

## Additional spectral properties of the fourth-order Bessel-type differential equation

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*Dedicated to the achievements and memory of F. V. Atkinson (1916–2002)*

This paper discusses the spectral properties of the self-adjoint differential operator generated by the fourth-order Bessel-type differential expression, as defined by Everitt and Markett in 1994, in a Lebesgue–Stieltjes Hilbert function space. This space involves functions defined on the real line; the Lebesgue–Stieltjes measure is locally absolutely continuous on the real line, with the origin removed; the origin itself has strictly positive measure. It is shown that there is a unique such self-adjoint operator; this operator has no eigenvalues but has a continuous spectrum on the positive half-line of the spectral plane.

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### 1 Introduction

This is the second paper on the spectral properties of the fourth-order Bessel-type linear ordinary differential equation, following the results in the paper [2]. To make this present paper reasonably self contained we reproduce part of Section 1 of the first paper; this reproduction also serves to indicate the origins of the fourth-order Bessel-type differential equation, and the structured connection with classical Bessel functions and the fourth-order differential equations of the Legendre, Jacobi, Laguerre-type orthogonal polynomials.

The fourth-order Bessel-type differential equation takes the form

$$(xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \quad \text{for all } x \in (0, \infty) \quad (1.1)$$

where  $M > 0$  is a positive parameter and  $\Lambda \in \mathbb{C}$ , the complex field, is a spectral parameter. The differential equation (1.1) is derived in the paper [7, Section 1, (1.10a)], by Everitt and Markett.

This linear, ordinary differential equation on the interval  $(0, \infty) \subset \mathbb{R}$ , the real field, is written in Lagrange symmetric (formally self-adjoint) form, or equivalently Naimark form, see [11, Chapter V] and [1, Appendix 2].

The structured Bessel-type functions, and their associated linear differential equations of all even-order, were introduced in the paper [7, Section 1] through linear combinations of, and limit processes applied to, the Laguerre and Laguerre-type orthogonal polynomials. This process is best illustrated through the following diagram, see

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[7, Section 1, p. 328] (for the first two lines of this table see the earlier work of Koornwinder [9] and Market [10]):

$$\left\{ \begin{array}{ll}
 \begin{array}{l}
 \text{Jacobi polynomials} \\
 k(\alpha, \beta)(1-x)^\alpha(1+x)^\beta \\
 \downarrow \\
 \text{Laguerre polynomials} \\
 k(\alpha)x^\alpha \exp(-x) \\
 \downarrow \\
 \text{Bessel functions} \\
 \kappa(\alpha)x^{2\alpha+1}
 \end{array}
 & \longrightarrow \begin{array}{l}
 \text{Jacobi-type polynomials} \\
 k(\alpha, \beta)(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1) \\
 \downarrow \\
 \text{Laguerre-type polynomials} \\
 k(\alpha)x^\alpha \exp(-x) + N\delta(x) \\
 \downarrow \\
 \text{Bessel-type functions} \\
 \kappa(\alpha)x^{2\alpha+1} + M\delta(x)
 \end{array}
 \end{array} \right. \quad (1.2)$$

The symbol entry (here  $k$  and  $\kappa$  are positive numbers depending only on the parameters  $\alpha$  and  $\beta$ ) under each special function indicates a non-negative (generalized) “weight”, on the interval  $(-1, 1)$  or  $(0, \infty)$ , involved in:

- (a) the orthogonality property of the special functions,
- (b) the weight coefficient in the associated differential equations.

It is important to note in this diagram that:

- (i) a horizontal arrow  $\longrightarrow$  indicates a definition process either by a linear combination of special functions of the same type but of different orders, or by a linear-differential combination of special functions of the same type and order.
- (ii) a vertical arrow  $\downarrow$  indicates a confluent limit process of one special function to give another special function.
- (iii) the use of the symbol  $M\delta(\cdot)$  is a notational device to indicate that the monotonic function on the real line  $\mathbb{R}$  defining the weight has a jump at an end-point of the interval concerned, of magnitude  $M > 0$ .
- (iv) the combination of any vertical arrow  $\downarrow$  with a horizontal arrow  $\longrightarrow$  must give a consistent single entry.

Information about the Jacobi-type and Laguerre-type orthogonal polynomials, and their associated differential equations, is given in the Everitt and Littlejohn survey paper [5]; see in particular the references in this paper to the introduction of the fourth-order Laguerre-type differential equation by H. L. and A. M. Krall, and by Littlejohn. The general Laguerre-type differential equation is introduced in the paper [8] by Koekoek and Koekoek; the order of this linear differential equation is determined by  $4 + 2\alpha$  with  $\alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ .

It is significant that the general order Bessel-type functions also satisfy a linear differential equation of order  $4 + 2\alpha$ , being an inheritance from the order of the general Laguerre-type equation.

The purpose of this paper is to initiate the study of properties of the Bessel-type linear differential in the special case when  $\alpha = 0$ , as given in the bottom right-hand corner of the diagram; this is the fourth-order Bessel-type differential equation (1.1) and involves the weight coefficient  $\kappa(0)x$ ; its solutions should, in some sense, have orthogonality properties with respect to the generalized weight function  $\kappa(0)x + M\delta(0)$ , where  $M > 0$  is the parameter appearing in the differential equation (1.1); see [7, Section 4].

