

THE FOURTH-ORDER BESSEL-TYPE DIFFERENTIAL EQUATION

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ABSTRACT. The Bessel-type functions, structured as extensions of the classical Bessel functions, were defined by Everitt and Markett in 1994. These special functions are derived by linear combinations and limit processes from the classical orthogonal polynomials, classical Bessel functions and the Krall Jacobi-type and Laguerre-type orthogonal polynomials. These Bessel-type functions are solutions of higher-order linear differential equations, with a regular singularity at the origin and an irregular singularity at the point of infinity of the complex plane.

There is a Bessel-type differential equation for each even-order integer; the equation of order two is the classical Bessel differential equation. These even-order Bessel-type equations are not formal powers of the classical Bessel equation.

When the independent variable of these equations is restricted to the positive real axis of the plane they can be written in the Lagrange symmetric (formally self-adjoint) form of the Glazman-Naimark type, with real coefficients. Embedded in this form of the equation is a spectral parameter; this combination leads to the generation of self-adjoint operators in a weighted Hilbert function space. In the second-order case one of these associated operators has an eigenfunction expansion that leads to the Hankel integral transform.

This paper is devoted to a study of the spectral theory of the Bessel-type differential equation of order four; considered on the positive real axis this equation has singularities at both end-points. In the associated Hilbert function space these singular end-points are classified, the minimal and maximal operators are defined and all associated self-adjoint operators are determined, including the Friedrichs self-adjoint operator. The spectral properties of these self-adjoint operators are given in explicit form.

From the properties of the domain of the maximal operator, in the associated Hilbert function space, it is possible to obtain a virial theorem for the fourth-order Bessel-type differential equation.

There are two solutions of this fourth-order equation that can be expressed in terms of classical Bessel functions of order zero and order one. However it appears that additional, independent solutions essentially involve new special functions not yet defined. The spectral properties of the self-adjoint operators suggest that there is an eigenfunction expansion similar to the Hankel transform, but details await a further study of the solutions of the differential equation.

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1. INTRODUCTION

The fourth-order Bessel-type differential equation takes the form

$$(1.1) \quad (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \text{ for all } x \in (0, \infty)$$

where $M > 0$ is a positive parameter and $\Lambda \in \mathbb{C}$, the complex field, is a spectral parameter. The differential equation (1.1) is derived in the paper [16, Section 1, (1.10a)], by Everitt and Markett.

This linear, ordinary differential equation on the interval $(0, \infty) \subset \mathbb{R}$, the real field, is written in Lagrange symmetric (formally self-adjoint) form, or equivalently Naimark form, see [25, Chapter V] and [1, Appendix 2].

The structured Bessel-type functions, and their associated linear differential equations of all even-order, were introduced in the paper [16, Section 1] through linear combinations of, and limit processes applied to, the Laguerre and Laguerre-type orthogonal polynomials. This process is best illustrated through the following diagram, see [16, Section 1, Page 328] (for the first two lines of this table see the earlier work of Koornwinder [23] and Markett [24]):

$$(1.2) \quad \left\{ \begin{array}{ll} \begin{array}{l} \text{Jacobi polynomials} \\ k(\alpha, \beta)(1-x)^\alpha(1+x)^\beta \\ \downarrow \\ \text{Laguerre polynomials} \\ k(\alpha)x^\alpha \exp(-x) \\ \downarrow \\ \text{Bessel functions} \\ \kappa(\alpha)x^{2\alpha+1} \end{array} & \longrightarrow \begin{array}{l} \text{Jacobi-type polynomials} \\ k(\alpha, \beta)(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1) \\ \downarrow \\ \text{Laguerre-type polynomials} \\ k(\alpha)x^\alpha \exp(-x) + N\delta(x) \\ \downarrow \\ \text{Bessel-type functions} \\ \kappa(\alpha)x^{2\alpha+1} + M\delta(x) \end{array} \end{array} \right.$$

The symbol entry (here k and κ are positive numbers depending only on the parameters α and β) under each special function indicates a non-negative (generalised) “weight”, on the interval $(-1, 1)$ or $(0, \infty)$, involved in:

- (a) the orthogonality property of the special functions
- (b) the weight coefficient in the associated differential equations.

It is important to note in this diagram that:

- (i) a horizontal arrow \longrightarrow indicates a definition process either by a linear combination of special functions of the same type but of different orders, or by a linear-differential combination of special functions of the same type and order
- (ii) a vertical arrow \downarrow indicates a confluent limit process of one special function to give another special function

- (iii) the use of the symbol $M\delta(\cdot)$ is a notational device to indicate that the monotonic function on the real line \mathbb{R} defining the weight has a jump at an end-point of the interval concerned, of magnitude $M > 0$
- (iv) the combination of any vertical arrow \downarrow with a horizontal arrow \longrightarrow must give a consistent single entry.

Information about the Jacobi-type and Laguerre-type orthogonal polynomials, and their associated differential equations, is given in the Everitt and Littlejohn survey paper [14]; see in particular the references in this paper to the introduction of the fourth-order Laguerre-type differential equation by H.L. and A.M. Krall, and by Littlejohn. The general Laguerre-type differential equation is introduced in the paper [22] by Koekoek and Koekoek; the order of this linear differential equation is determined by $4 + 2\alpha$ with $\alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

It is significant that the general order Bessel-type functions also satisfy a linear differential equation of order $4 + 2\alpha$, being an inheritance from the order of the general Laguerre-type equation.

The purpose of this paper is to initiate the study of properties of the Bessel-type linear differential in the special case when $\alpha = 0$, as given in the bottom right-hand corner of the diagram; this is the fourth-order differential equation (1.1) and involves the weight coefficient $\kappa(0)x$; its solutions should, in some sense, have orthogonality properties with respect to the generalized weight function $\kappa(0)x + M\delta(0)$, where $M > 0$ is the parameter appearing in the differential equation (1.1); see [16, Section 4].

However, in this paper we restrict attention to the spectral properties of the differential equation (1.1) in the classical weighted Hilbert function space (for notation see Section 2 below) of Lebesgue measurable functions with the property

$$L^2((0, \infty); x) := \left\{ f : (0, \infty) \rightarrow \mathbb{C} : \int_0^\infty x |f(x)|^2 dx < \infty \right\}.$$

Previous studies of fourth-order differential equations generating the Legendre-type, Jacobi-type and Laguerre-type orthogonal polynomials, see [13], [15] and [12], have shown that an initial study of the spectral properties of the differential equation in the classical Hilbert function space is essential to the subsequent study of spectral properties in the jump weighted Hilbert space.

Information about the higher even-order Bessel-type differential equations is given in [16]; in particular the explicit Lagrange symmetric forms of the sixth-order and eighth-order differential equations are given in [16, Section 1, (1.10b) and (1.10c)], and in the general case in [16, Section 2, (2.17)]. However the spectral theory of these higher order differential equations can be expected to follow the properties of the special case when $\alpha = 0$ and the order of the equation is four.

The classical Bessel differential equation, with order $\alpha = 0$, written in a form comparable to the fourth-order equation (1.1), is best taken from the left-hand bottom corner of the diagram (1.2); from [16, Section 1, (1.2)] with $\alpha = 0$ we obtain

$$(1.3) \quad -(xy'(x))' = \lambda xy(x) \text{ for all } x \in (0, \infty);$$

here $\lambda \in \mathbb{C}$ is the spectral parameter. The Bessel special function solutions of this equation are $J_0(x\sqrt{\lambda})$ and $Y_0(x\sqrt{\lambda})$, for all $x \in (0, \infty)$; the spectral theory for this equation, in the weighted Hilbert space $L^2((0, \infty); x)$, can be developed using the properties of these solutions, see [29, Chapter IV]. However it is possible to develop the spectral properties of (1.3) without reference to these solutions and these methods have to be adopted in the study of the fourth-order equation (1.1), since there is incomplete information concerning the solutions of this differential equation (1.3).

The comparison between the Bessel-type equation (1.1) and the classical Bessel equation (1.3) may seem surprising; nevertheless this Bessel-type equation is the true structured inheritor of the

classical Bessel equation; in particular this can be seen in the table of results (1.2); also in the comparable spectral properties of the differential operators, generated by the corresponding differential expressions, in the same Hilbert function space $L^2((0, \infty); x)$, as is shown below. The same type of comparison can be made in considering the inheritance of the Jacobi-type and Laguerre-type differential equations from the corresponding classical Jacobi and Laguerre differential equations, see again the table (1.2).

Our knowledge of the special function solutions of the Bessel-type differential equation (1.1) is, at present, limited. However, the results in [16, Section 1, (1.8a)], with $\alpha = 0$, show that the function defined by

$$(1.4) \quad J_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x) \text{ for all } x \in (0, \infty),$$

where

- (i) the parameter $M > 0$
- (ii) the parameter $\lambda \in \mathbb{C}$
- (iii) the spectral parameter Λ and the parameter M , in the equation (1.1), and the parameters M and λ , in the definition (1.4), are connected by the relationship

$$(1.5) \quad \Lambda \equiv \Lambda(\lambda, M) = \lambda^2(\lambda^2 + 8M^{-1}) \text{ for all } \lambda \in \mathbb{C} \text{ and all } M > 0$$

- (iv) J_0 and J_1 are the classical Bessel functions (of the first kind), see [30, Chapter III],

is a solution of the differential equation (1.1), for all $\lambda \in \mathbb{C}$, and hence for all $\Lambda \in \mathbb{C}$ and all $M > 0$. Similar arguments to the methods given in [16] show that the function defined by

$$(1.6) \quad Y_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]Y_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}Y_1(\lambda x) \text{ for all } x \in (0, \infty),$$

is also a solution of the differential equation (1.1), for all $\lambda \in \mathbb{C}$, and hence for all $\Lambda \in \mathbb{C}$ and all $M > 0$; here, again, Y_0 and Y_1 are classical Bessel functions (of the second kind), see [30, Chapter III].

Some properties of the combined classical Bessel functions (1.4) and (1.6) are considered by Ismail and Muldoon [20]; in particular there are results on the existence of real zeros of these special functions.

It had been hoped that with this knowledge of the two independent solutions $J_\lambda^{0,M}$ and $Y_\lambda^{0,M}$, the method of Rofe-Beketov, as given in the paper (written in Russian) [28] (see also the later paper of Dobrokhotov [8]), would yield two additional linearly independent solutions of the equation (1.1); however it is shown below, see Remark 3.2, that the two solutions $J_\lambda^{0,M}$ and $Y_\lambda^{0,M}$ do not allow the use of the Rofe-Beketov method.

The contents of the paper are: some notations are given in Section 2; the fourth-order Bessel-type differential equation and the corresponding differential expression are considered in Sections 3 and 4; the minimal, maximal and self-adjoint operators generated by the fourth-order Bessel-type differential expression, in the space $L^2((0, \infty); x)$, are defined and discussed in Sections 5, 6 and 7; the Sections 8, 9 and 10 are devoted to the special properties that appertain to the singular endpoint 0 of the differential equation; Liouville-type transformations are discussed in Section 11; the spectral properties of the classical Bessel and the fourth-order Bessel-type differential equations are considered and compared in Sections 12 and 13; in Section 14 a virial-type theorem is established and, finally, in Section 15 the properties of the Friedrichs self-adjoint extension are considered.

2. NOTATIONS

The real and complex fields are represented by \mathbb{R} and \mathbb{C} respectively; the positive and non-negative integers by \mathbb{N} and \mathbb{N}_0 respectively; open and compact intervals of \mathbb{R} are denoted by (a, b) and $[a, b]$; half-open intervals by $[a, b)$ or $(a, b]$.

L and AC denote Lebesgue integration and absolute continuity; the use of “loc” restricts a property to compact intervals of an open or half-open interval.

For intervals with the left-hand endpoint $a \geq 0$ the symbol $L^2(a, b; x)$ represents the weighted integrable-square space of Lebesgue measurable functions with the property

$$(2.1) \quad L^2((a, b); x) := \left\{ f : (a, b) \rightarrow \mathbb{C} : \int_a^b x |f(x)|^2 dx < +\infty \right\}.$$

This space is a function space and may be regarded as a Hilbert space if equivalent classes of functions are introduced; the inner-product and norm are then defined by

$$(2.2) \quad (f, g) := \int_a^b x f(x) \bar{g}(x) dx \quad \text{and} \quad \|f\| := (f, f)^{1/2} \quad \text{for all } f, g \in L^2((a, b); x).$$

The special case for this paper is when $a = 0$ and $b = +\infty$, to give $L^2((0, \infty); x)$.

