0^x f(t)\sqrt{t}\ dt = \int_0^\infty g(\lambda)\left(\int_0^x J^M_\lambda(t)\sqrt{t}\ dt\right) dn(\lambda),$$

which proves (22).

Finally, let $\Lambda > 0$ and replace g by $g_\Lambda := g\chi_{(0,\Lambda]}$. With the corresponding f_Λ we then have

$$\int_0^x f_\Lambda(t)\sqrt{t}\;dt = \int_0^\Lambda g(\lambda)\left(\int_0^x J_\lambda^M(t)\sqrt{t}\;dt\right) dn(\lambda).$$

On the r.h.s. we can now differentiate under the integral sign, since for a given $h_0 \in (0,x)$

$$\left|\frac{1}{h}\int_x^{x+h} J_\lambda^M(t)\sqrt{t}\;dt\right|$$

can be estimated by a number that is independent of $\lambda \in [0,\Lambda]$ and $0 < |h| \le h_0$. Hence

$$f_\Lambda(x) = \int_0^\Lambda J_\lambda^M(x)g(\lambda)\;dn(\lambda)$$

f.a.a. $x > 0$. From (27) we infer

$$\|f - f_\Lambda\|^2_{L^2((0,\infty);x)} = \|g - g_\Lambda\|_n^2 - \frac{1}{2}M\left|\int_0^\infty (g - g_\Lambda)\;dn\right|^2. \tag{29}$$

Since

$$f_\Lambda(0) := \int_0^\infty g_\Lambda\;dn \tag{30}$$

tends to $f(0)$ as $\Lambda \to \infty$, (29) and (30) yield (23).

b) Let $f \in L^2([0,\infty);m)$ and $\left(\varphi_i^{(1)}\right), \left(\varphi_i^{(2)}\right)$ be two sequences in $C_0^\infty((0,\infty))$. For $l \in \{1,2\}$ and $i \in \mathbb{N}$ we define

$$\gamma_i^{(l)}(\lambda) := \int_0^\infty J_\lambda^M(y)\varphi_i^{(l)}(y)y\;dy + \frac{1}{2}Mf(0)\;(\lambda > 0).$$

Let $\Lambda > 0$. From

$$\int_0^\Lambda \gamma_i^{(l)}\;dn = \int_0^\Lambda J_\lambda^M(0)\left(\int_0^\infty J_\lambda^M(y)\varphi_i^{(l)}(y)y\;dy\right)dn(\lambda) + \frac{1}{2}Mf(0)\int_0^\Lambda dn$$

we conclude

$$\int_0^\infty \gamma_i^{(l)}\;dn = f(0) \tag{31}$$

by virtue of Theorem 3.2 *b*). This theorem (or Proposition 2.5) also shows that the term in braces in

$$\int_0^\Lambda \left|\gamma_i^{(1)} - \gamma_k^{(2)}\right|^2\;dn = \int_0^\infty \left\{\int_0^\Lambda J_\lambda^M(x)\left(\int_0^\infty J_\lambda^M(y)[\varphi_i^{(1)} - \varphi_k^{(2)}](y)y\;dy\right)dn(\lambda)\right\} \times \overline{\left[\varphi_i^{(1)} - \varphi_k^{(2)}\right]}(x)x\;dx$$

tends to $\varphi_i^{(1)}(x) - \varphi_k^{(2)}(x)$ as $\Lambda \to \infty$. Hence

$$\left\| \gamma_i^{(1)} - \gamma_k^{(2)} \right\|_n = \left\| \varphi_i^{(1)} - \varphi_k^{(2)} \right\|_{L^2((0,\infty);x)}. \tag{32}$$

In particular, using this relationship for a single sequence (φ_i) with $\|f - \varphi_i\|_{L^2((0,\infty:x)} \to 0$ as $i \to \infty$, we find that the corresponding sequence (γ_i) converges to some $g \in L^2((0,\infty);n)$. Moreover

$$\lim_{i\to\infty} \int_0^\infty |\gamma_i - g| \ dn = 0$$

by inequality (28). In conjunction with (31) this establishes (24), and (21) results from (32).

For $\lambda > 0$ and $i \in \mathbb{N}$ we have

$$\int_0^\lambda \frac{\gamma_i(\mu)\sqrt{\mu}}{1+M(\mu/2)^2} d\mu = \int_0^\infty \varphi_i(x) \left(\int_0^\lambda J_\mu^M(x) \frac{\sqrt{\mu}}{1+M(\mu/2)^2} d\mu \right) x \ dx + \tfrac{1}{2} M f(0) \int_0^\lambda \frac{\sqrt{\mu}}{1+M(\mu/2)^2} d\mu.$$

Since the inner integral is in $L^2((0,\infty);x)$ by Lemma 2.2, we can perform the limit $i \to \infty$, and (25) follows by differentiation.