In the first paper [2] attention was restricted to the spectral properties of the differential equation (1.1) in the classical weighted Hilbert function space (for notation see Section 2 below) of equivalence classes of Lebesgue measurable functions with the property

$$L^2((0, \infty); x) := \left\{ f : (0, \infty) \rightarrow \mathbb{C} : \int_0^\infty x |f(x)|^2 dx < \infty \right\} \quad (1.3)$$

with norm and scalar product given by

$$\|f\|^2 = \int_0^\infty x |f(x)|^2 dx \quad \text{and} \quad (f, g) = \int_0^\infty x f(x) \bar{g}(x) dx. \quad (1.4)$$

Previous studies of fourth-order differential equations generating Legendre-type, Jacobi-type and Laguerre-type orthogonal polynomials, see [3], [4] and [6], have shown that an initial study of the spectral properties of

the differential equation in the classical Hilbert function space is essential to the subsequent study of spectral properties in the jump weighted Hilbert space.

In this second paper we consider the spectral properties of the fourth-order Bessel-type differential equation (1.1) in the jump weighted Hilbert space  $L^2(\mathbb{R}; m_k)$ , see the definition in Section 2 below. In a third paper, to follow, we shall discuss the distributional and eigenpacket orthogonality of both the classical Bessel functions and Bessel-type functions, and the generalized Hankel transform generated by the Bessel-type functions.

Information about the higher even-order Bessel-type differential equations is given in [7]; in particular the explicit Lagrange symmetric forms of the sixth-order and eighth-order differential equations are given in [7, Section 1, (1.10b) and (1.10c)], and in the general case in [7, Section 2, (2.17)]. However the spectral theory of these higher order differential equations can be expected to follow the properties of the special case when  $\alpha = 0$  and the order of the equation is four.

The classical Bessel differential equation, with order  $\alpha = 0$ , written in a form comparable to the fourth-order equation (1.1), is best taken from the left-hand bottom corner of the diagram (1.2); from [7, Section 1, (1.2)] with  $\alpha = 0$  we obtain

$$-(xy'(x))' = \lambda^2 xy(x) \quad \text{for all } x \in (0, \infty); \tag{1.5}$$

here  $\lambda \in \mathbb{C}$  is the spectral parameter. It is to be observed that, formally, if the fourth-order Bessel-type equation (1.1) is multiplied by the parameter  $M > 0$  and then  $M$  tends to zero, we obtain essentially the classical Bessel equation of order zero (1.5), on using the spectral relationship (1.7) between the parameters  $\lambda$  and  $\Lambda$ .

This comparison and connection between the Bessel-type equation (1.1) and the classical Bessel equation (1.5) may seem surprising; nevertheless this Bessel-type equation is the true structured inheritor of the classical Bessel equation; in particular this can be seen in the table of results (1.2); also in the comparable spectral properties of the differential operators, generated by the corresponding differential expressions, in the same Hilbert function space  $L^2((0, \infty); x)$ , as is shown below. The same type of comparison can be made in considering the inheritance of the Jacobi-type and Laguerre-type differential equations from the corresponding classical Jacobi and Laguerre differential equations, see again the table (1.2).

Our knowledge of the special function solutions of the Bessel-type differential equation (1.1) is, at present, limited. However, the results in [7, Section 1, (1.8a)], with  $\alpha = 0$ , show that the function defined by

$$J_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x) \quad \text{for all } x \in (0, \infty), \tag{1.6}$$

where

- (i) the parameter  $M > 0$ ,
- (ii) the parameter  $\lambda \in \mathbb{C}$ ,
- (iii) the spectral parameter  $\Lambda$  and the parameter  $M$ , in Eq. (1.1), and the parameters  $M$  and  $\lambda$ , in the definition (1.6), are connected by the relationship

$$\Lambda \equiv \Lambda(\lambda, M) = \lambda^2(\lambda^2 + 8M^{-1}) \quad \text{for all } \lambda \in \mathbb{C} \text{ and all } M > 0, \tag{1.7}$$

- (iv)  $J_0$  and  $J_1$  are the classical Bessel functions (of the first kind), see [13, Chapter III],

is a solution of the differential equation (1.1), for all  $\lambda \in \mathbb{C}$ , and hence for all  $\Lambda \in \mathbb{C}$  and all  $M > 0$ .

Similar arguments to the methods given in [7] show that the function defined by

$$Y_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]Y_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}Y_1(\lambda x) \quad \text{for all } x \in (0, \infty), \tag{1.8}$$

is also a solution of the differential equation (1.1), for all  $\lambda \in \mathbb{C}$ , and hence for all  $\Lambda \in \mathbb{C}$  and all  $M > 0$ ; here, again,  $Y_0$  and  $Y_1$  are classical Bessel functions (of the second kind), see [13, Chapter III].

From arguments given in [2, Section1] it appears that additional solutions of the fourth-order Bessel-type equation (1.1), linearly independent of the two known solutions  $J_\lambda^{0,M}$  and  $Y_\lambda^{0,M}$ , have to be defined separately; such additional solutions cannot be obtained through quadrature arguments based on the solutions  $J_\lambda^{0,M}$  and  $Y_\lambda^{0,M}$ .

## 2 The Lebesgue–Stieltjes Hilbert function space

This is a Lebesgue–Stieltjes Hilbert function space required for the study of the spectral properties of the fourth-order Bessel-type differential equation (1.1). This space is associated with a regular, non-negative Baire measure defined on the Borel  $\sigma$ -algebra of the real line  $\mathbb{R}$ , see the text [12, Chapter 12, Section 3].

The jump space  $L^2(\mathbb{R}; m_k)$  is suggested by the results in [7, Section 4].