3. THE DIFFERENTIAL EQUATION

Consider the differential equation (1.1)

$$(3.1) \quad (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \quad \text{for all } x \in (0, \infty)$$

with, recalling (1.5),

$$(3.2) \quad \Lambda(\lambda, M) \equiv \Lambda = \lambda^2(\lambda^2 + 8M^{-1}) \quad \text{for all } \lambda \in \mathbb{C} \text{ and } M > 0.$$

We have adopted the symbol Λ as the spectral parameter for the differential equation (3.1) in view of the use of the symbol λ in [16] for purposes involving the theory of special functions.

On the interval $(0, \infty) \subset \mathbb{R}$ and considered in the space $L_{\text{loc}}^1(0, \infty)$, the Lagrange symmetric differential equation (3.1) is *regular* at each point of $(0, \infty)$; the equation is *singular* at 0^+ and at $+\infty$. For these notations see the book by Naimark [25, Chapter V], and the paper by Everitt and Markus [17]. It should be noted that the classical Bessel differential equation (1.3) has the same $L_{\text{loc}}^1(0, \infty)$ classification when considered on the interval $(0, \infty) \subset \mathbb{R}$.

In the form (3.1) Λ is now the normal spectral parameter, but dependent upon the parameter λ ; note that λ is not a spectral parameter. This notation has been adopted as λ is introduced into the analysis of [16] for other than spectral purposes.

For the need to apply the Frobenius series method of solution we also consider the differential equation (3.1) on the complex plane \mathbb{C} :

$$(3.3) \quad w^{(4)}(z) + 2z^{-1}w^{(3)}(z) - (9z^{-2} + 8M^{-1})w''(z) + (9z^{-3} - 8M^{-1}z^{-1})w'(z) - \Lambda w(z) = 0$$

for all $z \in \mathbb{C}$. In this form the equation has a *regular singularity* at the origin 0, and an *irregular singularity* at the point at infinity ∞ of the complex plane \mathbb{C} ; all other points of the plane are *regular* or *ordinary* points for the differential equation; see Copson [7, Sections 10.1 and 10.12]. It should be noted that the classical Bessel differential equation (1.3) has the same classification when considered in the complex plane \mathbb{C} .

A calculation shows that the Frobenius indicial roots, see the text of Ince [19, Chapter XVI, Page 397], for the regular singularity of the differential equation (3.3) at the origin 0, are $\{4, 2, 0, -2\}$. The application of the Frobenius series method, using the computer program [4], yields four linearly

independent series solutions of (3.3), each with infinite radius of convergence in the complex plane \mathbb{C} . If these solutions are labelled to hold for the Bessel-type differential equation (3.1) then we have four solutions $\{y_r(\cdot, \Lambda, M) : r = 4, 2, 0, -2\}$, to accord with the indicial roots, to give the theorem:

Theorem 3.1. *For all $\Lambda \in \mathbb{C}$ and all $M > 0$, the differential equation (3.1) has four linearly independent solutions $\{y_r(\cdot, \Lambda, M) : r = 4, 2, 0, -2\}$, defined on $(0, \infty) \times \mathbb{C}$, with the following series properties as $x \rightarrow 0^+$, where the O -terms depend upon the complex spectral parameter Λ and the parameter M ,*

$$(3.4) \quad \begin{cases} y_4(x, \Lambda, M) &= x^4 + \frac{1}{3}M^{-1}x^6 + O(x^8) \\ y_2(x, \Lambda, M) &= kx^2 + O(x^4 |\ln(x)|) \\ y_0(x, \Lambda, M) &= l + O(x^4 |\ln(x)|) \\ y_{-2}(x, \Lambda, M) &= mx^{-2} + O(|\ln(x)|). \end{cases}$$

Here the fixed numbers $k, l, m \in \mathbb{R}$ and are independent of the parameters Λ and M ; these numbers are produced by the Frobenius computer program [4] and have the explicit values:

$$k = -(27720)^{-1}, \quad l = (174636000)^{-1}, \quad m = -(9779616000)^{-1}.$$

An additional analysis based on this Frobenius solution basis shows that the solution $J_\lambda^{0,M}$ of the equation (3.1), as defined by (1.4), is represented in terms of this basis by

$$(3.5) \quad J_\lambda^{0,M}(x) = \alpha(\Lambda, M)y_4(x, \Lambda, M) + \beta(\Lambda, M)y_2(x, \Lambda, M) + \gamma(\Lambda, M)y_0(x, \Lambda, M),$$

for all $x \in (0, \infty)$, all $\Lambda \in \mathbb{C}$ and all $M > 0$. The explicit form of the coefficients in this representation is given by

$$(3.6) \quad \alpha(\Lambda, M) = \frac{315M^3\lambda^6 + 7381M^2\lambda^4 + 43928M\lambda^2 + 40320}{120960M^2}$$

$$(3.7) \quad \beta(\Lambda, M) = \frac{3465\lambda^2(M\lambda^2 + 8)}{4}$$

$$(3.8) \quad \gamma(\Lambda, M) = 174636000.$$

Note that from the definition (1.4) we obtain, from the explicit formulae for J_0 and J_1 , the result $J_\lambda^{0,M}(0) = 1$; this result is consistent with (3.4), (3.5) and (3.8).

Remark 3.1. (i) The Frobenius series solution method predicts that there could be a $\ln^2(\cdot)$ term in the series solution $y_0(\cdot, \Lambda, M)$; the Frobenius program [4] correctly indicates that this solution is free from such a $\ln^2(\cdot)$ term.

(ii) On examination of the explicit form of the solution $J_\lambda^{0,M}(\cdot)$ it is seen that this solution is an analytic (entire) function on the complex plane \mathbb{C} ; in particular it is free from any Frobenius series contributions involving $\ln(\cdot)$ and $\ln^2(\cdot)$ terms.

(iii) Although the two solutions $y_2(\cdot, \Lambda, M)$ and $y_0(\cdot, \Lambda, M)$ both have logarithmic terms of the form

$$y_2(x, \Lambda, M) \rightarrow \ln(x)(\alpha x^4 + \beta x^6 + \dots)$$

and

$$y_0(x, \Lambda, M) \rightarrow \ln(x)(\gamma x^4 + \delta x^6 + \dots),$$

the two series in parenthesis are linearly dependent. This result is essential since otherwise it would be impossible for the solution $J_\lambda^{0,M}(\cdot)$, see the representation (3.5), to be free of a $\ln(\cdot)$ term.

There is a similar representation for the solution $Y_\lambda^{0,M}$, given by (1.6); however in this case the solution $y_{-2}(\cdot, \Lambda, M)$ has a definite part in the representation.

Remark 3.2. This remark is to show that the Rofe-Beketov method of producing two additional linearly independent solutions of the differential equation (3.1), by quadrature on the two known solutions $J_{\lambda}^{0,M}$ and $Y_{\lambda}^{0,M}$, is not possible; in fact for this method to work it is necessary to show that for all $\lambda \in \mathbb{R}$ (for the introduction of the symplectic form $[f, g](x)$ see the notation in (4.6) below)

$$(3.9) \quad [J_{\lambda}^{0,M}(\cdot, \Lambda, M), Y_{\lambda}^{0,M}(\cdot, \Lambda, M)](x) = 0 \text{ for all } x \in (0, \infty).$$

If $\lambda \in \mathbb{R}$ then, from the relationship (1.5), $\Lambda \in [0, \infty)$ and an application of the Green's formula (4.5) it is possible to show that the the function $[J_{\lambda}^{0,M}(\cdot, \Lambda, M), Y_{\lambda}^{0,M}(\cdot, \Lambda, M)](\cdot) : (0, \infty) \rightarrow \mathbb{R}$ is constant on $(0, \infty)$. However a calculation shows that this constant value is not zero so that the condition (3.9) is not satisfied; in fact, we have

$$(3.10) \quad [J_{\lambda}^{0,M}(\cdot, \Lambda, M), Y_{\lambda}^{0,M}(\cdot, \Lambda, M)](x) = \frac{16}{\pi M} [1 + M(\lambda/2)^2]^{\delta} \neq 0 \text{ for all } x \in (0, \infty) \text{ and } \lambda \in \mathbb{R},$$

where the number $\delta > 0$ (and could be calculated).

This negative result for the application of the Rofe-Beketov method indicates that it may not be possible to obtain explicit information about the two additional solutions of the fourth-order Bessel-type differential equation (3.1), in terms of the classical Bessel functions.

4. THE DIFFERENTIAL EXPRESSION L_M

We define the differential expression L_M with domain $D(L_M)$ as follows:

$$(4.1) \quad D(L_M) := \{f : (0, \infty) \rightarrow \mathbb{C} : f^{(r)} \in AC_{\text{loc}}(0, \infty) \text{ for } r = 0, 1, 2, 3\},$$

and for all $f \in D(L_M)$

$$(4.2) \quad L_M[f](x) := (xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' \quad (x \in (0, \infty));$$

it follows that

$$(4.3) \quad L_M : D(L_M) \rightarrow L_{\text{loc}}^1(0, \infty).$$

For spectral purposes we study the differential expression L_M of (4.1) and (4.2), and the differential equation of (3.1) in the weighted Hilbert function space $L^2((0, +\infty); x)$, *i.e.*

$$(4.4) \quad L^2((0, \infty); x) := \left\{ f : (0, +\infty) \rightarrow \mathbb{C} : \int_0^{\infty} x |f(x)|^2 dx < +\infty \right\}.$$

The Green's formula for L_M on any compact interval $[\alpha, \beta] \subset (0, +\infty)$ takes the form

$$(4.5) \quad \int_{\alpha}^{\beta} \left\{ \bar{g}(x)L_M[f](x) - f(x)\overline{L_M[g]}(x) \right\} dx = [f, g](x)|_{\alpha}^{\beta}$$

where the symplectic form $[\cdot, \cdot](\cdot) : D(L_M) \times D(L_M) \times (0, +\infty) \rightarrow \mathbb{C}$ is given explicitly by

$$(4.6) \quad \begin{aligned} [f, g](x) &= \bar{g}(x)(xf''(x))' - (x\bar{g}''(x))'f(x) \\ &\quad - x(\bar{g}'(x)f''(x) - \bar{g}''(x)f'(x)) \\ &\quad - (9x^{-1} + 8M^{-1}x)(\bar{g}(x)f'(x) - \bar{g}'(x)f(x)). \end{aligned}$$

The Dirichlet formula for L_M on any compact interval $[\alpha, \beta] \subset (0, +\infty)$ takes the form

$$(4.7) \quad \begin{aligned} & \int_{\alpha}^{\beta} \{x f''(x) \bar{g}''(x) + (9x^{-1} + 8M^{-1}x) f'(x) \bar{g}'(x)\} dx \\ &= [f, g]_D(x)|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} L_M[f](x) \bar{g}(x) dx \end{aligned}$$

where the Dirichlet form $[\cdot, \cdot]_D : D(L_M) \times D_0(L_M) \times (0, +\infty) \rightarrow \mathbb{C}$ is given by, for $f \in D(L_M)$ and $g \in D_0(L_M)$, with

$$(4.8) \quad D_0(L_M) := \{g : (0, +\infty) \rightarrow \mathbb{C} : g^{(r)} \in AC_{\text{loc}}(0, +\infty) \text{ for } r = 0, 1\}$$

and

$$(4.9) \quad [f, g]_D(x) := -\bar{g}(x) (x f''(x))' + \bar{g}'(x) x f''(x) + \bar{g}(x) (9x^{-1} + 8M^{-1}x) f'(x).$$

5. BASIC DIFFERENTIAL OPERATORS IN $L^2((0, \infty); x)$

The maximal and the minimal differential operators, denoted respectively T_1 and T_0 , generated by the differential expression L_M in the Hilbert function space $L^2((0, \infty); x)$ are defined as follows, see [17] and [25, Chapter V, Section 17]:

(i) $T_1 : D(T_1) \subset L^2((0, \infty); x) \rightarrow L^2((0, \infty); x)$ by

$$(5.1) \quad D(T_1) := \{f \in D(L_M) : f, x^{-1} L_M(f) \in L^2((0, \infty); x)\}$$

and

$$(5.2) \quad T_1 f := x^{-1} L_M(f) \text{ for all } f \in D(T_1).$$

From the Green's formula (4.5) it follows that the limits

$$(5.3) \quad [f, g](0^+) := \lim_{x \rightarrow 0} [f, g](x) \text{ and } [f, g](\infty) := \lim_{x \rightarrow \infty} [f, g](x)$$

both exist and are finite in \mathbb{C} for all $f, g \in D(T_1)$.

