Math 2311 Quiz #7

SPRING SEMESTER 2008

Name SOLUTIONS

1. Let $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$.

(a) Find all eigenvalues of A.

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (4-\lambda)\big((5-\lambda)(2-\lambda)+2\big)-1(4-2\lambda+2)-1(2-5+\lambda)$$

$$= (4-\lambda)(\lambda^2-7\lambda+12)-(6-2\lambda)-(\lambda-3)$$

$$=(4-\lambda)(\lambda-3)(\lambda-4)+2(\lambda-3)-(\lambda-3) = (4-\lambda)(\lambda-3)(\lambda-4)+(\lambda-3)$$

$$= (\lambda-3)\big((4-\lambda)(\lambda-4)+1\big) = (\lambda-3)(-\lambda^2+8\lambda-15)$$

$$= -(\lambda-3)(\lambda^2-8\lambda+15) = -(\lambda-3)(\lambda-3)(\lambda-5) = -(\lambda-3)^2(\lambda-5)$$

$\therefore$ the eigenvalues are $\lambda = 3, 3, 5$.

(b) Find a basis for the eigenspace of each eigenvalue of A.

$\underline{\lambda=3}$ | $A-3I = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix}$ so $x+y-z=0$

Take $\{(1,0,1),(0,1,1)\}$ as a basis for the eigenspace of $\lambda=3$.

$\underline{\lambda=5}$ | $A-5I = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{pmatrix} \xrightarrow[R_1+R_3]{2R_1+R_2} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{pmatrix} \xrightarrow{2R_1} \begin{pmatrix} -2 & 2 & -2 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{pmatrix}$

$\xrightarrow[-R_2+R_3]{-R_2+R_1} \begin{pmatrix} -2 & 0 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\frac{1}{2}R_2]{-\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ so $x-z=0$, $y-2z=0$

Take $\{(1,2,1)\}$ as a basis for eigenspace of $\lambda=5$

(c) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix. What is this diagonal matrix?

Take $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ so $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$