Let the monotonic non-decreasing function  $\widehat{m}_k : \mathbb{R} \rightarrow \mathbb{R}$  be defined by, where  $k > 0$  is a real parameter,

$$\widehat{m}_k(x) = \begin{cases} -k & \text{for all } x \in (-\infty, 0], \\ x^2/2 & \text{for all } x \in (0, +\infty). \end{cases}$$

Then  $\widehat{m}_k$  generates a Baire measure  $m_k$  on the  $\sigma$ -algebra  $\mathcal{B}$  of Borel sets on the real line  $\mathbb{R}$ , see [12, Chapter 12, Section 3]; in turn this measure generates a Lebesgue–Stieltjes integral for Borel measurable functions.

The Hilbert function space  $L^2(\mathbb{R}; m_k)$  is defined on all equivalence classes of functions with the properties:

- (i)  $f : \mathbb{R} \rightarrow \mathbb{C}$  and is Borel measurable on  $\mathbb{R}$ ,
- (ii)  $f(0) \in \mathbb{C}$ ,
- (iii)  $\int_0^{+\infty} x |f(x)|^2 dx < +\infty$ .

The norm and inner-product in  $L^2(\mathbb{R}; m_k)$  are defined by

$$\|f\|_{m_k}^2 := \int_{(-\infty, +\infty)} |f(x)|^2 dm_k(x) = k |f(0)|^2 + \int_0^{+\infty} x |f(x)|^2 dx$$

and

$$(f, g)_{m_k} := \int_{(-\infty, +\infty)} f(x)\overline{g(x)} dm_k(x) = kf(0)\overline{g(0)} + \int_0^{+\infty} xf(x)\overline{g(x)} dx. \tag{2.1}$$

Note that the first integrals in both these definitions are Lebesgue–Stieltjes integrals taken over the set  $(-\infty, +\infty)$ , whilst the second integrals can be taken as Lebesgue integrals.

## 3 Properties of the maximal operator in $L^2((0, \infty); x)$

In this section we recall the definition and properties of the maximal operator  $T_1$  generated by the differential expression  $L_M$  in the Hilbert function space  $L^2((0, \infty); x)$ ; see Eqs. (1.3) and (1.4).

As in the paper [2, Section 4] we define the fourth-order Bessel-type differential expression  $L_M$  with domain  $D(L_M)$  by

$$D(L_M) := \left\{ f : (0, \infty) \rightarrow \mathbb{C} : f^{(r)} \in AC_{loc}(0, \infty) \text{ for } r = 0, 1, 2, 3 \right\}, \tag{3.1}$$

and then, for all  $f \in D(L_M)$ ,

$$L_M[f](x) := (xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' \quad (x \in (0, \infty)); \tag{3.2}$$

it follows that

$$L_M : D(L_M) \longrightarrow L^1_{loc}(0, \infty). \tag{3.3}$$

The Green’s formula for  $L_M$  on any compact interval  $[\alpha, \beta] \subset (0, +\infty)$  takes the form

$$\int_{\alpha}^{\beta} \left\{ \overline{g(x)}L_M[f](x) - f(x)\overline{L_M[g](x)} \right\} dx = [f, g](x) \Big|_{\alpha}^{\beta} \tag{3.4}$$

where the symplectic form  $[\cdot, \cdot](\cdot) : D(L_M) \times D(L_M) \times (0, +\infty) \rightarrow \mathbb{C}$  is given explicitly by

$$\begin{aligned} [f, g](x) &= \overline{g(x)}(xf''(x))' - (x\overline{g''(x)})'f(x) \\ &\quad - x(\overline{g'(x)}f''(x) - \overline{g''(x)}f'(x)) \\ &\quad - (9x^{-1} + 8M^{-1}x)(\overline{g(x)}f'(x) - \overline{g'(x)}f(x)). \end{aligned} \tag{3.5}$$

The Dirichlet formula for  $L_M$  on any compact interval  $[\alpha, \beta] \subset (0, +\infty)$  takes the form

$$\begin{aligned} & \int_{\alpha}^{\beta} \{xf''(x)\overline{g}''(x) + (9x^{-1} + 8M^{-1}x)f'(x)\overline{g}'(x)\} dx \\ &= [f, g]_D(x)|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} L_M[f](x)\overline{g}(x) dx \end{aligned} \tag{3.6}$$

where the Dirichlet form  $[\cdot, \cdot]_D(\cdot) : D(L_M) \times D_0(L_M) \times (0, +\infty) \rightarrow \mathbb{C}$  is given by, for  $f \in D(L_M)$  and  $g \in D_0(L_M)$ , with

$$D_0(L_M) := \left\{ g : (0, +\infty) \rightarrow \mathbb{C} : g^{(r)} \in AC_{loc}(0, +\infty) \text{ for } r = 0, 1 \right\} \tag{3.7}$$

and

$$[f, g]_D(x) := -\overline{g}(x)(xf''(x))' + \overline{g}'(x)xf''(x) + \overline{g}(x)(9x^{-1} + 8M^{-1}x)f'(x). \tag{3.8}$$

The maximal operator  $T_1$  generated by the differential expression in the space  $L^2((0, \infty); x)$  is defined, see [2, Section 5], as  $T_1 : D(T_1) \subset L^2((0, \infty); x) \rightarrow L^2((0, \infty); x)$  with

$$D(T_1) := \{f \in D(L_M) : f, x^{-1}L_M(f) \in L^2((0, \infty); x)\} \tag{3.9}$$

and

$$T_1f := x^{-1}L_M(f) \text{ for all } f \in D(T_1). \tag{3.10}$$

From the Green's formula (3.4) it follows that the limits

$$[f, g](0^+) := \lim_{x \rightarrow 0^+} [f, g](x) \quad \text{and} \quad [f, g](\infty) := \lim_{x \rightarrow \infty} [f, g](x) \tag{3.11}$$

both exist and are finite in  $\mathbb{C}$  for all  $f, g \in D(T_1)$ .