(ii) $T_0 : D(T_0) \subset L^2((0, \infty); x) \rightarrow L^2((0, \infty); x)$ by

$$(5.4) \quad D(T_0) := \{f \in D(T_1) : \lim_{x \rightarrow 0} [f, g](x) = 0 \text{ and } \lim_{x \rightarrow \infty} [f, g](x) = 0 \text{ for all } g \in D(T_1)\}$$

and

$$(5.5) \quad T_0 f := x^{-1} L_M(f) \text{ for all } f \in D(T_0).$$

From standard results we have the operator properties

$$(5.6) \quad T_0 \subseteq T_1, T_0^* = T_1 \text{ and } T_1^* = T_0,$$

and that T_0 is a closed, symmetric operator in $L^2((0, \infty); x)$.

6. END-POINT CLASSIFICATION IN $L^2((0, \infty); x)$

For the differential equation (3.1) the singular end-point 0^+ is limit-3 in $L^2((0, +\infty); x)$; this result can be proved by using the Frobenius (infinite series) method, see [15, Chapter XVI], at the regular singularity 0 in the complex plane, see Theorem 3.1. The indicial roots at 0 are $\{4, 2, 0, -2\}$; this information, allowing for the fact that these four roots all differ by integers thereby introducing logarithmic factors into the Frobenius solutions, yields 3 linearly independent solutions in the space $L^2((0, 1]; x)$ to give the limit-3 classification.

The singular end-point $+\infty$ is Dirichlet and strong limit-2 in $L^2((0, +\infty); x)$; this result can be proved by using the methods of Race [26]. The strong limit-2 result implies that, see (4.9),

$$(6.1) \quad \lim_{x \rightarrow \infty} [f, g]_D(x) = 0 \text{ for all } f, g \in D(T_1);$$

the Dirichlet result implies that for all $f \in D(T_1)$

$$(6.2) \quad \int_1^\infty \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx < \infty.$$

This last result implies that $f^{(r)} \in L^2[1, \infty)$ for $r = 0, 1, 2$ and a straightforward argument then gives

$$(6.3) \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} f'(x) = 0 \text{ for all } f \in D(T_1).$$

Note that (4.6) and (4.7), together with (6.1), imply that, for the symplectic form $[\cdot, \cdot]$,

$$(6.4) \quad \lim_{x \rightarrow \infty} [f, g](x) = 0 \text{ for all } f, g \in D(T_1).$$

Thus the “local” deficiency indices of the symmetric operator T_0 at 0^+ in, say, $L^2((0, 1]; x)$ are $d_0^- = d_0^+ = 3$; the “local” deficiency indices of T_0 at $+\infty$ in, say, $L^2([1, \infty); x)$ are $d_\infty^- = d_\infty^+ = 2$. From known results, see [25, Section 17.5 Equation (47)], it then follows that the deficiency indices $d^\pm(T_0)$ of T_0 in $L^2((0, \infty); x)$ are given by

$$(6.5) \quad d^\pm(T_0) = d_0^\pm + d_\infty^\pm - 4 = 1.$$

This last result implies, see the general results in [25, Chapter IV, Section 14], the following dimensional result for the quotient space

$$(6.6) \quad \dim D(T_1)/D(T_0) = 2.$$

7. SELF-ADJOINT OPERATORS IN $L^2((0, \infty); x)$

The self-adjoint extensions of the closed symmetric operator T_0 are determined by the GKN-theorem on boundary conditions as given in [25, Chapter V] and in [17]. In particular, for the operators T_0 and T_1 any self-adjoint operator $T = T^*$ generated by L_M in $L^2((0, \infty); x)$ is a one-dimensional extension of T_0 or, equivalently, a one-dimensional restriction of T_1 . If we take the restriction of T_1 the domain $D(T)$ is determined by

$$(7.1) \quad D(T) := \{f \in D(T_1) : [f, \varphi](\infty) - [f, \varphi](0) = 0\}$$

where the function $\varphi \in D(T_1)$ is non-null in the quotient space $D(T_1)/D(T_0)$ and satisfies the symmetry condition

$$[\varphi, \varphi](\infty) - [\varphi, \varphi](0) = 0.$$

From (6.3) it follows that $[\varphi, \varphi](\infty) = 0$ so that the symmetry condition reduces to

$$(7.2) \quad [\varphi, \varphi](0) = 0.$$

In Section 9 below the explicit form of all possible boundary condition functions are given.

8. PROPERTIES OF $D(T_1)$ AT 0^+

The following properties near 0^+ of the elements of the maximal domain $D(T_1)$ are required; the proof uses the methods given in the papers [12] and [15]:

Theorem 8.1. *Let $f \in D(T_1)$; then the values of f, f', f'' can be defined at the point 0 so that the following results hold:*

- (i) $f \in AC[0, 1]$
- (ii) $f' \in AC[0, 1]$ and $f'(0) = 0$
- (iii) $f'' \in AC_{loc}(0, 1]$ and $f'' \in C[0, 1]$
- (iv) $f^{(3)} \in AC_{loc}(0, 1]$ and $\lim_{x \rightarrow 0^+}(xf^{(3)}(x)) = 0$.

Remark 8.1. The results of this theorem are essential to obtaining the explicit forms of the boundary conditions at 0^+ ; see Sections 9 and 10 below.

Proof. Let $f \in D(T_1)$; since the proof concerns only the values of f in the neighbourhood of 0^+ we "patch" f , using the Naimark patching lemma [25, Chapter V, Section 17.3, Lemma 2], to zero on the interval $[1, \infty)$ but maintaining $f \in D(T_1)$; this step implies that $f^{(r)}(1) = 0$ for $r = 0, 1, 2, 3, 4$, which properties are used below.

From (4.2) we have

$$(8.1) \quad (xf''(x))' - ((9x^{-1} + 8M^{-1}x)f'(x))' = L_M[f](x) \quad (x \in (0, 1]).$$

Since $x^{-1}L_M[f] \in L^2((0, 1); x)$ we have

$$(8.2) \quad \left(\int_0^1 |L_M[f](x)| dx \right)^2 \leq \int_0^1 x dx \int_0^1 x |x^{-1}L_M[f](x)|^2 dx < \infty$$

and so $L_M[f] \in L^1(0, 1)$. Then on integration of (8.1) we have, for all $x \in (0, 1]$ and defining $\Phi(\cdot; f) : (0, 1] \rightarrow \mathbb{C}$,

$$(8.3) \quad \Phi(x; f) := (xf''(x))' - (9x^{-1} + 8M^{-1}x)f'(x) = - \int_x^1 L_M[f](x) dx;$$

from (8.2) it follows that if we define $\Phi(0; f) := \lim_{x \rightarrow 0^+} \Phi(x; f)$ then

$$(8.4) \quad \Phi(\cdot; f) \in AC[0, 1].$$

Consider now the second-order differential equation

$$(8.5) \quad -(xy'(x))' + (9x^{-1} + 8M^{-1}x)y(x) = 0 \text{ for all } x \in (0, 1].$$

This equation on $(0, 1]$ has a singular point at 0^+ ; considered as a differential equation on the complex plane \mathbb{C} , compare with the results in Section 3, the equation has a regular singularity at the origin $0 \in \mathbb{C}$. An application of the Frobenius series solution method yields indicial roots $\{3, -3\}$; this implies the existence of a solution φ of (8.5) with the properties

$$(8.6) \quad \varphi(x) = x^3 + O(x^4) \text{ and } \varphi'(x) = 3x^2 + O(x^3) \text{ as } x \rightarrow 0^+.$$

For this solution we assume $\varphi(x) > 0$ for all $x \in (0, 1]$; if this is not the case then the proof has to be worked on a smaller interval $(0, c]$ for some $c \in (0, 1)$. A second solution ψ of (8.5) is then defined by

$$\psi(x) := -\varphi(x) \int_x^1 \frac{1}{t\varphi(t)^2} dt \text{ for all } x \in (0, 1]$$

which yields the results

$$(8.7) \quad \psi(x) = x^{-3}/6 + O(x^{-2}) \text{ and } \psi'(x) = -x^{-4}/2 + O(x^{-3}) \text{ as } x \rightarrow 0^+.$$

A calculation then shows that the Wronskian of the pair of solutions $\{\varphi, \psi\}$, which is constant on $(0, 1]$, satisfies

$$(8.8) \quad x(\varphi(x)\psi'(x) - \varphi'(x)\psi(x)) = -1 \text{ for all } x \in (0, 1].$$

With $\Phi(\cdot; f)$ defined in (8.3) now define $\Psi(\cdot; f) : (0, 1] \rightarrow \mathbb{C}$ by

$$(8.9) \quad \Psi(x; f) := \psi(x) \int_0^x \varphi(t)\Phi(t; f) dt + \varphi(x) \int_x^1 \psi(t)\Phi(t; f) dt \text{ for all } x \in (0, 1].$$

From the definition (8.9) and the result (8.4) it follows that $\Psi(\cdot; f) \in AC_{\text{loc}}(0, 1]$ and that $x\Psi'(\cdot; f) \in AC_{\text{loc}}(0, 1]$. A calculation now shows that $\Psi(\cdot; f)$ satisfies the non-homogeneous differential equation

$$(8.10) \quad -(x\Psi'(x; f))' + (9x^{-1} + 8M^{-1}x)\Psi(x; f) = -\Phi(x; f) \text{ for all } x \in (0, 1].$$

Taking the modulus on both sides of (8.9), using (8.4) and the solution properties (8.6) and (8.7), and integrating it follows that $\lim_{x \rightarrow 0^+} \Psi(x; f) = 0$; finally define $\Psi(0; f) := 0$. Thus we have the properties of $\Psi(\cdot; f)$

$$(8.11) \quad \Psi(\cdot; f) \in AC_{\text{loc}}(0, 1], \quad \Psi(\cdot; f) \in C[0, 1] \text{ and } \Psi(0; f) = 0.$$

If we now compare the functional relationship (8.3) for the function f with the differential equation (8.10) then we obtain the existence of $\alpha, \beta \in \mathbb{C}$ to give the representation

$$(8.12) \quad f'(x) = \alpha\varphi(x) + \beta\psi(x) + \Psi(x; f) \text{ for all } x \in (0, 1].$$

Integrating this last result gives, recall that $f(1) = 0$,

$$(8.13) \quad -f(x) = \alpha \int_x^1 \varphi(t) dt + \beta \int_x^1 \psi(t) dt + \int_x^1 \Psi(t; f) dt \text{ for all } x \in (0, 1];$$

thus

$$(8.14) \quad -xf(x) = \alpha x \int_x^1 \varphi(t) dt + \beta x \int_x^1 \psi(t) dt + x \int_x^1 \Psi(t; f) dt \text{ for all } x \in (0, 1].$$

From $f \in L^2((0, 1); x)$ it follows that

$$\left(\int_0^1 |xf(x)| dx \right)^2 \leq \int_0^1 x dx \int_0^1 x |f(x)|^2 dx < \infty$$

so that $xf \in L^1(0, 1)$. On examination of the terms in (8.14) all but the middle term on the right-hand side are seen to belong to $L^1(0, 1)$; thus we have to conclude that $\beta = 0$.

Hence we have the two representations

$$(8.15) \quad -f(x) = \alpha \int_x^1 \varphi(t) dt + \int_x^1 \Psi(t; f) dt \text{ for all } x \in (0, 1]$$

$$(8.16) \quad f'(x) = \alpha\varphi(x) + \Psi(x; f) \text{ for all } x \in (0, 1].$$

From the definition (4.1) it follows that $f \in AC_{\text{loc}}(0, 1]$ and if we define, using (8.11) and (8.15), $f(0) := \lim_{x \rightarrow 0^+} f(x)$ then

$$(8.17) \quad f \in AC[0, 1].$$

If now we form the difference quotient $(f(x) - f(0))/x$ for $x > 0$ then we find, on using the definition of $f(0)$, (8.15) and then (8.11),

$$(8.18) \quad f'(0) := \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \Psi(0; f) = 0.$$

Further from (8.16) it follows that $\lim_{x \rightarrow 0^+} f'(x) = 0$ and so $f' \in C[0, 1]$.