Finally, let $X > 0$. If g_X is the transform that belongs to $f_X := f\chi_{[0,X]}$, then

$$g_X(\lambda) = \int_0^X J_\lambda^M(x) f(x) x \ dx + \frac{1}{2} M f(0) = \int_{[0,X]} J_\lambda^M(x) f(x) \ dm(x)$$

f.a.a. $\lambda > 0$, since the differentiation in (25) can be performed under the integral sign when f is replaced by f_X. If $\left(\varphi_i^{(1)}\right), \left(\varphi_i^{(2)}\right)$ are sequences which converge in $L^2((0,\infty);x)$ to f and f_X, respectively, then $\left(\gamma_i^{(1)}\right), \left(\gamma_i^{(2)}\right)$ converge to g and g_X in $L^2((0,\infty);n)$ and from (32) we obtain

$$\|g - g_X\|_n = \left\| (1 - \chi_{[0,X]}) f \right\|_{L^2((0,\infty);x)}$$

which proves (26). □

Remark 1 We have $U^{-1} = T$, since these operators coincide on the dense set $C_0^3((0,\infty))$.

Remark 2 From the known connection between the norm $\|\cdot\|$ and inner-product in abstract Hilbert space it follows from (21) that if f_1, f_2 and g_1, g_2 are as defined in Theorem 4.1 then

$$(f_1, f_2)_m = (g_1, g_2)_n.$$

We conclude with an analogue of Theorem 2.3.

THEOREM 4.2 *Let S_M be the self-adjoint operator generated by $x^{-1}L_M$ in $L^2([0,\infty);m)$. For $\lambda \geq 0$ let t_λ be the unique non-negative number with $t_\lambda^2(t_\lambda^2 + 8M^{-1}) = \lambda$. Then the functions defined by*

$$\phi_\lambda(x) := \begin{cases} \displaystyle\int_0^{t_\lambda} \frac{\sqrt{t}}{1+M(t/2)^2} J_t^M(x) \ dt & \text{if } \lambda \geq 0 \\ 0 & \text{if } \lambda < 0 \end{cases} \quad (x \geq 0)$$

form an eigenpacket of S_M. It is complete in the following sense:

If $f \in L^2([0,\infty);m)$ is a function with

$$(f, \phi_\lambda)_m = 0 \; (\lambda \in \mathbb{R}), \tag{33}$$

then $f(0) = 0$ and $f(x) = 0$ f.a.a. $x > 0$. Moreover, if $0 < \lambda_1 < \lambda_2 < \infty, 0 < \mu_1 < \mu_2 < \infty$ and the intervals $I_1 := [t_{\lambda_1}, t_{\lambda_2}], I_2 := [t_{\mu_1}, t_{\mu_2}]$ have at most one point in common, then

$$(\phi_{\lambda_2} - \phi_{\lambda_1}, \phi_{\mu_2} - \phi_{\mu_1})_m = 0. \tag{34}$$

Proof Let $\lambda, x > 0$. We start with the observation that

$$\begin{aligned}\frac{1}{x}\left(L_M\phi_\lambda\right)(x) &= \int_0^{t_\lambda} \frac{\sqrt{t}}{1+M(t/2)^2}\frac{1}{x}\left(L_M J_t^M\right)(x)\; dt \\ &= \int_0^{t_\lambda} \frac{\Lambda(t)\sqrt{t}}{1+M(t/2)^2} J_t^M(x)\; dt\end{aligned}$$

where

$$\Lambda(t) := t^2(t^2 + 8M^{-1}) \; (t \geq 0).$$

On the other hand

$$\begin{aligned}\int_0^\lambda \phi_\mu(x)\; d\mu &= \int_0^{t_\lambda}\left(\int_{\Lambda(t)}^\lambda \frac{\sqrt{t}}{1+M(t/2)^2} J_t^M(x)\; d\mu\right) dt \\ &= \lambda\phi_\lambda(x) - \int_0^{t_\lambda} \frac{\Lambda(t)\sqrt{t}}{1+M(t/2)^2} J_t^M(x)\; dt,\end{aligned}$$

which proves

$$\frac{1}{x}(L_M\phi_\lambda)(x) = \lambda\phi_\lambda(x) - \int_0^\lambda \phi_\mu(x)\; d\mu.$$

Owing to Lemma 2.2, the r.h.s. is in $L^2((0,\infty);x)$.

The power series representation of the Bessel functions [15, p. 15] gives us

$$J_t^M(x) = 1 - \frac{1}{4}(tx)^2(1 + \frac{1}{8}Mt^2) + \mathcal{O}((tx)^4),$$

when t and x vary in compact sets. Hence

$$J_t^{M\prime}(0) = 0, J_t^{M\prime\prime}(0) = -\frac{1}{2}t^2(1 + \frac{1}{8}Mt^2),$$

$$\left.\frac{1}{x}(L_M\phi_\lambda)(x)\right|_{x=0} = \Lambda(t) = -\frac{16}{M}J_t^{M\prime\prime}(0)$$

for $t \geq 0$. This proves $\phi_\lambda \in D(S_M)$.

The L^2-continuity in λ of ϕ_λ follows from Lemma 2.2. Hence, $(\phi_\lambda)_{\lambda\in\mathbb{R}}$ is therefore an eigenpacket of S_M.

If f satisfies (33), then (25) implies

$$g(\lambda) = \frac{1}{2} M f(0)$$

f.a.a. $\lambda > 0$. The Parseval relation (21) then yields

$$\int_0^\infty |f(x)|^2 \, x \; dx = 0$$

and so $f = 0$ a.e. on $(0, \infty)$. Now (33) provides us with the information that $f(0) = 0$.

To establish (34), we can follow the last step in the proof of Theorem 2.3, however substituting Proposition 2.1 by Proposition 2.5. □

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