The following properties of the elements of the maximal domain  $D(T_1)$  are required, see [2, Section 8].

**Theorem 3.1** *Let  $f \in D(T_1)$ ; then the values of  $f, f'$  and  $f''$  can be defined at the point 0, by taking appropriate limits, so that the following results hold:*

- (i)  $f \in AC[0, 1]$ ,
- (ii)  $f' \in AC[0, 1]$  and  $f'(0) = 0$ ,
- (iii)  $f'' \in AC_{loc}(0, 1]$  and  $f'' \in C[0, 1]$ ,
- (iv)  $f^{(3)} \in AC_{loc}(0, 1]$  and  $\lim_{x \rightarrow 0^+} (xf^{(3)}(x)) = 0$ .

*Proof.* See [2, Section 8, Theorem 8.1]. □

**Corollary 3.2** *For all  $f, g \in D(T_1)$ , see (3.5) and (3.8):*

- (i)  $\lim_{x \rightarrow 0^+} [f, g](x) = 8(f(0)\overline{g}''(0) - f''(0)\overline{g}(0))$ ,
- (ii)  $\lim_{x \rightarrow 0^+} [f, g]_D(x) = 8f''(0)\overline{g}(0)$ .

*Proof.* See [2, Section 9, Lemma 9.3]. □

**Theorem 3.3** *For all  $f, g \in D(T_1)$ :*

- (i)  $\lim_{x \rightarrow +\infty} [f, g](x) = 0$ ,
- (ii)  $\lim_{x \rightarrow +\infty} [f, g]_D(x) = 0$ .

*Proof.* See [2, Section 6]. □

**Corollary 3.4** *For all  $f, g \in D(T_1)$*

- (i)  $x^{1/2}f''$  and  $x^{-1/2}f' \in L^2(0, \infty)$ ,

(ii)

$$\int_0^\infty \{x f''(x) \overline{g}''(x) + (9x^{-1} + 8M^{-1}x) f'(x) \overline{g}'(x)\} dx = \int_0^\infty \{x^{-1} L_M[f](x) \overline{g}(x)\} x dx - 8f''(0) \overline{g}(0),$$

(iii)

$$\int_0^\infty \{\overline{g}(x) x^{-1} L_M[f](x) - f(x) x^{-1} \overline{L_M[g]}(x)\} x dx = 8\{f''(0) \overline{g}(0) - f(0) \overline{g}''(0)\}.$$

Proof. See [2, Corollary 9.3 and Corollary 9.4]. □

#### 4 Properties of self-adjoint operators in $L^2((0, \infty); x)$

The differential expression  $L_M$  generates a continuum of self-adjoint operators in the space  $L^2((0, \infty); x)$ ; the properties of these operators are fully discussed in [2, Sections 10 and 13]. We state here a theorem which gives all the required spectral properties of these self-adjoint operators, where  $\sigma_{\text{ess}}(T)$  and  $\sigma_{\text{cont}}(T)$  represent, respectively, the essential and the continuous spectrum of the self-adjoint operator  $T$ , see [14, Chapter 7, pp. 202 and 209] for these spectral definitions:

**Theorem 4.1** 1. *Let  $T$  be any self-adjoint operator generated by the differential expression  $L_M$  in the Hilbert space  $L^2((0, \infty); x)$ . Then*

(i) *The essential spectrum  $\sigma_{\text{ess}}(T)$  is given by*

$$\sigma_{\text{ess}}(T) = \sigma_{\text{cont}}(T) = [0, \infty).$$

(ii) *There are no embedded eigenvalues of  $T$  in the essential spectrum.*

(iii)  *$T$  has at most one eigenvalue; if this eigenvalue is present then it is simple and lies in the interval  $(-\infty, 0)$ .*

2. *Every point  $\mu \in (-\infty, 0)$  is the eigenvalue of some unique self-adjoint operator  $T$ .*

Proof. See [2, Theorem 13.2]. □

#### 5 The self-adjoint operator $S_k$ in $L^2(\mathbb{R}; m_k)$

In this section, given any  $k \in (0, \infty)$ , we define the operator  $S_k$  generated by the differential expression  $L_M$  in the Hilbert function space  $L^2(\mathbb{R}; m_k)$ , where this space is defined in Section 2 above.