Now differentiating (8.16) and using the definition (8.9) we find

$$(8.19) \quad f''(x) = \alpha \varphi'(x) + \psi'(x) \int_0^x \varphi(t) \Phi(t; f) dt + \varphi'(x) \int_x^1 \psi(t) \Phi(t; f) dt \text{ for all } x \in (0, 1];$$

thus $f'' \in AC_{\text{loc}}(0, 1]$. Taking the modulus of both sides and estimating the integrals gives the existence of a positive number $K(f)$ such that $|f''(x)| \leq K(f)$ for all $x \in (0, 1]$, and thus $f'' \in L^2(0, 1)$. This last result implies that, using also (8.16),

$$(8.20) \quad f' \in AC[0, 1] \text{ and } \Psi(\cdot; f) \in AC[0, 1]$$

Now consider the integral terms on the right-hand side of (8.19), written in the form, for $x > 0$,

$$x^4 \psi'(x) \times \frac{\int_0^x \varphi(t) \Phi(t; f) dt}{x^4} + x^{-2} \varphi'(x) \times \frac{\int_x^1 \psi(t) \Phi(t; f) dt}{x^{-2}};$$

then use of the properties (8.6) and (8.7), and applying the l'Hôpital rule to the quotient factors, shows that this terms tends to the limit $\Phi(0; f)/8$ as $x \rightarrow 0^+$, *i.e.*

$$(8.21) \quad \lim_{x \rightarrow 0^+} f''(x) = \Phi(0; f)/8.$$

To show that $f''(0)$ exists we start with the quotient, again with $x > 0$, using (8.18) and (8.16),

$$(8.22) \quad \frac{f'(x) - f'(0)}{x} = x^{-1} f'(x) = \alpha x^{-1} \varphi(x) + x^{-1} \Psi(x; f) \text{ for all } x \in (0, 1].$$

Using the same arguments as for the limit results that gave (8.21) we find that

$$(8.23) \quad f''(0) = \lim_{x \rightarrow 0^+} \frac{f'(x) - f'(0)}{x} = \Phi(0; f)/8.$$

The results (8.19), (8.21) and (8.23) show that $f''(0)$ exists and that

$$(8.24) \quad f'' \in AC_{\text{loc}}(0, 1] \text{ and } f'' \in C[0, 1].$$

Differentiating (8.19) gives, on using the Wronskian property (8.8), for all $x \in (0, 1]$

$$(8.25) \quad x f^{(3)}(x) = \alpha x \varphi''(x) + x \psi''(x) \int_0^x \varphi(t) \Phi(t; f) dt + x \varphi''(x) \int_x^1 \psi(t) \Phi(t; f) dt + \Phi(x; f).$$

On taking the limit of the right-hand side of (8.25), using again the methods given above, it follows the $\lim_{x \rightarrow 0^+} x f^{(3)}(x)$ exists and is finite; if this limit is not zero then a straightforward argument, on integrating, gives a contradiction to the result that $f'' \in C[0, 1]$; thus we have

$$(8.26) \quad f^{(3)} \in AC_{\text{loc}}(0, 1] \text{ and } \lim_{x \rightarrow 0^+} x f^{(3)}(x) = 0.$$

The required results (i), (ii), (iii) and (iv) of Theorem 8.1 now follow from the statements in (8.17), (8.18) and (8.20), (8.24) and (8.26) respectively. \square

Remark 8.2. Some of the ideas in the proof of the crucial results in Theorem 8.1 follow from the adoption of methods in the earlier papers [12] and [15].

The results given in Theorem 8.1 seem to be best possible; as in the paper [12] it is remarkable that the singular differential expression L_M defines a maximal domain $D(T_1)$ in $L^2((0, \infty); x)$ with such smooth properties at the singular end-point 0.

9. BOUNDARY PROPERTIES AT 0^+

In this section we consider the functions $1, x, x^2$ on the interval $[0, 1]$ but “patched”, see the Naimark patching lemma [25, Chapter V, Section 17.3, Lemma 2], to zero on $[1, \infty)$ in such a manner that the patched functions belong to the domain $D(L_M)$, see (4.1); we continue to use the symbols $1, x, x^2$ for the patched functions.

A calculation shows that the results given in the next lemma are satisfied:

Lemma 9.1. *The patched functions $1, x, x^2$ have the following limit properties in respect of the symplectic form $[\cdot, \cdot]$ and the maximal domain $D(T_1)$:*

- (i) $1, x^2 \in D(T_1)$ but $x \notin D(T_1)$
- (ii) $[1, 1](0^+) = [x, x](0^+) = [x^2, x^2](0^+) = 0$
- (iii) $[x, x^2](0^+) = 0$ and $[1, x^2](0^+) = 16$
- (iv) $[1, x](0^+)$ does not exist.

The results of Theorem 8.1 and Lemma 9.1 now provide a basis for the two-dimensional quotient space $D(T_1)/D(T_0)$;

$$(9.1) \quad D(T_1)/D(T_0) = \text{span}\{1, x^2\} = \{a + bx^2 : a, b \in \mathbb{C}\}.$$

The linear independence of the basis $\{1, x^2\}$ within the the quotient space follows from the property $[1, x^2](0^+) = 16 \neq 0$.

A calculation now gives, recall Theorem 8.1,

Lemma 9.2. *Let $f \in D(T_1)$; then the following identities hold:*

- (i) $[f, 1](0^+) = -8f''(0)$
- (ii) $[f, x^2](0^+) = 16f(0)$.

Similarly we have

Lemma 9.3. *Let $f, g \in D(T_1)$; then*

- (i) $[f, g](0^+) = 8[f(0)\bar{g}''(0) - f''(0)\bar{g}(0)]$
- (ii) $[f, g]_D(0^+) = 8f''(0)\bar{g}(0)$.

We have the corollaries:

Corollary 9.1. *The domain of the minimal operator T_0 is determined explicitly by*

$$(9.2) \quad D(T_0) = \{f \in D(T_1) : f(0) = 0 \text{ and } f''(0) = 0\}.$$

Proof. This follows from the definition (5.4), the result (6.4) and (i) of Lemma 9.3. □

Corollary 9.2. *For all $f \in D(T_1)$,*

$$(9.3) \quad \int_0^1 \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx < \infty.$$

Proof. This result follows from (ii) of Lemma 9.3 together with an application of the Dirichlet formula (4.7) on the interval $(0, 1]$. □

Corollary 9.3. *For all $f \in D(T_1)$ we have the complete Dirichlet integral for the differential expression L_M*

$$(9.4) \quad \int_0^\infty \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx < \infty.$$

Proof. This result follows from the two results (6.2) and (9.3). □

Corollary 9.4. *For all $f, g \in D(T_1)$ the Dirichlet formula takes the form*

$$(9.5) \quad (T_1 f, g) = 8f''(0)\bar{g}(0) + \int_0^\infty \{x f''(x)\bar{g}''(x) + (9x^{-1} + 8M^{-1}x) f'(x)\bar{g}'(x)\} dx.$$

Proof. In the Dirichlet formula (4.7) let $\alpha \rightarrow 0^+$ and $\beta \rightarrow \infty$, use the limit result (6.1) and then (ii) of Lemma 9.3. \square

10. BOUNDARY CONDITION FUNCTIONS AT 0^+

We can now determine all forms of the boundary condition function φ satisfying the symmetry condition (7.2) to determine the domain of all self-adjoint extensions T of the minimal operator T_0 .

Lemma 10.1. *All self-adjoint extensions T of T_0 generated by the differential expression L_M in $L^2((0, \infty; x))$ are determined by, using the patched functions $1, x^2$,*

$$(10.1) \quad D(T) := \{f \in D(T_1) : [f, \varphi](0^+) = 0 \text{ where } \begin{array}{l} (i) \quad \varphi(x) = \alpha + \beta x^2 \\ (ii) \quad \alpha, \beta \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 \neq 0 \end{array}\}.$$

and

$$(10.2) \quad (Tf)(x) := x^{-1}L_M(f)(x) \text{ for all } x \in (0, \infty) \text{ and all } f \in D(T).$$

There is an equivalent form of this last result, using the results of Lemma 9.2:

Lemma 10.2. *All self-adjoint extensions T of T_0 generated by the differential expression L_M in $L^2((0, \infty; x))$ are determined by*

$$(10.3) \quad D(T) := \{f \in D(T_1) : \begin{array}{l} (i) \quad -\alpha f''(0) + 2\beta f(0) = 0 \\ (ii) \quad \alpha, \beta \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 \neq 0 \end{array}\}.$$

and

$$(10.4) \quad (Tf)(x) := x^{-1}L_M(f)(x) \text{ for all } x \in (0, \infty) \text{ and all } f \in D(T).$$

Remark 10.1. We note the two special cases:

- (i) When $\alpha = 0$ the boundary condition is $f(0) = 0$; this boundary condition plays a special role, as is given below in Section 15 on the Friedrichs extension of T_0 .
- (ii) When $\beta = 0$ the boundary condition is $f''(0) = 0$.

11. TRANSFORMATION THEORY

As remarked in the paper [16, Section 1] the classical Bessel differential equation can be written in a number of different, but equivalent in transformation theory, forms. Two forms of this equation are given in [16, Section 1, (1.2) and (1.2a)], viz

$$(11.1) \quad -(x^{2\alpha+1}y'(x))' = \lambda x^{2\alpha+1}y(x) \text{ for all } x \in (0, \infty)$$

and

$$(11.2) \quad -y''(x) - (2\alpha + 1)x^{-1}y'(x) = \lambda y(x) \text{ for all } x \in (0, \infty),$$

where there is a change of parameter from λ^2 to λ . Solutions of these differential equations, in terms of the classical Bessel functions, are $x^{-\alpha}J_\alpha(x\sqrt{\lambda})$ and $x^{-\alpha}Y_\alpha(x\sqrt{\lambda})$, for all $\alpha \in \mathbb{R}$.

A canonical form of these differential equations is obtained by the application of the Liouville transformation and results in the differential equation, see [29, Chapter IV],

$$(11.3) \quad -y''(x) + \left(\alpha^2 - \frac{1}{4}\right)x^{-2}y(x) = \lambda y(x) \text{ for all } x \in (0, \infty),$$

with solutions $x^{1/2}J_\alpha(x\sqrt{\lambda})$ and $x^{1/2}Y_\alpha(x\sqrt{\lambda})$.

The canonical form (11.3) has the advantage that the second derivative y'' is free from a variable coefficient and that there is no term in the first derivative y' .

The Lagrange symmetric (formally self-adjoint) differential equations (11.1) and (11.3) are unitarily equivalent when considered in their respective Hilbert function spaces, *i.e.*

$L^2((0, \infty); x^{2\alpha+1})$ and $L^2(0, \infty)$, and so have the same spectral properties.

For this paper the special case of (11.1) and (11.3) when $\alpha = 0$ is of importance; in this case the Liouville form of the differential equation, see (1.3),

$$-(xy'(x))' = \lambda xy(x) \text{ for all } x \in (0, \infty)$$

becomes

$$(11.4) \quad -y''(x) - \frac{1}{4}x^{-2}y(x) = \lambda y(x) \text{ for all } x \in (0, \infty),$$

with solutions $x^{1/2}J_0(x\sqrt{\lambda})$ and $x^{1/2}Y_0(x\sqrt{\lambda})$.

For the fourth-order Bessel-type differential equation, see Section 1 above, in particular (1.1),

$$(11.5) \quad (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \text{ for all } x \in (0, \infty),$$

there is a transformation which may be regarded as similar to the Liouville transformation for the second-order classical Sturm-Liouville differential equations. The required transformation of (11.5) gives new independent and dependent variables X and $Y(\cdot)$, respectively,

$$(11.6) \quad X(x) := x \text{ and } Y(X) := x^{1/2}y(x) \text{ for all } x \in (0, \infty).$$

For the use of this notation see the general theory in [11, Section 2].

The application of the transformation (11.6) to the differential equation yields the (canonical) differential equation, for all $X \in (0, \infty)$ and to be studied in the Hilbert function space $L^2(0, \infty)$,

$$(11.7) \quad Y^{(4)}(X) - \left(\left(\frac{15}{2}X^{-2} + 8M^{-1}\right)Y'(X)\right)' + \left(-\frac{135}{16}X^{-4} - 2M^{-1}X^{-2}\right)Y(X) = \Lambda Y(X).$$

As for the classical Bessel differential equation, the transformation (11.6) gives an isomorphic isometric mapping, see [11], between the spaces $L^2((0, \infty); x)$ and $L^2(0, \infty)$, under which mapping there is a unitary equivalence between associated self-adjoint operators generated by the respective differential expressions, in these two spaces.

It is to be noted that in this transformed equation (11.7) there is now no term in the third derivative $Y^{(3)}$; however, in general, in the study of this type of fourth-order differential equations there are no transformations which remove all the transformed derivatives $Y^{(3)}$, Y'' and Y' and yield the required unitary equivalence of self-adjoint operators.

The differential equation (11.7), when considered in the complex plane, has a regular singularity at the origin $0 \in \mathbb{C}$. The indicial roots for this singularity are $\{9/2, 5/2, 1/2, -3/2\}$.