**Definition 5.1** Let  $k \in (0, \infty)$  be given; then the operator  $S_k$

$$S_k : D(S_k) \subset L^2(\mathbb{R}; m_k) \longrightarrow L^2(\mathbb{R}; m_k) \tag{5.1}$$

is defined by (see (3.9) and Theorem 3.1 for the definition and properties of the domain  $D(T_1) \subset L^2((0, \infty); x)$ )

(i)  $D(S_k) := D(T_1)$ ,

(ii) for all  $f \in D(S_k)$

$$(S_k f)(x) := \begin{cases} -8k^{-1} f''(0) & \text{for } x = 0, \\ x^{-1} L_M[f](x) & \text{for all } x \in (0, \infty). \end{cases} \tag{5.2}$$

**Theorem 5.2** For all  $k \in (0, \infty)$ :

(i) *The linear manifold  $D(S_k)$  is dense in  $L^2(\mathbb{R}; m_k)$ .*

(ii) *The operator  $S_k$  is hermitian in  $L^2(\mathbb{R}; m_k)$ .*

(iii) *The operator  $S_k$  is symmetric in  $L^2(\mathbb{R}; m_k)$ .*

(iv) The operator  $S_k$  is bounded below in  $L^2(\mathbb{R}; m_k)$

$$(S_k f, f)_{m_k} \geq 0 \text{ for all } f \in D(S_k). \tag{5.3}$$

**Proof.** We have:

(i) Let  $f \in L^2(\mathbb{R}; m_k)$  and choose  $\varepsilon > 0$ . Then there exists  $\varphi \in C_0^\infty(0, \infty) \subset D(S_k)$  such that

$$\int_0^\infty x |f(x) - \varphi(x)|^2 dx < \varepsilon;$$

let  $\text{supp}(\varphi) = [\delta, \Delta]$  so that  $0 < \delta < \Delta < \infty$ .

Choose  $\eta \in (0, \delta)$  so that

$$\int_0^\eta x |f(x)|^2 dx < \varepsilon.$$

Now consider the Frobenius solution  $y_0(\cdot, 0, M)$  of the differential equation  $L_M[y] = 0$  as defined in [2, Section 3, (3.4)]; this solution has the property that  $y_0(0, 0, M) = l \neq 0$ . Choose  $\eta_1 \in (0, \eta)$  so that

$$\int_0^{\eta_1} x |f(0)l^{-1}y_0(x, 0, M)|^2 dx < \varepsilon$$

and define  $\psi_1 : [0, \eta_1] \rightarrow \mathbb{C}$  by

$$\psi_1(x) := f(0)l^{-1}y_0(x, 0, M) \text{ for all } x \in [0, \eta_1],$$

noting that  $\psi_1(0) = f(0)$ .

**Remark 5.3** The solution  $J_\lambda^{0,M}(\cdot)$ , see (1.6), could also have been used at this stage instead of the Frobenius solution  $y_0(\cdot)$ .

Using the Naimark patching lemma, see [11, Chapter V, Section 17.3, Lemma 2], define  $\psi_2 : [\eta_1, \eta] \rightarrow \mathbb{C}$  so that

- (a)  $\psi_2 \in D(L_M)$  on the interval  $[\eta_1, \eta]$ ,
- (b)  $\psi_1$  and  $\psi_2$  have the same quasi-derivatives at  $\eta_1$ ,
- (c) all the quasi-derivatives of  $\psi_2$  are zero at  $\eta$ .

Now choose  $\varphi_2 \in C_0^\infty(\eta_1, \eta)$  so that

$$\int_{\eta_1}^\eta x |\psi_2(x) - \varphi_2(x)|^2 dx < \varepsilon.$$

Finally define  $\psi : [0, \infty) \rightarrow \mathbb{C}$  by

$$\psi(x) := \begin{cases} \psi_1(x) & \text{for all } x \in [0, \eta_1], \\ \psi_2(x) - \varphi_2(x) & \text{for all } x \in [\eta_1, \eta], \\ 0 & \text{for all } x \in [\eta, \delta], \\ \varphi(x) & \text{for all } x \in [\delta, \Delta], \\ 0 & \text{for all } x \in [\Delta, \infty). \end{cases}$$

Then  $\psi \in D(T_1) = D(S_k)$  and in the norm of  $L^2(\mathbb{R}; m_k)$ ,

$$\begin{aligned} \|f - \psi\|_{m_k}^2 &= k|f(0) - \psi(0)|^2 + \int_0^\infty x|f(x) - \psi(x)|^2 dx \\ &= \int_0^\eta x|f(x) - \psi(x)|^2 dx + \int_\delta^\Delta x|f(x) - \psi(x)|^2 dx \\ &\leq 2 \int_0^{\eta_1} x|f(x)|^2 dx + 2 \int_0^{\eta_1} x|\psi_1(x)|^2 dx + 2 \int_{\eta_1}^\eta x|f(x)|^2 dx \\ &\quad + 2 \int_{\eta_1}^\eta |\psi_2(x) - \varphi_2(x)|^2 dx + \int_\delta^\Delta x|f(x) - \varphi(x)|^2 dx \\ &\leq 9\varepsilon. \end{aligned}$$

This last result gives the required density of  $D(S_k)$  in the Hilbert space  $L^2(\mathbb{R}; m_k)$ .

(ii) To prove that the operator  $S_k$  on  $D(S_k)$  is hermitian in  $L^2(\mathbb{R}; m_k)$  we note, for all  $f, g \in D(S_k)$  and recalling the Defs. (2.1) and (5.2),

$$\begin{aligned} (S_k f, g)_{m_k} &= k(-8k^{-1}f''(0))\bar{g}(0) + \int_0^\infty (S_k f)(x)\bar{g}(x)x dx \\ &= -8f''(0)\bar{g}(0) + \int_0^\infty x^{-1}L_M[f](x)\bar{g}(x)x dx \end{aligned}$$

and, likewise,

$$(f, S_k g)_{m_k} = -8f(0)\bar{g}''(0) + \int_0^\infty f(x)x^{-1}\overline{L_M[g]}(x)x dx.$$

Thus, recalling (iii) of Corollary 3.4,

$$\begin{aligned} (S_k f, g)_{m_k} - (f, S_k g)_{m_k} &= 8(f(0)\bar{g}''(0) - f''(0)\bar{g}(0)) \\ &\quad + \int_0^\infty \{x^{-1}L_M[f](x)\bar{g}(x) - f(x)x^{-1}\overline{L_M[g]}(x)\} x dx \\ &= 8(f(0)\bar{g}''(0) - f''(0)\bar{g}(0)) + 8\{f''(0)\bar{g}(0) - f(0)\bar{g}''(0)\} \\ &= 0 \end{aligned}$$

to give the required hermitian property of  $S_k$  in  $L^2(\mathbb{R}; m_k)$ .

(iii) The symmetry of  $S_k$  in  $L^2(\mathbb{R}; m_k)$  now follows from the results in items (i) and (ii) above; see [1] and [11].

(iv) Let  $f \in D(S_k)$ ; then, using Eq. (2.1), Def. 5.1 and item (ii) of Corollary 3.4, we have

$$\begin{aligned} (S_k f, f)_{m_k} &= k(S_k f)(0)\bar{f}(0) + \int_0^\infty x(S_k f)(x)\bar{f}(x) dx \\ &= k(-8k^{-1}f''(0))\bar{f}(0) + \int_0^\infty x(x^{-1}L_M[f](x))\bar{f}(x) dx \\ &= -8f''(0)\bar{f}(0) + 8f''(0)\bar{f}(0) \\ &\quad + \int_0^\infty \{x|f''(x)|^2 + (9x^{-1} + 8M^{-1}x)|f'(x)|^2\} dx \\ &\geq 0. \end{aligned}$$

These results complete the proof of Theorem 5.2. □

**Theorem 5.4** *Let  $k \in (0, \infty)$  be given; then the symmetric operator  $S_k$  on the domain  $D(S_k)$  is self-adjoint in the Hilbert function space  $L^2(\mathbb{R}; m_k)$ .*

*Proof.* We use the result, see [1, Section 46, Theorem 2], that if  $A$  on the domain  $D(A)$  is a symmetric operator in a Hilbert space  $H$  with the property that the range of  $A$  is the whole space  $H$ , i.e., if

$$\{Af : f \in D(A)\} = H, \quad (5.4)$$

then  $A$  is self-adjoint in  $H$ .

In this proof we write, for convenience,  $L^2$  for the space  $L^2((0, \infty); x)$  and  $L_k^2$  for the space  $L^2(\mathbb{R}; m_k)$ ; see Sections 1 and 2 above.

Define the self-adjoint operator  $T : D(T) \subset L^2 \rightarrow L^2$  by, see [2, Section 10],

$$D(T) := \{f \in D(T_1) : f''(0) = 0\} \quad \text{and} \quad Tf := x^{-1}L_M[f] \quad \text{for all } f \in D(T). \quad (5.5)$$

From [2, Section 13, Theorem 13.2 and Section 14, Corollary 14.2] we obtain the spectrum result, where  $\sigma_{\text{ess}}(T)$  and  $\sigma_{\text{cont}}(T)$  represent, respectively, the essential and the continuous spectrum of the self-adjoint operator  $T$ , see [14, Chapter 7, pp. 202 and 209],

$$\sigma(T) = \sigma_{\text{ess}}(T) = \sigma_{\text{cont}}(T) = [0, \infty), \quad (5.6)$$

so that the point  $-1 \in \mathbb{R}$  is in the resolvent set  $\rho(T)$  of the operator  $T$ .

From [11, Chapter IV, Section 14.10, Corollary 2], see also [2, Section 13, Theorem 13.1], the differential equation

$$L_M[y](x) \equiv (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = -xy(x) \quad \text{for all } x \in (0, \infty) \quad (5.7)$$

has a non-null, real-valued solution  $\varphi(\cdot)$  with the properties

$$\varphi \in L^2, \quad \varphi \in D(T_1) \quad \text{and} \quad x^{-1}L_M[\varphi](x) = -\varphi(x) \quad \text{for all } x \in [0, \infty); \quad (5.8)$$

this solution  $\varphi$  has to satisfy  $\varphi''(0) \neq 0$  since otherwise  $-1$  is an eigenvalue of the operator  $T$  in contradiction to the result (5.6). Without loss of generality we take, for use below,

$$\varphi''(0) = 1. \quad (5.9)$$

We use  $I$  for the identity operator in both  $L^2$  and  $L_k^2$ .

Let  $R_{-1}(T)$  be the resolvent operator, of the self-adjoint operator  $T$  in  $L^2$ , at the point  $-1 \in \rho(T)$ , i.e.,

$$(T + I)R_{-1}(T)f = f \quad \text{for all } f \in L^2, \quad (5.10)$$

and

$$R_{-1}(T) : L^2 \xrightarrow{\text{onto}} D(T) \quad (5.11)$$

is a bounded operator in this space. We note that if  $f \in L^2$  then  $R_{-1}(T)f \in D(T)$  and so

$$(R_{-1}(T)f)''(0) = 0. \quad (5.12)$$

The operator  $S_k : D(S_k) \rightarrow L_k^2$  is defined in Def. 5.1 above; define the operator  $S'_k : D(S'_k) \rightarrow L_k^2$  by

$$D(S'_k) := D(S_k) = D(T_1) \quad \text{and} \quad S'_k := S_k + I. \quad (5.13)$$

We show below that  $S'_k$  is self-adjoint in  $L_k^2$  which implies that  $S_k$  is self-adjoint in  $L_k^2$ .