12. SPECTRAL PROPERTIES OF THE CLASSICAL BESSEL EQUATION WITH $\alpha = 0$.

The spectral theory, in the space $L^2((0, \infty); x)$, of the classical Bessel differential equation of order zero

$$(12.1) \quad -(xy'(x))' = \lambda xy(x) \text{ for all } x \in (0, \infty)$$

is well known; for the analysis in the Liouville form of this equation, see (11.4), in the space $L^2(0, \infty)$,

$$(12.2) \quad -y''(x) - \frac{1}{4}x^{-2}y(x) = \lambda y(x) \text{ for all } x \in (0, \infty)$$

see the early account in [29, Chapter IV], which is based on the properties of the solutions $x^{1/2}J_0(x\sqrt{\lambda})$ and $x^{1/2}Y_0(x\sqrt{\lambda})$ defined on the half-line $x \in (0, \infty)$.

If we work in the space $L^2((0, \infty); x)$ for (12.1) then the maximal and minimal operators T_1 and T_0 are defined as follows:

$$(i) \quad T_1 : D(T_1) \subset L^2((0, \infty); x) \rightarrow L^2((0, \infty); x) \text{ by}$$

$$(12.3) \quad D(T_1) := \{f : (0, \infty) \rightarrow \mathbb{C} : f, x^{-1}(xf')' \in L^2((0, \infty); x)\}$$

and

$$T_1 f := x^{-1}(xf')' \text{ for all } f \in D(T_1).$$

$$(ii) \quad T_0 : D(T_0) \subset L^2((0, \infty); x) \rightarrow L^2((0, \infty); x), \text{ with appropriate definition of the form } [\cdot, \cdot]_0$$

by

$$[f, g]_0(x) := x(f(x)\overline{g}'(x) - f'(x)\overline{g}(x)) \text{ for all } x \in (0, \infty) \text{ and } f, g \in D(T_1),$$

with

$$D(T_0) := \{f \in D(T_1) : \lim_{x \rightarrow 0} [f, g]_0(x) = 0 \text{ and } \lim_{x \rightarrow \infty} [f, g]_0(x) = 0 \text{ for all } g \in D(T_1)\}$$

and

$$T_0 f := x^{-1}(xf')' \text{ for all } f \in D(T_0).$$

The end-point classification of the equation (12.1) is:

- (i) the singular end-point $+\infty$ is strong limit-point and Dirichlet in $L^2((0, \infty); x)$
- (ii) the singular end-point 0^+ is limit-circle non-oscillatory in $L^2((0, \infty); x)$

There are similar properties for the maximal domain $D(T_1)$ near 0^+ of (12.1), given in (12.3), as for the maximal domain of the fourth-order Bessel-type equation, detailed in Theorem 8.1; in detail we have:

Theorem 12.1. *The following results hold:*

- (i) if 1 represents the unit function on $(0, \infty)$ patched to zero at infinity then $1 \in D(T_1)$
- (ii) if $f \in D(T_1)$ then $f = O(|\ln(x)|)$ as $x \rightarrow 0^+$ and $f \in L^2(0, 1)$
- (iii) if $f, g \in D(T_1)$ and $\lim_{x \rightarrow 0^+} [f, 1]_0(x) = \lim_{x \rightarrow 0^+} [g, 1]_0(x) = 0$ then
 - (a) $\lim_{x \rightarrow 0^+} x f'(x) = 0$, $x^{1/2} f' \in L^2(0, 1)$, $f' \in L^2(0, 1)$ and $f \in AC[0, 1]$
 - (b) $\lim_{x \rightarrow 0^+} [f, g](x) = 0$.

Proof. We omit the proof of these results but they follow the same lines as the proof of Theorem 8.1. \square

We can now state

Theorem 12.2. *The following results hold:*

- (i) The operator T_0 is bounded below in $L^2((0, \infty); x)$ by the null operator O , and has a continuum of self-adjoint extensions T that satisfy $T_0 \subset T = T^* \subset T_1$; the essential spectrum of any such extension T satisfies (for the definition of the essential spectrum of a general closed operator in Hilbert space see Dunford and Schwartz [9, Chapter XIII, Section 6.1, Definition 1])

$$\sigma_{\text{ess}}(T) = [0, \infty);$$

T has no eigenvalues in $[0, \infty)$ but may have at most one simple eigenvalue in the interval $(-\infty, 0)$.

- (ii) Every point $\mu \in (-\infty, 0)$ is the eigenvalue of some unique self-adjoint extension T of T_0 .

(iii) If the operator $T_F : D(T_F) \subset L^2((0, \infty); x) \rightarrow L^2((0, \infty); x)$ is defined by

$$(12.4) \quad D(T_F) := \{f \in D(T_1) : [f, 1](0^+) = 0\}$$

and

$$(12.5) \quad (T_F f)(x) := -x^{-1}(xf'(x))' \text{ for all } x \in (0, \infty) \text{ and all } f \in D(T_F)$$

then:

- (a) T_F is self-adjoint
- (b) the spectrum $\sigma(T_F)$ of T_F satisfies $\sigma(T_F) = \sigma_{\text{ess}}(T_F) = [0, \infty)$
- (c) T_F has no eigenvalues
- (d) T_F is the self-adjoint Friedrichs extension of the minimal operator T_0 .

Proof. We omit the proof of these known results; the methods follow the analysis given in the proof of Theorem 13.2 below. \square

Remark 12.1. We emphasize here that these properties, the proof details for which follow the proof in the next Section 13, can be made to be independent of the properties of the solutions J_0 and Y_0 ; whilst this independence is not necessary for this classical Bessel differential equation it is essential to adopt this procedure for the Bessel-type differential equation, where we do not have complete information concerning solutions and their properties.

Remark 12.2. For a classical proof of the spectral properties of the self-adjoint operator T_F , given (12.4) and (12.5), and the associated eigenfunction expansion giving the Hankel transform theory, see the account of Bessel examples in Titchmarsh [29, Chapter IV].

Remark 12.3. For the definition and properties of the Friedrichs extension of any closed symmetric operator that is bounded below in Hilbert space, see Section 15 below.

13. SPECTRAL PROPERTIES OF THE FOURTH-ORDER BESSEL-TYPE EQUATION

Theorem 13.1. *The minimal operator T_0 , defined in (5.4) and (5.5), is bounded below in the space $L^2((0, \infty); x)$ by the null operator O , i.e.*

$$(13.1) \quad (T_0 f, f) \geq 0 \text{ for all } f \in D(T_0).$$

Proof. Since T_0 is a restriction of the maximal operator T_1 the result of Corollary 9.4 can be applied to give, using also Corollary 9.1,

$$\begin{aligned} (T_0 f, f) &= 8f''(0)\bar{f}(0) + \int_0^\infty \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx \\ &= \int_0^\infty \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx \geq 0 \text{ for all } f \in D(T_0). \end{aligned}$$

\square

Remark 13.1. There is a concise account of the spectral properties of symmetric operators in Hilbert space, with finite and equal deficiency indices, in [25, Chapter IV, Section 14.9].

Theorem 13.2. (1) *Let T be a self-adjoint extension of T_0 ; then:*

- (i) *The essential spectrum $\sigma_{\text{ess}}(T)$ is given by*

$$(13.2) \quad \sigma_{\text{ess}}(T) = [0, \infty).$$

- (ii) *There are no embedded eigenvalues of T in the essential spectrum.*

(iii) T has at most one eigenvalue; if this eigenvalue is present then it is simple and lies in the interval $(-\infty, 0)$.

(2) Every point $\mu \in (-\infty, 0)$ is the eigenvalue of some unique self-adjoint extension T of T_0 .

Proof. Proof of item 1.

Since the closed symmetric operator T_0 has equal and finite deficiency indices $d^\pm(T_0) = 1$, see (6.5), all self-adjoint extensions of T_0 have the same essential spectrum, see [25, Chapter IV, Section 14.9, Theorem 9]. Thus to prove that the given self-adjoint extension T has essential spectrum given by (13.2) it is sufficient to prove that any one self-adjoint extension, say T' , of T_0 has the property that

$$(13.3) \quad \sigma_{ess}(T') = [0, \infty).$$

We use the ‘‘splitting method’’ discussed in [25, Chapter VII, Section 24.1, Theorem 1] to construct such an operator T' . The interval $(0, \infty)$ is split into the two abutting intervals $(0, 1]$ and $[1, \infty)$, and then we construct self-adjoint operators T_L and T_R in the spaces $L^2((0, 1]; x)$ and $L^2([1, \infty); x)$, respectively, that are self-adjoint extensions of the minimal closed symmetric operators $T_{0,L}$ and $T_{0,R}$ generated by the differential expression L_M , respectively, in these two Hilbert function spaces.

For both these intervals the end-point 1 is a regular point for the differential expression L_M , and gives local deficiency indices $(4, 4)$ on both sides of the splitting point 1. From the earlier results on local deficiency indices for L_M at the singular end-points 0 and ∞ it follows that, in the associated L^2 spaces:

- (i) the deficiency indices for L_M on $(0, 1]$ are $(3, 3)$
- (ii) the deficiency indices for L_M on $[1, \infty)$ are $(2, 2)$.

To determine self-adjoint operators in the split spaces, using separated boundary conditions, requires:

- (i) on $(0, 1]$ one symmetric condition at 0^+ and two symmetric conditions at 1^-
- (ii) on $[1, \infty)$ two symmetric conditions at 1^+ only.

For the symmetric boundary condition at 0^+ choose a boundary condition function φ as determined in (10.1).

For the symmetric boundary conditions at 1 take two real-valued functions $\{\varphi_1, \varphi_2\}$, with four continuous derivatives in an open interval neighbourhood of 1 and patched down to zero in the neighbourhoods of 0^+ and $+\infty$; then $\varphi_r \in D(T_1)$ for $r = 1, 2$. Now choose the values of φ_1 and φ_2 at the point 1 as follows:

- (i) $\varphi_1(1) = 1$ and $\varphi_1^{(r)}(1) = 0$ for $r = 1, 2, 3$
- (ii) $\varphi_2(1) = \varphi_2'(1) = 0$ and $\varphi_2''(1) = 1$, $\varphi_2^{(3)}(1) = -1$.

With the symplectic form $[\cdot, \cdot]$ for L_M given by (4.6) it may be shown that

$$(13.4) \quad [\varphi_r, \varphi_s](1^\pm) = 0 \text{ for all } r, s = 1, 2$$

so that the pair $\{\varphi_1, \varphi_2\}$ form an independent, symmetric base for boundary conditions at both 1^\pm ; see [25, Section 18.3].

We now define self-adjoint operators T_L and T_R in $L^2((0, 1]; x)$ and $L^2([1, \infty); x)$, respectively, by

$$(13.5) \quad \begin{cases} D(T_L) := \{f \in D(L_M) & : f, x^{-1}L_M(f) \in L^2((0, 1]; x) \\ & [f, \varphi](0^+) = 0 \\ & [f, \varphi_r](1^-) = 0 \text{ for } r = 1, 2 \} \end{cases}$$

$$(13.6) \quad \begin{cases} D(T_R) := \{f \in D(L_M) & : f, x^{-1}L_M(f) \in L^2([1, \infty); x) \\ & [f, \varphi_r](1^+) = 0 \text{ for } r = 1, 2\}, \end{cases}$$

and

$$(13.7) \quad T_L f := x^{-1}L_M(f) \text{ for all } f \in D(T_L)$$

$$(13.8) \quad T_R f := x^{-1}L_M(f) \text{ for all } f \in D(T_R).$$

To prove (i) of item 1 we prove that the following two results hold:

$$(13.9) \quad \sigma_{ess}(T_L) = \emptyset.$$

and

$$(13.10) \quad \sigma_{ess}(T_R) = [0, \infty).$$

To prove both (13.9) and (13.10) we make use of the transformation of the differential equation (3.1) to canonical form, as given in Section 11. This canonical equation has singularities at the end-points 0^+ and ∞ ; as for the fourth-order Bessel-type equation the splitting method, as described above, can be applied to this canonical equation, and to match the application given above we again choose to split the interval at the interior point 1 of the interval $(0, \infty)$. This leads to consideration of self-adjoint differential operators generated by the canonical differential expression in the two spaces $L^2(0, 1)$ and $L^2(1, \infty)$.

In view of the unitary (isometric isomorphic) mapping (11.6) between the associated self-adjoint operators generated by the differential equations (3.1) and (11.7) in, respectively, the spaces $L^2((0, \infty); x)$ and $L^2(0, \infty)$, the spectral properties of these operators, and the split operators, are invariant under this mapping (11.6).