A computation shows the  $S'_k$  is hermitian in  $L_k^2$  from which it follows that  $S'_k$  is a symmetric operator in the space  $L_k^2$ . Thus to prove that  $S'_k$  is self-adjoint in  $L_k^2$  it is sufficient to show that, see Eq. (5.4),

$$\{S'_k g : g \in D(S'_k)\} = L_k^2. \quad (5.14)$$

Let  $f \in L_k^2$ , and let the parameters  $K \in \mathbb{R}$  and  $\alpha \in \mathbb{C}$ , both to be chosen later, with  $\alpha$  to be dependent upon the function  $f$ . Now define  $g : [0, \infty) \rightarrow \mathbb{C}$  by, see Eqs. (5.7) and (5.8) for the definition of  $\varphi$ ,

$$g(x) := (Kf(0) + \alpha)\varphi(x) + (R_{-1}(T)f)(x) \quad \text{for all } x \in [0, \infty); \quad (5.15)$$

then  $g \in D(T_1) = D(S'_k)$  and

$$S'_k g = (Kf(0) + \alpha)S'_k \varphi + S'_k(R_{-1}(T)f). \quad (5.16)$$

For  $x \in (0, \infty)$  we have, using Eqs. (5.2), (5.8) and (5.10),

$$\begin{aligned} (S'_k g)(x) &= ((S_k + I)g)(x) \\ &= (Kf(0) + \alpha)(x^{-1}L_M[\varphi](x) + \varphi(x)) + (x^{-1}L_M + I)(R_{-1}(T)f)(x) \\ &= 0 + (T + I)(R_{-1}(T)f)(x) \\ &= f(x). \end{aligned} \quad (5.17)$$

For  $x = 0$  we have, on using Eqs. (5.2) and (5.9),

$$\begin{aligned} (S'_k g)(0) &= ((S_k + I)g)(0) \\ &= (Kf(0) + \alpha)((S_k \varphi)(0) + \varphi(0)) + S_k(R_{-1}(T)f)(0) + (R_{-1}(T)f)(0) \\ &= (Kf(0) + \alpha)(-8k^{-1}\varphi''(0) + \varphi(0)) - 8k^{-1}(R_{-1}(T)f)''(0) + (R_{-1}(T)f)(0) \\ &= -8k^{-1}Kf(0) + Kf(0)\varphi(0) + \alpha(-8k^{-1}\varphi''(0) + \varphi(0)) + 0 + (R_{-1}(T)f)(0). \end{aligned} \quad (5.18)$$

We now choose

$$K = -k/8 \in \mathbb{R} \quad (5.19)$$

and

$$\alpha = \frac{Kf(0)\varphi(0) + (R_{-1}(T)f)(0)}{8k^{-1}\varphi''(0) - \varphi(0)} \in \mathbb{C}, \quad (5.20)$$

assuming, for the moment, that

$$\varphi(0) - 8k^{-1}\varphi''(0) \neq 0. \quad (5.21)$$

Then, on using Eqs. (5.18)–(5.20), a computation shows that

$$(S'_k g)(0) = f(0). \quad (5.22)$$

To establish the validity of Eq. (5.21) it suffices to observe that  $\varphi(0)$  and  $\varphi''(0)$  have opposite signs. This result, however, follows from inserting the solution  $\varphi(\cdot)$  from Eq. (5.8), for both  $f$  and  $g$ , into the identity (ii) of Corollary 3.4, re-arranging the terms to give the result that

$$-8\varphi''(0)\varphi(0) > 0.$$

From the two results (5.17) and (5.22) it now follows that given any  $f \in L^2_k \equiv L^2(\mathbb{R}; m_k)$  we can construct an element  $g \in D(S'_k)$ , where  $g$  depends upon  $f$ , such that

$$(S'_k g)(x) = f(x) \quad \text{for all } x \in [0, \infty), \quad \text{i.e., } S'_k g = f.$$

From the method indicated by the requirement (5.4) this last result implies that the symmetric operator  $S'_k$  is self-adjoint in the space  $L^2(\mathbb{R}; m_k)$ ; this result in turn implies, from the Def. (5.13), that the original operator  $S_k$  is self-adjoint in the space  $L^2(\mathbb{R}; m_k)$ .  $\square$

**Theorem 5.5** *The operator  $S_k$  in  $L^2(\mathbb{R}; m_k)$  is the unique self-adjoint operator in this space generated by the fourth-order Bessel-type differential expression  $L_M$ , as defined by Eqs. (3.1) and (3.2).*

*Proof.* Any such self-adjoint operator in  $L^2(\mathbb{R}; m_k)$  has to be defined on the interval  $(0, \infty)$  using the maximal domain  $D(T_1)$  of Eq. (3.9), together with the addition of a complex number to complete the definition at the point  $0 \in [0, \infty)$ . The only possible choice of this number, to give symmetry in the space  $L^2(\mathbb{R}; m_k)$  (see (iii) of Theorem 5.2), is to choose this number as  $-8k^{-1}$ , see Eq. (5.2).  $\square$

## 6 The spectrum of the self-adjoint operator $S_k$

**Theorem 6.1** For any  $k \in (0, \infty)$  let the self-adjoint operator  $S_k$ , in the space  $L^2(\mathbb{R}; m_k)$ , be defined as in Def. 5.1 above; then the spectrum  $\sigma(S_k)$  of  $S_k$  has the following properties:

- (i)  $S_k$  has no eigenvalues,
- (ii) the essential spectrum of  $S_k$  is given by

$$\sigma_{\text{ess}}(S_k) = \sigma_{\text{cont}}(S_k) = [0, \infty) \quad (6.1)$$

**Proof.** From item (iv) of Theorem 5.2 we note that the operator  $S_k$  is bounded below in  $L^2(\mathbb{R}; m_k)$ , i.e.,  $(S_k f, f)_{m_k} \geq 0$  for all  $f \in D(S_k)$ . Thus the spectrum of  $S_k$  satisfies  $\sigma(S_k) \subseteq [0, \infty)$  and  $(-\infty, 0) \subseteq \rho(S_k)$ , the resolvent set of  $S_k$ .