Thus to prove (13.9) it is sufficient to prove that any self-adjoint operator S generated by (11.7) in $L^2(0, 1)$ has the property that $\sigma_{ess}(S) = \emptyset$; from the unitary mapping this result implies that the self-adjoint operator T_L has this same property.

The result $\sigma_{ess}(S) = \emptyset$ is obtained by showing that the spectrum $\sigma(S)$ of S is discrete in $L^2(0, 1)$; this result forces the essential spectrum $\sigma_{ess}(S)$ to be empty and so gives (13.9). Incidentally the proof also shows that the operator S is bounded below in $L^2(0, 1)$.

Likewise to prove (13.10) it is sufficient to prove that any self-adjoint operator S generated by (11.7) in $L^2(1, \infty)$ has the property that $\sigma_{ess}(S) = [0, \infty)$.

To prove these results it is convenient to define the Lagrange symmetric differential expression K_M by

$$(13.11) \quad K_M[F](X) := F^{(4)}(X) - \left(\left(\frac{15}{2}X^{-2} + 8M^{-1} \right) F'(X) \right)' + \left(-\frac{135}{16}X^{-4} - 2M^{-1}X^{-2} \right) F(X)$$

for all $X \in (0, \infty)$ and all $F \in D(K_M)$, where the domain of K_M is defined by

$$(13.12) \quad D(K_M) := \{F : (0, \infty) \rightarrow \mathbb{C} : F^{(r)} \in AC_{loc}(0, \infty) \text{ for } r = 0, 1, 2, 3\}.$$

The basic minimal and maximal differential operators S_0 and S_1 generated by K_M , in the space $L^2(0, \infty)$, are defined as in Sections 4 and 5 above, with corresponding results for the split spaces $L^2(0, 1)$ and $L^2(1, \infty)$. Self-adjoint operators S generated by K_M , in either $L^2(0, 1)$ or $L^2(1, \infty)$, then satisfy $S_0 \subset S \subset S_1$; any S is determined by applying separated boundary conditions at the end-points of $(0, 1]$ or $[1, \infty)$, as is the case.

To prove that the self-adjoint operator S , generated by K_M in $L^2(0, 1)$, has a discrete spectrum that is bounded below in $L^2(0, 1)$ we follow the method given in [2, Section 11]; this method has been used extensively by Hinton and Lewis [18].

Taking the definitions, notations, lemmas, remarks and theorems given in [18, Section 2] and [2, Section 11] we follow the trail of these items but now applied to the fourth-order differential expression K_M .

Given the compact interval $[a, b] \subset (0, \infty)$ define

$$(13.13) \quad A_1[a, b] := \{f : [a, b] \rightarrow \mathbb{R} : f \in AC[a, b], f' \in L^2(a, b) \text{ and } f(a) = f(b) = 0\}$$

Let $f \in A_1[a, b]$ with $f \neq 0$; then for all $\alpha \in \mathbb{R}$ but with $\alpha \neq -1$ the following inequality is valid

$$(13.14) \quad \int_a^b x^\alpha f(x)^2 dx < \frac{4}{(1+\alpha)^2} \int_a^b x^{\alpha+2} f'(x)^2 dx.$$

The proof of this integral inequality is given in [18, Section 2, (2.3)].

We require the following special cases of the inequality (13.14), all for $f \in A_1[a, b]$:

(i) $\alpha = -4$

$$(13.15) \quad \int_a^b x^{-4} f(x)^2 dx < \frac{4}{9} \int_a^b x^{-2} f'(x)^2 dx$$

(ii) $\alpha = -2$

$$(13.16) \quad \int_a^b x^{-2} f(x)^2 dx < 4 \int_a^b f'(x)^2 dx$$

(iii) $\alpha = 0$

$$(13.17) \quad \int_a^b f(x)^2 dx < 4 \int_a^b x^2 f'(x)^2 dx.$$

Now define, again for any compact interval $[a, b] \subset (0, \infty)$,

$$(13.18) \quad A_2[a, b] := \{F : [a, b] \rightarrow \mathbb{R} : \begin{array}{l} \text{(i) } F, F' \in AC[a, b] \text{ and } F'' \in L^2(a, b) \\ \text{(ii) } F(a) = F(b) = F'(a) = F'(b) = 0 \end{array}\}.$$

The critical result to prove for the results in [18] to be applied to the self-adjoint S in $L^2(0, 1)$ concerns the functional $I(\cdot) : A_2[a, b] \rightarrow \mathbb{R}$ where, for any $[a, b] \subset (0, \infty)$ and recalling the coefficients of the differential expression K_M as given in (13.11),

$$(13.19) \quad I(F) := \int_a^b \{F''(X)^2 + [\frac{15}{2}X^{-2} + 8M^{-1}] F'(X)^2 + [-\frac{135}{16}X^{-4} - 2M^{-1}X^{-2} - \Lambda] F(X)^2\} dX.$$

Theorem 13.3. *Given the self-adjoint operator S in $L^2(0, 1)$, as determined above, then S has a discrete spectrum which is bounded below, if for any $\Lambda \in \mathbb{R}$ there exists $\delta \equiv \delta(\Lambda) \in (0, 1)$ such that for all compact intervals $[a, b] \subset (0, \delta)$ the functional $I(\cdot)$, defined in (13.19), satisfies*

$$(13.20) \quad I(F) > 0 \text{ for all } F \in A_2[a, b] \text{ with } F \neq 0.$$

For the proof of this theorem see [18, Theorems 0.1 and 0.2] and [2, Theorem 11.5].

To apply this Theorem 13.3 to $I(\cdot)$ of (13.19) we note that since $F \in A_2[a, b]$ implies that $F \in A_1[a, b]$, the inequalities (13.15), (13.16) and (13.17) can be applied to F ; if these inequalities are then substituted into (13.19) a calculation shows that, for all $F \in A_2[a, b]$,

$$(13.21) \quad I(F) \geq \int_a^b \{F''(X)^2 + [\frac{15}{4}X^{-2} - 4\Lambda X^2] F'(X)^2\} dX.$$

If $\Lambda \leq 0$ then it follows from (13.21) that the conditions of Theorem 13.3 are satisfied for any $\delta \in (0, 1)$.

If $\Lambda > 0$ then we observe that

$$(13.22) \quad \frac{15}{4}X^{-2} - 4\Lambda X^2 \geq \frac{15}{4}X^{-2} - 4|\Lambda|X^2 \text{ for all } X \in (0, 1];$$

if now we choose $\delta \in (0, 1)$ so that $\delta^4 < \frac{15}{16}|\Lambda|^{-1}$ then it is clear that for any $[a, b] \subset (0, \delta)$ the condition (13.20) is satisfied.

Thus for the self-adjoint operator S in $L^2(0, 1)$ we have that the spectrum $\sigma(S)$ is discrete and bounded below; thus $\sigma_{ess}(S) = \emptyset$.

Finally then from the Naimark text it follows that all self-adjoint extensions of the minimal operator from K_M have empty essential spectrum, *i.e.* $\sigma_{ess}(T_L) = \emptyset$ and (13.9) is established. (For an alternative method of proving this last result see the paper of Read [27, Section 3].)

To prove that a self-adjoint operator S , generated by K_M in $L^2(1, \infty)$, has the property $\sigma_{ess}(S) = [0, \infty)$ we appeal to the theorem in [3, Theorem 1.1]; the conditions of this theorem are satisfied by the differential expression K_M , see (13.11) on the interval $[1, \infty)$. The limit form of K_M , as $X \rightarrow \infty$, is

$$(13.23) \quad K_{M,\infty}[F](X) = F^{(4)}(X) - 8M^{-1}F''(x) \text{ for all } X \in [1, \infty).$$

From the proof of [3, Theorem 1.1] the essential spectrum $\sigma_{ess}(S)$ is identical with the essential spectrum of any self-adjoint operator generated by the differential expression $K_{M,\infty}$ in $L^2(1, \infty)$; moreover this essential spectrum is determined by the range of the polynomial

$$(i\Lambda)^4 - 8M^{-1}(i\Lambda)^2 = \Lambda^4 + 8M^{-1}\Lambda^2$$

for all $\Lambda \in \mathbb{R}$; thus $\sigma_{ess}(S) = [0, \infty)$ and so, again quoting [25, Chapter IV, Section 14.9, Theorem 9], it follows that

$$\sigma_{ess}(T_R) = [0, \infty),$$

as required for the establishment of (13.10).

Finally, for the proof of (i) of item 1 we appeal to the general splitting method as given in [25, Chapter VII, Section 24.1, Theorem 1]. Let the operator T' be defined by the direct sum

$$T' := T_L \oplus T_R;$$

now define the self-adjoint operator T'' in $L^2((0, \infty); x)$ as the unitary map of T' from the direct sum space

$$L^2((0, 1); x) \oplus L^2((1, \infty); x)$$

onto the space $L^2((0, \infty); x)$. Then the essential spectrum $\sigma_{ess}(T'')$ of this defined operator T'' , which is an extension of the minimal operator T_0 in $L^2((0, \infty); x)$, is the essential spectra of the operator T' ; thus from (13.9) and (13.10) we find

$$\sigma_{ess}(T'') = \sigma_{ess}(T') = \sigma_{ess}(T_L) \cup \sigma_{ess}(T_R) = \emptyset \cup [0, \infty) = [0, \infty).$$

This last result implies, see [25, Chapter IV, Section 14.9, Theorem 9], that all self-adjoint operators T , that are extensions of T_0 in $L^2((0, \infty); x)$, satisfy the required result

$$\sigma_{ess}(T) = [0, \infty).$$

These statements complete the proof of (i) of item 1 of Theorem 13.2. above.

Proof of (ii) of item 1.

Suppose that $f : (0, \infty) \rightarrow \mathbb{R}$ with $f \in L^2((0, \infty); x)$, and for some $\Lambda > 0$ satisfies the differential equation

$$(13.24) \quad (xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' = \Lambda xf(x) \text{ for all } x \in (0, \infty);$$

these conditions imply that f is an eigenfunction, with eigenvalue Λ , for some self-adjoint operator T ; here $D(T)$ is determined by the boundary conditions satisfied by f at 0, see Section 10 above. We prove that $f(x) = 0$ for all $x \in (0, \infty)$.

Define $Y(x) := x^{1/2}f(x)$; then Y satisfies the transformed differential equation (11.7) which, in system form is written on $(0, \infty)$ as

$$(13.25) \quad \begin{bmatrix} Y \\ Y' \\ Y'' \\ Y''' \end{bmatrix}' = \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 8M^{-1} & 0 & 1 \\ \Lambda & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} Y \\ Y' \\ Y'' \\ Y''' \end{bmatrix},$$

or equivalently, in vector form,

$$(13.26) \quad \mathbf{Y}'(x) = \{A + R(x)\} \mathbf{Y}(x) \text{ for all } x \in (0, \infty).$$

Here the column vector \mathbf{Y} is determined by

$$\mathbf{Y}(x) = [Y(x) \quad Y'(x) \quad Y''(x) \quad Y'''(x)]^T \text{ for all } x \in (0, \infty),$$

and the scalar coefficients p and q by

$$(13.27) \quad p(x) = \frac{15}{2}x^{-2} \text{ and } q(x) = \frac{135}{16}x^{-4} + 2M^{-1}x^{-2} \text{ for all } x \in (0, \infty).$$

To determine the asymptotic form of the eigenfunction f at $+\infty$ we consider the system (13.27) on the interval $[1, \infty)$, where we note

$$(13.28) \quad \int_1^\infty |p(x)| dx < \infty \text{ and } \int_1^\infty |q(x)| dx < \infty.$$

The matrix A has diagonal Jordan form; the eigenvalues of A are the roots of

$$r^4 - 8M^{-1}r^2 - \Lambda = 0;$$

these roots are given explicitly by

$$r = \pm \sqrt{\frac{4}{M} \pm \sqrt{\Lambda + \frac{16}{M^2}}}.$$

Since $\Lambda > 0$ there are two real roots $\pm\alpha$ and two imaginary roots $\pm i\beta$ where, with positive square-roots,

$$(13.29) \quad \alpha = \left(\left(\Lambda + \frac{16}{M^2} \right)^{1/2} + \frac{4}{M} \right)^{1/2} \text{ and } \beta = \left(\left(\Lambda + \frac{16}{M^2} \right)^{1/2} - \frac{4}{M} \right)^{1/2}.$$

We now apply standard (Levinson) asymptotic analysis to the system (13.26) using (13.28) and the methods given in [5, Chapter 3, Section 8, Theorem 8.1]; this application gives four linearly independent vector solutions $\{\mathbf{Y}_s : s = 1, 2, 3, 4\}$ with the asymptotic properties

$$(13.30) \quad \mathbf{Y}_s(x) = \exp(r_s x) [v_s + o(1)] \text{ for } s = 1, 2, 3, 4 \text{ as } x \rightarrow +\infty,$$

where $r_1 = \alpha, r_2 = -\alpha, r_3 = i\beta$ and $r_4 = -i\beta$ and the $\{v_s\}$ are the linearly independent eigenvectors of the matrix A , corresponding to the eigenvalues $\{r_s\}$, with $\alpha, \beta > 0$.