(i) If  $\Lambda$  is an eigenvalue of  $S_k$  then  $\Lambda \in [0, \infty)$  and there exists a real-valued non-null element  $f$  of  $D(S_k)$  such that

$$(S_k f)(0) \equiv -8k^{-1} f''(0) = \Lambda f(0) \quad (6.2)$$

and

$$x^{-1} L_M[f](x) = \Lambda f(x) \quad \text{for all } x \in (0, \infty). \quad (6.3)$$

Taken together these two results imply that  $f$  is an eigenfunction, with eigenvalue  $\Lambda$ , of the self-adjoint restriction  $T$  of the operator  $T_1$  in the Hilbert space  $L^2((0, \infty); x)$ , determined by, see [2, Section 10, Lemmata 10.1 and 10.2],

$$D(T) := \{f \in D(T_1) : 8k^{-1} f''(0) + \Lambda f(0)\} = 0. \quad (6.4)$$

However this is a contradiction on the result in [2, Section 13, Theorem 13.2] that any value  $\Lambda \in [0, \infty)$  cannot be an eigenvalue of any such self-adjoint extension  $T$  of  $T_0$  in  $L^2((0, \infty); x)$ .

(ii) Let  $\nu \in [0, \infty)$ ; then from (i) above either  $\nu \in \sigma_{\text{ess}}(S_k)$  or  $\nu \in \rho(S_k)$ . We suppose that  $\nu \in \rho(S_k)$  and seek a contradiction.

We use the notations

$$L^2 \equiv L^2((0, \infty); x) \quad \text{with norm } \|\cdot\|$$

and

$$L_k^2 \equiv L^2(\mathbb{R}; m_k) \quad \text{with norm } \|\cdot\|_k.$$

Since the operator  $S_k$  is self-adjoint in  $L_k^2$ , see Theorem 5.4, it follows that the resolvent operator  $R_\nu(S_k) \equiv (S_k - \nu I)^{-1}$ , where  $I$  is the identity operator, has the properties, see [1, Chapter 4, Section 49], [11, Chapter IV, Section 12.4] and [14]:

1.  $R_\nu(S_k) : L_k^2 \xrightarrow{\text{onto}} D(S_k)$ ,
2. There exists  $K \in (0, \infty)$  such that

$$\|R_\nu(S_k) f\|_k^2 \leq K \|f\|_k^2 \quad \text{for all } f \in L_k^2. \quad (6.5)$$

Let  $g \in D(S_k)$  then  $g = R_\nu(S_k)(S_k - \nu I)g$  and so, taking  $f = (S_k - \nu I)g$  in Eq. (6.5),

$$\|(S_k - \nu I)g\|_k^2 \geq K^{-1} \|g\|_k^2 \geq K^{-1} (k |g(0)|^2 + \|g\|^2) \geq K^{-1} \|g\|^2. \quad (6.6)$$

Thus for all  $g \in D(S_k) = D(T_1)$  we have

$$\begin{aligned} \|(S_k - \nu I)g\|_k^2 &= k |(S_k - \nu I)g(0)|^2 + \|(T_1 - \nu I)g\|^2 \\ &= k |-8k^{-1} g''(0) - \nu g(0)|^2 + \|(T_1 - \nu I)g\|^2. \end{aligned} \quad (6.7)$$

We now define a self-adjoint restriction  $T$  of  $T_1$  in  $L^2$ , following item (i) above, by

$$D(T) := \{g \in D(T_1) : 8k^{-1} g''(0) + \nu g(0)\} = 0. \quad (6.8)$$

Then for all  $g \in D(T)$  we have, from (6.6) and (6.7),

$$\|(T - \nu I)g\|^2 \geq K^{-1} \|g\|^2. \quad (6.9)$$

From the spectral result in [2, Section 13, Theorem 13.2] it follows that this number  $\nu \in [0, \infty)$  satisfies  $\nu \in \sigma_{\text{cont}}(T)$ ; then from [1, II, Chapter 7, Section 93, Theorem 1] there exists a sequence  $\{g_n \in D(T) : n \in \mathbb{N}\}$  with the properties

$$\begin{cases} \text{(i)} & \|g_n\| = 1 \quad \text{for all } n \in \mathbb{N}, \\ \text{(ii)} & \lim_{n \rightarrow \infty} \|(T - \nu I)g_n\| = 0. \end{cases} \quad (6.10)$$

However, the two results (6.9) and (6.10) are in contradiction and it is impossible for the number  $\nu \in [0, \infty)$  to satisfy  $\nu \in \rho(S_k)$ .

Thus for any number  $\nu \in [0, \infty)$  it follows that  $\nu \in \sigma_{\text{ess}}(S_k) = \sigma_{\text{cont}}(S_k)$ , as required.

These results complete the proof of Theorem 6.1.  $\square$

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