Of the four solutions $\{\mathbf{Y}_s\}$ only \mathbf{Y}_2 tends to zero at $+\infty$; the components of this solution satisfy the order properties

$$Y_2^{(s-1)}(x) = O(\exp(-\alpha x)) \text{ for } s = 1, 2, 3, 4 \text{ as } x \rightarrow +\infty;$$

this solution Y_2 of the equation (11.7) is the only solution, for the given Λ , with the property $Y_2 \in L^2(0, \infty)$. Inverting the transformation (11.6) gives a solution y of the fourth-order Bessel-type equation (11.5) where $y(x) = x^{-1/2}Y_2(x)$ with the properties $y^{(s-1)}(x) = O(x^{-1/2} \exp(-\alpha x))$ for $s = 1, 2, 3, 4$ as $x \rightarrow +\infty$; up to linear independence this solution is unique with these properties.

Returning now to the original eigenfunction f satisfying (13.24) we see that, for some real number $k \neq 0$, f has the representation $f(x) = kx^{-1/2}Y_2(x)$ for all $x \in (0, \infty)$ and so satisfies the order results

$$(13.31) \quad f^{(s-1)}(x) = O\left(x^{-1/2} \exp(-\alpha x)\right) \text{ for } s = 1, 2, 3, 4 \text{ as } x \rightarrow +\infty.$$

Now multiply (13.24) by $xf'(x)$ and integrate over $(0, \infty)$ to give

$$(13.32) \quad \int_0^\infty \left[(xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' \right] xf'(x) dx = \Lambda \int_0^\infty x^2 f'(x)f(x) dx$$

where both integrals are absolutely integrable from the asymptotic properties (13.31).

From Theorem 8.1, since $f \in D(T_1)$, the following properties hold:

$$(13.33) \quad \begin{cases} \lim_{x \rightarrow 0^+} f(x) \text{ and } \lim_{x \rightarrow 0^+} f''(x) \text{ both exist and are finite} \\ \lim_{x \rightarrow 0^+} f'(x) \text{ and } \lim_{x \rightarrow 0^+} xf'''(x) \text{ both exist and are zero.} \end{cases}$$

We now integrate both sides of (13.32) by parts, repeatedly where required, and use the boundary conditions on f given by (13.31) and (13.33) to obtain

$$\begin{aligned} \int_0^\infty x^2 f'(x)f(x) dx &= \int_0^\infty \frac{1}{2}x^2 (f(x)^2)' dx = - \int_0^\infty xf(x)^2 dx \\ \int_0^\infty (xf''(x))'' xf'(x) dx &= \int_0^\infty x (f''(x))^2 dx \\ - \int_0^\infty ((9x^{-1} + 8M^{-1}x)f'(x))' xf'(x) dx &= \int_0^\infty (9x^{-1} + 8M^{-1}x)f'(x)(xf''(x) + f'(x)) dx \\ &= \int_0^\infty \left\{ (9 + 8M^{-1}x^2)\frac{1}{2}(f'(x)^2)' + \right. \\ &\quad \left. + (9x^{-1} + 8M^{-1}x)f'(x)^2 \right\} dx \\ &= \int_0^\infty (-16M^{-1}\frac{1}{2}x + 9x^{-1} + 8M^{-1}x)f'(x)^2 dx \\ &= \int_0^\infty 9x^{-1}f'(x)^2 dx. \end{aligned}$$

Substituting these last three results in (13.32) we obtain

$$(13.34) \quad \int_0^\infty [xf''(x)^2 + 9x^{-1}f'(x)^2 + \Lambda xf(x)^2] dx = 0;$$

since all three terms under this integral sign are non-negative it follows that $f(x) = 0$ for all $x \in (0, \infty)$; thus if $\Lambda > 0$ then there are no eigenvalues for any self-adjoint extensions T of the minimal operator T_0 .

Suppose now that $f : (0, \infty) \rightarrow \mathbb{R}$, $f \in L^2((0, \infty); x)$ and satisfies the differential equation

$$(13.35) \quad (xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' = 0 \text{ for all } x \in (0, \infty),$$

i.e. we now take $\Lambda = 0$. In this case the standard (Levinson) method to determine the asymptotic form of solutions of (13.35) in $L^2((1, \infty); x)$ are no longer effective; the reason being that in the vector form

$$\mathbf{Y}'(x) = \{A + R(x)\} \mathbf{Y}(x) \text{ for all } x \in (0, \infty)$$

of the transformed differential equation (11.7), the matrix A does not have a diagonal Jordan form. Instead we work directly with solutions of (13.35), noting that this equation has one solution $f_1(x) = 1$ for all $x \in (0, \infty)$. To determine properties of three other independent solutions $\{f_r : r = 2, 3, 4\}$ carry out the differentiations in (13.35), divide by x and set $g(x) = f'(x)$ for $x \in (0, \infty)$. Then g satisfies the third-order equation

$$(13.36) \quad g^{(3)}(x) + 2x^{-1}g''(x) - (9x^{-2} + 8M^{-1})g'(x) + (9x^{-3} - 8(Mx)^{-1})g(x) = 0 \text{ for all } x \in (0, \infty).$$

We can apply the standard asymptotic analysis to this last differential equation, on the interval $[1, \infty)$; following again the method in [5, Chapter 3, Section 8, Theorem 8.1] we find that the characteristic polynomial for the constant part of (13.36) is

$$r^3 - 8M^{-1}r = 0$$

which has three distinct roots $0, s, -s$ where $s = +\sqrt{8M^{-1}}$. The Levinson asymptotic method now applies and (13.36) has three independent solutions $\{g_r : r = 2, 3, 4\}$ with asymptotic form, as $x \rightarrow \infty$,

$$g_2(x) = [1 + o(1)], \quad g_3(x) = \exp(-sx)[1 + o(1)], \quad g_4(x) = \exp(sx)[1 + o(1)].$$

These results now give, using l'Hôpital's rule, the three independent solutions $\{f_r : r = 2, 3, 4\}$ of (13.35) with the asymptotic form, as $x \rightarrow \infty$,

$$f_2(x) = x[1 + o(1)], \quad f_3(x) = s^{-1} \exp(-sx)[1 + o(1)], \quad f_4(x) = s^{-1} \exp(sx)[1 + o(1)].$$

Thus for $\Lambda = 0$, as with the case when $\Lambda > 0$, there is only one non-null solution f_3 on $(0, \infty)$ of (13.35) in the space $L^2((1, \infty); x)$; this solution, and its derivatives, is exponentially small at $+\infty$.

Following the analysis that gave the result (13.34) we find that

$$\int_0^\infty [x f_3''(x)^2 + 9x^{-1} f_3'(x)^2] \, dx = 0;$$

since both terms under this integral sign are non-negative it follows that $f_3(x) = 0$ for all $x \in (0, \infty)$; thus if $\Lambda = 0$ then there are no eigenvalues for any self-adjoint extensions T of the minimal operator T_0 .

Taken together these results show that there are no eigenvalues in the essential spectrum $\sigma_{ess}(T) = [0, \infty)$ for any self-adjoint extension T of the minimal operator T_0 .

These results complete the proof of the statement made in (ii) of item 1 of Theorem 13.2.

Proof of (iii) of item 1.

Since the closed symmetric operator T_0 , with finite deficiency indices $(1, 1)$, is bounded below by the null operator O in $L^2((0, \infty); x)$, we can quote from [25, Chapter IV, Section 14.11, Theorem 16] that the negative part of the spectrum of the given self-adjoint operator T can consist only of a finite number of negative eigenvalues; the sum of the multiplicities of these eigenvalues is at most equal to 1. For the operator T this result implies that there can be at most one eigenvalue and that its multiplicity is 1, *i.e.* if there is an eigenvalue then it is simple.

This completes the proof of statement (iii) of item 1 of the Theorem 13.2 above.

Proof of item 2.

We begin with

Lemma 13.1. *Given $\Lambda < 0$ the differential equation*

$$(13.37) \quad (xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \text{ for all } x \in (0, \infty)$$

has exactly one (up to linear independence), real-valued, non-null solution in the space $L^2((0, \infty); x)$.

For the proof of this lemma consider the minimal operator T_0 in $L^2((0, \infty); x)$ generated by the differential expression L_M . This closed, symmetric operator is bounded below in the space $L^2((0, \infty); x)$ by the null operator O ; see Theorem 13.1 above. From this property it follows that all spectral points $\Lambda < 0$ are points of regular type of T_0 ; thus all points $\Lambda < 0$ are in the domain of regularity of T_0 (see the definitions required here in [25, Chapter IV, Section 9]).

It is known that for any $\Lambda \in \mathbb{C} \setminus \mathbb{R}$ the deficiency index of T_0 , say $d(T_0; \Lambda)$, satisfies

$$d(T_0; \Lambda) = d^\pm(T_0) = 1;$$

see the notation and results in Section 6 and [25, Chapter IV, Section 14.10, Theorem 12]. Thus for $\Lambda < 0$ in the domain of regularity of T_0 and using [25, Chapter IV, Section 14.10, Corollary 2], it then follows that $d(T_0; \Lambda) = 1$. This result implies that the dimension of the eigenspace of the maximal operator T_1 satisfies

$$\dim\{f \in D(T_1) : T_1 f = \Lambda f\} = \dim\{f \in D(T_1) : L_M[f] = \Lambda x f\} = 1.$$

Thus the differential equation (13.37) has exactly one non-null solution in $L^2((0, \infty); x)$; since $\Lambda \in \mathbb{R}$ this solution can be taken, without loss of generality, to be real-valued on $(0, \infty)$.

This completes the proof of Lemma 13.1.

With $\Lambda < 0$ let f denote the solution of (13.37) determined by Lemma 13.1. This solution $f \in D(T_1)$ and so from the results of Theorem 8.1 the two real numbers $f(0)$ and $f''(0)$ are well-defined.

Now substitute $g = f$ in (9.5) of Corollary 9.4 to obtain, recalling $T_1 f = \Lambda f$,

$$(13.38) \quad \Lambda \int_0^\infty x f(x)^2 dx = 8f(0)f''(0) + \int_0^\infty \{x f''(x)^2 + (9x^{-1} + 8M^{-1}x) f'(x)^2\} dx;$$

this result shows, since $\Lambda < 0$, that $f(0) \neq 0$ and $f''(0) \neq 0$.

If we now determine two real numbers α, β such that

$$\alpha^2 + \beta^2 > 0 \text{ and } 2\beta f(0) - \alpha f''(0) = 0$$

then the results of Lemma 10.2 show that if the domain of the self-adjoint extension T of T_0 is determined by

$$D(T) = \{f \in D(T_1) : 2\beta f(0) - \alpha f''(0) = 0\},$$

then the solution f is an eigenfunction with eigenvalue Λ , of the operator T .

This result completes the proof of item 2, and hence the proof of Theorem 13.2. \square

Remark 13.2. There is an illuminating comparison to be made with the spectral properties of the classical Bessel differential operators, as given in items (i) and (ii) of Theorem 12.2, with the spectral properties of the fourth-order Bessel-type differential operators as given in Theorem 13.2. This comparison indicates the structured form of the definition made of the Bessel-type functions given in [16, Sections 1 and 2].

14. THE VIRIAL THEOREM

For a general discussion on results for differential operators under the general title of virial theorems see the book [10].

For the maximal operator T_1 , as defined in (5.1) and (5.2), the virial theorem takes the form:

Theorem 14.1. *Let $f \in D(T_1)$ and assume:*

(i) $f : (0, \infty) \rightarrow \mathbb{R}$

(ii) f satisfies the fourth-order Bessel-type differential equation, for some $\Lambda \in \mathbb{R}$, see (3.1),

$$(14.1) \quad (xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' = \Lambda xf(x) \text{ for all } x \in (0, \infty);$$

then

$$(14.2) \quad \int_0^\infty \{xf''(x)^2 + [9x^{-1} + 4M^{-1}x]f'(x)^2\} dx = -4f(0)f''(0).$$

Proof. It follows from items 1(iii) and 2 of Theorem 13.2 that the given conditions on the solution f imply that f is an eigenfunction for some self-adjoint extension T of T_0 , with eigenvalue Λ , and that $\Lambda < 0$.

The proof of this theorem falls into two cases.

1. Suppose $\Lambda < 0$ but $\Lambda \neq -16M^{-2}$.

We consider the transformed form (11.7) of the differential equation (14.1) on the interval $[1, \infty)$ and apply the asymptotic analysis used in the proof of item 1(ii) of Theorem 13.2. In this case the eigenvalues of the constant matrix A are again given by the roots of

$$r^4 - 8M^{-1}r^2 - \Lambda = 0$$

but now with $\Lambda < 0$; these four eigenvalues $\{\alpha_1 : r = 1, 2, 3, 4\}$ are distinct and two eigenvalues, say α_1 and α_2 , satisfy

$$\alpha_1 \neq \alpha_2 \text{ and } \operatorname{Re}(\alpha_r) < 0 \text{ for } r = 1, 2.$$

Following the analysis in item 1(ii) of Theorem 13.2 we obtain two linearly independent solutions $\mathbf{Y}_1(\cdot)$ and $\mathbf{Y}_2(\cdot)$ of the system (13.26) with the asymptotic form, as $x \rightarrow +\infty$,

$$\mathbf{Y}_r(x) = \exp(\alpha_r x)[v_r + o(1)] \text{ for } r = 1, 2.$$

If we define $\rho_r := -\operatorname{Re}(\alpha_r)$, for $r = 1, 2$, then we obtain two linearly independent solutions Y_r , with $r = 1, 2$, of the transformed equation (11.7) with the asymptotic form

$$Y_r^{(s-1)}(x) = O(\exp(-\rho_r x)) \text{ for } r = 1, 2 \text{ and } s = 1, 2, 3, 4 \text{ as } x \rightarrow +\infty,$$

and two independent solutions f_r , with $r = 1, 2$, of the equation (14.1) with the asymptotic form

$$(14.3) \quad f_r^{(s-1)}(x) = O(x^{-1/2} \exp(-\rho_r x)) \text{ for } r = 1, 2 \text{ and } s = 1, 2, 3, 4 \text{ as } x \rightarrow +\infty.$$

It is now clear that these two solutions satisfy $f_r \in L^2((1, \infty); x)$.

Some additional analysis shows that the two additional solutions of (14.1), which then give a basis of solutions, are asymptotically large at $+\infty$ and do not lie within the space $L^2((1, \infty); x)$.

Taken together these results show that the given solution f of (14.1) in Theorem 14.1 must be linearly dependent upon the pair $\{f_1, f_2\}$; thus we have for this given solution f the asymptotic property

$$(14.4) \quad f^{(s-1)}(x) = O(x^{-1/2} \exp(-\rho x)) \text{ for } s = 1, 2, 3, 4 \text{ as } x \rightarrow +\infty$$

where $\rho := \min\{\rho_1, \rho_2\}$ satisfies $\rho > 0$.

2. Suppose $\Lambda = -16M^{-2}$.

This case is more complicated due to the fact that now the constant matrix A no longer has a diagonal Jordan form (compare with the proof of Theorem 13.2 in the case when $\Lambda = 0$). However, it is still possible to prove that the differential equation, in this case,

$$(xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' + 16M^{-2}xf(x) = 0 \text{ for all } x \in (0, \infty),$$

has two linearly independent solutions $\{f_r : r = 1, 2\}$ with an asymptotic form at $+\infty$ similar to (14.4). The existence proof of these two solutions depends upon the theory of exponential dichotomies as given in Coppel [6, Page 10, and Proposition 1 on Page 34], and Ju and Wiggins [21, Proposition 2.1]; we omit the details in this paper.

All the results are now available for us to apply the analysis in Section 13 to show that the solution f given in Theorem 14.1 above satisfies the identity (13.34), to give

$$(14.5) \quad \int_0^\infty [xf''(x)^2 + 9x^{-1}f'(x)^2 + \Lambda xf(x)^2] dx = 0.$$

Likewise the conditions on the solution f in Theorem 14.1 shows that the identity (13.38) also holds to give

$$(14.6) \quad \Lambda \int_0^\infty xf(x)^2 dx = 8f(0)f''(0) + \int_0^\infty \{xf''(x)^2 + (9x^{-1} + 8M^{-1}x)f'(x)^2\} dx.$$

If we now eliminate the two terms involving the eigenvalue Λ between the two identities (14.5) and (14.6) then we obtain the required virial result

$$(14.7) \quad \int_0^\infty \{xf''(x)^2 + [9x^{-1} + 4M^{-1}x]f'(x)^2\} dx = -4f(0)f''(0).$$

□

For the self-adjoint differential operators T as defined in Sections 7 and 10, in terms of the two real parameters α and β , the virial theorem takes the form:

Corollary 14.1. *Let a self-adjoint operator T be defined as in Sections 7 and 10; let $f \in D(T)$ and otherwise let f satisfy the conditions of Theorem 14.1; let the parameter $\alpha \neq 0$; then*

$$(14.8) \quad \int_0^\infty \{xf''(x)^2 + [9x^{-1} + 4M^{-1}x]f'(x)^2\} dx = -8\frac{\beta}{\alpha}f(0)^2.$$

Proof. This result follows from (10.3) of Lemma 10.2, and (14.7) above. □

Corollary 14.2. *Let T be a self-adjoint extension of the minimal operator T_0 ; then T has no eigenvalues in the following cases:*

- (i) *When $\alpha = 0$, equivalently the boundary condition is $f(0) = 0$.*
- (ii) *When $\beta = 0$, equivalently the boundary condition is $f''(0) = 0$, or $\beta \neq 0$ and $\text{sign}(\beta) = \text{sign}(\alpha)$.*

Proof. These results follow from Theorem 14.1 and Corollary 14.1 above. □

15. THE FRIEDRICHS EXTENSION

Since the closed symmetric operator T_0 in $L^2((0, \infty); x)$ has deficiency indices $(1, 1)$, T_0 is itself not self-adjoint but does have self-adjoint extensions; these extensions are best defined as restrictions of the maximal operator $T_1 = T_0^*$ as given in Sections 7 and 10 above.

The operator T_0 is bounded below in $L^2((0, \infty); x)$, see (13.1), and the general theory of such operators implies the existence of a distinguished self-adjoint extension T_F , called the Friedrichs extension of T_0 .

Let S_0 be a closed symmetric operator in a Hilbert space H with norm $\|\cdot\|$ and inner-product (\cdot, \cdot) . Suppose S_0 is bounded below in H with lower bound zero, *i.e.*

$$(15.1) \quad \inf\{(S_0 f, f) : f \in D(S_0) \text{ with } \|f\| = 1\} = 0.$$

Then S_0 has equal deficiency indices and there exists at least one self-adjoint extension S of S_0 ; all such extensions satisfy

$$(15.2) \quad S_0 \subseteq S \subseteq S_0^*.$$

Definition 15.1. *Let S_0 in H satisfy the above properties. Define the linear manifold D_F by $f \in D_F$ if $f \in D(S_0^*)$ and there exists a sequence $\{f_n \in D(S_0) : n \in \mathbb{N}\}$ such that*

- (i) $\lim_{n \rightarrow \infty} f_n = f$ in H , *i.e.* $\lim_{n \rightarrow \infty} \|f - f_n\| = 0$
- (ii) $\lim_{m, n \rightarrow \infty} (S_0(f_m - f_n), f_m - f_n) = 0$.

From this definition there follows:

Theorem 15.1. *Let S_0 in H satisfy the above properties. Following Definition 15.1 above of the domain D_F , define the Friedrichs extension operator S_F in H by*

$$(15.3) \quad \begin{cases} (i) & D(S_F) := D_F \subseteq D(S_0^*) \\ (ii) & S_F f := S_0^* f. \end{cases}$$

Then;

- (a) S_F is a self-adjoint extension of S_0 , *i.e.* $D_F = D(S_F) \supseteq D(S_0)$ and $S_0 \subseteq S_F = S_F^* \subseteq S_0^*$
- (b) $\inf\{(S_F f, f) : f \in D(S_F) \text{ with } \|f\| = 1\} = 0$.

Proof. For the details of the Definition 15.1 and proof of Theorem 15.1 see [9, Chapter XII, Page 1240, Section 5]; see also the discussion in [31, Chapter 5, Section 5.5]. \square

For the minimal differential operator T_0 generated by the differential expression L_M defined in Section 5 above we have

Theorem 15.2. *The closed symmetric operator T_0 in $L^2((0, \infty); x)$ satisfies the properties (15.1) and (15.2) above. The Friedrichs extension T_F of T_0 has the explicit representation*

$$(15.4) \quad \begin{cases} (\alpha) & D(T_F) = \{f \in D(T_1) = D(T_0^*) : f(0) = 0\} \\ (\beta) & T_F f = x^{-1} L_M[f] \text{ for all } f \in D(T_F). \end{cases}$$

Proof. (α) Let $f \in D(T_F)$; then both conditions (i) and (ii) of Definition 15.1 are satisfied for $f \in D(T_1)$ and for the given sequence $\{f_n \in D(T_0) : n \in \mathbb{N}\}$.

From the Dirichlet formula (9.5) of Corollary 9.4 we obtain, since $f_n(0) = 0$ for all $n \in \mathbb{N}$ on using (9.2) of Corollary 9.1,

$$(T_0(f_m - f_n), f_m - f_n) = \int_0^\infty \left\{ x |f_m''(x) - f_n''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f_m'(x) - f_n'(x)|^2 \right\} dx.$$

Thus from condition (ii) of Definition 15.1

$$\lim_{m,n \rightarrow \infty} \int_0^\infty (9x^{-1} + 8M^{-1}x) |f'_m(x) - f'_n(x)|^2 dx = 0$$

and so

$$(15.5) \quad \lim_{m,n \rightarrow \infty} \int_0^\infty |f'_m(x) - f'_n(x)|^2 dx \leq \frac{1}{k} \lim_{m,n \rightarrow \infty} \int_0^\infty (9x^{-1} + 8M^{-1}x) |f'_m(x) - f'_n(x)|^2 dx = 0$$

where $k > 0$ is defined by $k := \min\{9x^{-1} + 8M^{-1}x : x \in (0, \infty)\}$. Since

$$f_m(x) - f_n(x) = \int_0^x (f'_m(x) - f'_n(x)) dx \text{ for all } x \in [0, 1]$$

we obtain

$$|f_m(x) - f_n(x)|^2 \leq \int_0^1 |f'_m(x) - f'_n(x)|^2 dx \text{ for all } x \in [0, 1].$$

Thus from (15.5) it follows that the sequence of functions $\{f_n \in C[0, 1] : n \in \mathbb{N}\}$ is uniformly convergent to, say, the function g where $g \in C[0, 1]$, and $g(0) = 0$ since $f_n(0) = 0$ for all $n \in \mathbb{N}$.

From condition (i) of Definition 15.1 we obtain

$$\lim_{n \rightarrow \infty} \int_0^1 x |f(x) - f_n(x)|^2 dx \leq \lim_{n \rightarrow \infty} \int_0^\infty x |f(x) - f_n(x)|^2 dx = 0;$$

thus, since the sequence of functions $\{f_n \in C[0, 1] : n \in \mathbb{N}\}$ is uniformly convergent to $g \in C[0, 1]$,

$$\int_0^1 x |f(x) - g(x)|^2 dx = \lim_{n \rightarrow \infty} \int_0^1 x |f(x) - f_n(x)|^2 dx = 0;$$

also from (i) of Theorem 8.1 we have $f \in C[0, 1]$. Hence $f(x) = g(x)$ for all $x \in [0, 1]$ and $f(0) = g(0) = 0$; this gives $f(0) = 0$ for all $f \in D(T_F)$, as required for result (α) above.

(β) Since now we have shown that

$$D(T_F) = \{f \in D(T_1) : f(0) = 0\}$$

it follows from (10.3) and (10.4) that T_F is a self-adjoint extension of T_0 and that

$$T_F f = x^{-1} L_M[f] \text{ for all } f \in D(T_F).$$

These results complete the proof of (β) and thus the proof of Theorem 15.2. □

Remark 15.1. We can now extend the content of Remark 13.2 above; there is a further comparison to be made in respect of the Friedrichs extension T_F of the classical Bessel minimal operator T_0 , see (iii) of Theorem 12.2, and the results given in Theorem 15.2 for the Friedrichs extension of the fourth-order Bessel-type minimal operator T_0 .

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